Midterm 2: solutions

Problem 1

(a) p is a direction of decrease at a point x_0 if $\exists\ \sigma > 0$ s.t. $f(x_0 + \lambda p) < f(x_0)\ \forall\ \lambda \in (0, \sigma]$

(b) p is a descent direction at a point x_0 if $\left[\nabla f(x_0)\right]^t p < 0$

(c) A point x^* is a stationary pt. if $\nabla f(x^*) = 0$

(d) $x^* \in \mathbb{R}^n$ is a local minimizer of f if there exists an open neighborhood around x^* s.t. $f(x^*) \leq f(x)\ \forall\ x$ in the given neighborhood.

(e) $\nabla f(x) = \begin{bmatrix} 2x_1 - 4 \\ 4x_2 \end{bmatrix}$, $\nabla f\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ 1st order necessary cond. satisfied.

(f) $H(x) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \lambda = 2, 4$, positive definite

$\Rightarrow$ 1st order necessary and 2nd order sufficient conditions satisfied.

$\Rightarrow$ local minimizer.

$\boxed{\text{Problem 2}}$

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function x_k = newton_P2( ε, maxIter, x_0 )
% here, g(x) and H(x) are defined beforehand

k = 0;
x_k = x_0;
while ( norm( g(x), 2 ) > ε && k < maxIter )
    h = - (H(x))^-1 · g(x);
    α = 1;
    while norm(g(x_k + αh), 2) ≥ norm(g(x_k), 2)
        α = α/2;
    end
    x_k = x_k + αh;
    k = k + 1;

end
```

Problem 3

$$(u - u_n, v) = 0 \quad \forall\, v \in G$$

$$\Rightarrow (u - u_n, v_i) = 0 \quad \text{for } v_1, v_2 \dots v_n$$

$$u_n = \sum_{j=1}^{n} \alpha_j v_j$$

$$\Rightarrow \left(u - \sum_{j=1}^{n} \alpha_j v_j, v_i\right) = 0 \quad \text{for } v_1, \dots v_n$$

$$\Rightarrow \left(\sum_{j=1}^{n} \alpha_j v_j, v_i\right) = (u, v_i) \quad \text{for } v_1 \dots v_n$$

$$\Rightarrow \begin{bmatrix} (v_1, v_1) & (v_2, v_1) & \cdots & (v_n, v_1) \\ (v_1, v_2) & & & \\ \vdots & & & \\ (v_1, v_n) & \cdots & \cdots & (v_n, v_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} (u, v_1) \\ (u, v_2) \\ \vdots \\ (u, v_n) \end{bmatrix}$$

Problem 4

x	0	1	2
$f(x)$	1	2	13

(a) $p(x) = \sum_{k=0}^{n} y_k \ell_k(x) = 1 \cdot \frac{(x-1)(x-2)}{2} + \frac{2x \cdot (x-2)}{-1} + \frac{13 \cdot x \cdot (x-1)}{2}$

(b)

0	1	1	5
1	2	11	
2	13		

$P(x) = 1 + x + 5x(x-1)$

(c) $p\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + 5\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \boxed{\frac{1}{4}}$

(d) $e(x) = f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i); \ \xi_x \in (a,b)$

where $[a,b]$ is interval containing nodes.

(e) $f^{(3)}(\xi_x) = (2x^3 - x^2 + 1)^{(3)}(\xi_x)$

$(2x^3 - x^2 + 1)^{(3)} = (6x^2 - 2x)^{(2)} = (12x - 2)' = 12$

$\Rightarrow |e(x)| = \left| \left(\frac{1}{3!}\right) \cdot 12 (x)(x-1)(x-2) \right|$ on $[0,2]$

$\leq 2 \, |x(x-2)| \, |(x-1)|$ " "

and since $|x-1| \leq 1$ and $|x(x-2)| \leq 1$ on " "

≤ 2