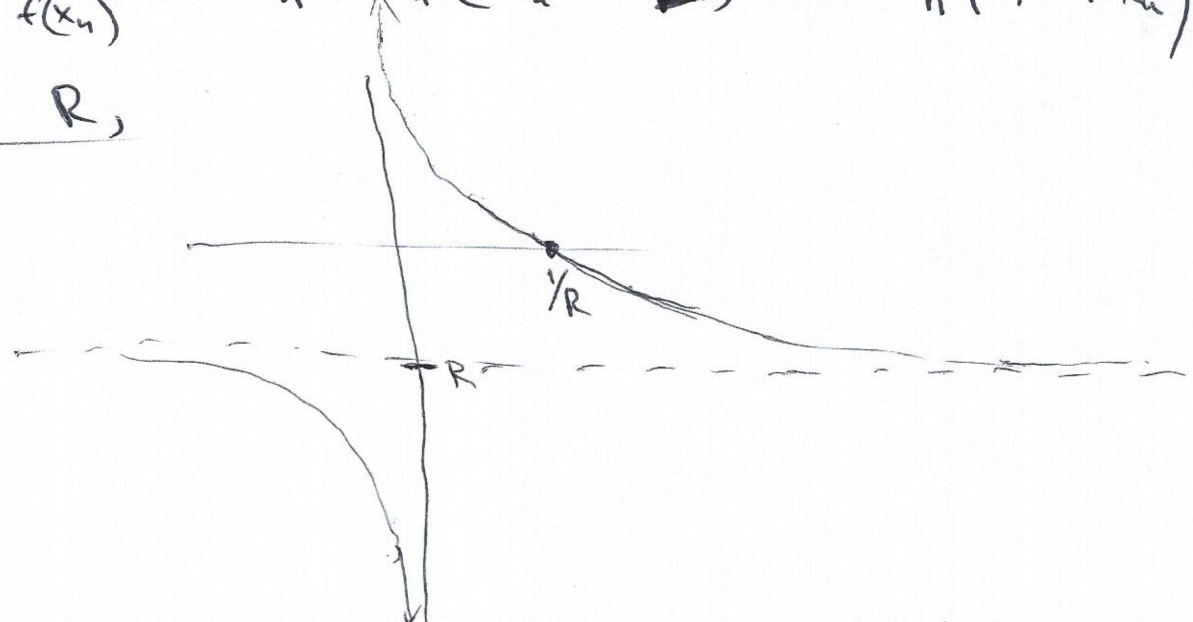
Section 3.2

#6 $f(x) = x^{-1} - R \quad f'(x) = -x^{-2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + x_n^2\left(\frac{1}{x_n} - R\right) = 2x_n(1 - Rx_n)$$

For Positive R,



Clearly, any starting pts ≤ 0 are not suitable, which can be shown by looking at f'.

For $x_0 \in (0, \frac{1}{R}]$, all starting pts are suitable, which can be seen by looking at f''.

for $x_0 \in (\frac{1}{R}, \infty)$, it can be shown $x_1 \in (-\infty, \frac{1}{R})$.

If x_1 is suitable then x_0 is also. And visa-versa.

$\Rightarrow x_1 = 2x_0 - Rx_0^2 > 0$; if x_0 is suitable

$\Rightarrow \frac{2}{R} > x_0$

3.2 #8 $p(x) = 4x^3 - 2x^2 + 3$; $p'(x) = 12x^2 - 4x$

$x_0 = -1$

$$x_1 = x_0 - \frac{p(x_0)}{p'(x_0)} \quad , \quad x_2 = x_1 - \frac{p(x_1)}{p'(x_1)}$$

. . . . calculate values.

#13 $f(x) = x^2 - 1$ $\quad f'(x) = 2x$ $\quad x_0 = 10^{10}$

$$x_{n+1} = x_n - \frac{x_n^2 - 1}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$$

Given the convexity of the function, it will approach the positive root, and thus

$x_n > 1 \quad \forall n$,

$$\Rightarrow \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) < \frac{1}{2}x_n + \frac{1}{2}$$

$$\Rightarrow x_n < \frac{1}{2}x_{n-1} + \frac{1}{2}$$

$$< \left(\frac{1}{2}\right)^2 x_{n-2} + \left(\frac{1}{2}^2 + \frac{1}{2}\right)$$

$$\vdots$$

$$< \left(\frac{1}{2}\right)^n x_0 + \sum_{i=1}^{n}\left(\frac{1}{2}\right)^i < \left(\frac{1}{2}\right)^n x_0 + 1$$

$$\Rightarrow e_n < \left(\frac{1}{2}\right)^n x_0$$

Find n s.t. $\left(\frac{1}{2}\right)^n x_0 < 10^{-8}$

3.2

#15 $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_0)}$

$$f(r) = f(x_n - e_n) = 0$$

Use Taylor $\Rightarrow f(x_n) - \dfrac{f'(\xi_n)}{1!} e_n = 0$

$$\Rightarrow f(x_n) = f'(\xi_n) e_n$$

$$\Rightarrow x_{n+1} - r = x_n - r - \frac{f'(\xi_n) e_n}{f'(x_0)}$$

$$\Rightarrow e_{n+1} = e_n\left(1 - \frac{f'(\xi_n)}{f'(x_0)}\right) \approx e_n \left(1 - \frac{f'(r)}{f'(x_0)}\right)$$

$s = 1$ $\qquad C =$

#18 $x_{n+1} = x_n - \alpha f(x_n)$

From previous problem we have $f(x_n) = f'(\xi_n) e_n$

$$\Rightarrow e_{n+1} = e_n - \alpha f'(\xi_n) e_n = e_n(1 - \alpha f'(\xi_n))$$

$$\approx e_n \underbrace{(1 - \alpha f'(r))}$$

Need to find when magnitude is < 1 for linear convergence.

3.2 #23

(a) start w/ $(x_1, x_2) = (0, 1)$

$$\begin{cases} 4x_1^2 - x_2^2 = 0 \\ 4x_1 x_2^2 - x_1 = 1 \end{cases}$$

$$F(x) = \begin{bmatrix} 4x_1^2 - x_2^2 \\ 4x_1x_2 - x_1 - 1 \end{bmatrix} \qquad F'(x) = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1x_2 \end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_1 = \vec{x}_0 - F'(\vec{x}_0)^{-1} F(\vec{x}_0)$$

Repeat 2 iterations and calculate values

(b) start w $(x, y) = (1, 1)$

$$F(\vec{x}) = \begin{bmatrix} xy^2 + x^2y + x^4 - 3 \\ x^3y^5 - 2x^5y - x^2 + 2 \end{bmatrix} \qquad F'(\vec{x}) = \begin{bmatrix} y^2 + 2xy + 4x^3 & 2yx + x^2 \\ 3x^2y^5 - 10x^4y - 2x & 5y^4x^3 - 2x^5 \end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_1 = \vec{x}_0 - F'(\vec{x}_0)^{-1} F(\vec{x}_0)$$

2 two iterations and calculate values

3.3

#5 $x_0 = 1 \quad x_1 = 2 \quad f(x_0) = 2 \quad f(x_1) = 1.5$

$$x_2 = x_1 - f(x_1)\left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] \Big\} \text{ Calculate value}$$

#7 $$x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)$$

$$= \frac{\cancel{x_n f(x_n)} - x_n f(x_{n-1}) - \cancel{f(x_n) x_n} + f(x_n) x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Efficiency can be gained in ~~secan~~ normal way of writting secant method by saving $x_n - x_{n-1}$ from previous step. Also one less multiply in each step.

3.4 ① $x_{n+1} = F(x_n) \quad x_0 \in [a, b]$

F is contractive $\Rightarrow |F(x) - F(y)| \le \lambda |x - y| \quad ; \lambda < 1$

$\Rightarrow |F(x_n) - F(s)| = |x_{n+1} - s| \le \lambda |x_n - s| \quad ; \lambda < 1$

$\Rightarrow |x_{n+1} - s| \le \lambda |x_n - s|$

$\Rightarrow |x_n - s| \le \lambda^n |x_0 - s|$

$|x_0 - s| < b - a$

3.4 #2

By MVT, $|F(x) - F(y)| = \underbrace{|F'(\xi)|}_{< 1 \text{ on } [a,b]} |x-y|$

$\Rightarrow$ holds $\forall\, x, y \in [a, b]$

$\Rightarrow$ contraction since $[a,b]$ is closed, and F' is continuous, thus $\sup\limits_{\forall x \in [a,b]} |F'(x)| < 1$

F may not have FP. Ex. $F(x) = 2$, $a = 0$, $b = 1$

#3 $F: [a, b] \to [a, b]$

suppose $F(a) \neq a$, $F(b) \neq b$ } in the case where this doesn't hold there is clearly a FP

Then $F(a) > a$, $F(b) < b$

$\Rightarrow$ $F(a) - a > 0$ and $F(b) - b < 0$

By continuity of $F(x)$, $\exists$ $k \in (a, b)$

s.t. $F(k) - k = 0 \Rightarrow F(k) = k$

For a general continuous function from $\mathbb{R} \to \mathbb{R}$ this doesn't alway hold.

ex. $f(x) = x + 50$