170B HW1

Problem 1.1

#5: (a) Since $f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(x+1)^{-n}$ and $f(0)=0$

$$\ln(x+1) = \sum_{i=1}^{n} \frac{(-1)^{i-1}(i-1)!\,(0+1)^{-i}}{i!}(x-0)^i + R_n(x)$$

$$= \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} x^i + R_n(x),$$

where $R_n(x) = \frac{1}{(n+1)!}(-1)^n n!\,(\xi+1)^{-(n+1)}(x-0)^{n+1}$

$$= \frac{(-1)^n}{n+1}(\xi+1)^{-(n+1)} x^{n+1}, \quad \xi \in [0, x].$$

or $R_n(x) = \frac{1}{n!}\int_0^x \frac{(-1)^n}{n+1}(t+1)^{-(n+1)}(x-t)^n\,dt.$

(b). For $\ln 1.5$, $x = 0.5$.

The error is actually reminder.

$$\text{Error} = |f(x) - T_n(x)| = |R_n(x)|$$

since $\xi \geq 0$

Consider $|R_n(x)| = \left|\frac{(-1)^n}{n+1}(\xi+1)^{-(n+1)} 0.5^{n+1}\right| \leq \frac{0.5^{n+1}}{n+1}$

Next find $n$ ~~$\frac{0.5^{n+1}}{n+1}$~~, s.t. $\frac{0.5^{n+1}}{n+1} \leq 10^{-8}$.

(c) Let $x = 0.6$,

Find $n$, s.t. $\frac{0.6^{n+1}}{n+1} \leq 10^{-10}$

#9. (a) $f$ is differentiable at $x$

$$\Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = \frac{1}{2}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{1}{2}\lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$$

$$= f'(x)$$

(b) Consider $f(x) = |x|$ at $x=0$

it's not differentiable,

but $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = 0$

#25 $\underline{n=2}$ $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$. use $c=0$

$$e^x = e^0 + x e^0 + \frac{x^2}{2!} e^0 + E_2(x)$$

$$= 1 + x + \frac{x^2}{2} + \frac{1}{3!} e^{(\xi)} x^3$$

$\frac{x^2}{2} + \frac{1}{3} e^{(\xi)} x^3 > 0 \quad \forall \; x > 0$, thus result clearly holds for all $x > 0$.

when $x=0$, we have $e^x = 1+x$

$\frac{d(e^x)}{dx} = e^x$, $\frac{d(1+x)}{dx} = 1$

when $x < 0$, $(e^x)' < 1 = (1+x)'$

$\Rightarrow$ Thus if equality holds at $x=0$, and $(e^x)' < (1+x)'$ $\forall\, x < 0$, then $e^x > 1+x \quad \forall\, x < 0$

Problem 1.2

#6. (a) False

(b) False

(c) False

(d) True

(e) False

#14. Look at $y$ as function of $x$, i.e $y(x)$

For $2x^3y^2 + x^2y + e^x = C$, take $\frac{d}{dx}$ on both sides.

$$\frac{d}{dx} 2x^3y^2 + \frac{d}{dx} x^2y + \frac{d}{dx} e^x = \frac{d}{dx} C$$

$$\Rightarrow 6x^2y^2 + 2x^3 2yy' + 2xy + x^2y' + e^x = 0$$

$$\Rightarrow y'(\cancel{2x^3}\, 4x^3y + x^2) = -(6x^2y^2 + 2xy + e^x)$$

$$\Rightarrow y' = -(6x^2y^2 + 2xy + e^x) / (4x^3y + x^2).$$

#20 Since $\cos x \geq 0$ on $[0, \pi/2]$,

By mean-value Theorem,

$$\int_0^{\pi/2} e^x \cos x \, dx = e^y \int_0^{\pi/2} \cos x \, dx$$

$$= e^y, \quad y \in [0, \pi/2]$$

#32. Show if $x_n = O(\alpha_n)$, then $cx_n = O(\alpha_n)$.

Since $x_n = O(\alpha_n)$,

$\exists\ \bar{C}$ and $n_0$, s.t. $|x_n| \leq \bar{C}|\alpha_n|$, $\forall n \geq n_0$.

Let $\hat{C} = \bar{C} \cdot |c|$, we have

$|cx_n| \leq |c|\bar{C}\,|\alpha_n|$, $\forall n \geq n_0$,

thus $cx_n = O(\alpha_n)$. $\square$

#36. Show if $x_n = O(\alpha_n)$, $y_n = O(\alpha_n)$

then $x_n + y_n = O(\alpha_n)$.

Since $\exists\ C_1, n_1$ s.t. $|x_n| \leq C_1|\alpha_n|$, $\forall n \geq n_1$

$\exists\ C_2, n_2$ s.t. $|y_n| \leq C_2|\alpha_n|$, $\forall n \geq n_2$

Let $C = C_1 + C_2$, $n_0 = \max\{n_1, n_2\}$,

s.t. $|x_n + y_n| \leq |x_n| + |y_n| \leq (C_1 + C_2)|\alpha_n|$, $\forall n \geq n_0$.

thus ~~$\phi$~~ $x_n + y_n$ ~~is~~ $= O(\alpha_n)$. $\square$

Problem 3.1

#2. (a) The width of interval at nth step

is $(b_0 - a_0)\, 2^{-n}$

(b) At nth step $a_n \quad c_n \quad b_n$, the root $r$ is at most at boundary $a_n$ or $b_n$

$\Rightarrow$ Distance $\leq (b_0 - a_0)\, 2^{-(n+1)}$

#7. (a) Abs error $\le (b_0-a_0)2^{-(n+1)}$

Find $n$, s.t. $(b_0-a_0)2^{-(n+1)}$

$= 2^{-(n+1)} \le 10^{-6}$

(b). Relative error $\dfrac{|r-c_n|}{|r|} \le \dfrac{(b_0-a_0)2^{-(n+1)}}{3} = \dfrac{2^{-(n+1)}}{3}$

then find $n$, s.t. $\dfrac{2^{-(n+1)}}{3} \le 10^{-6}$