Exam 1 Solutions

① $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(c)(x-c)^k$

$f(x) = \cos x \quad f'(x) = -\sin x \quad f''(x) = -\cos x \quad f'''(x) = \sin x$

Evaluating the function and its derivatives at 0, and noticing the pattern in the derivatives, we have:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) x^k = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

(b) $E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) x^{n+1}$, ξ is between 0 & x or alternatively I accepted the $(n+1)^{th}$ non-zero term in which case $2n$ or $2n+1$ can be used rather than $n+1$. $f^{(n+1)}$ can be found using the pattern above.

(c) $|E_n(1)| = \left|\frac{1}{(n+1)!} f^{(n+1)}(\xi) 1^{(n+1)}\right| = \left|\frac{1}{(n+1)!} f^{(n+1)}(\xi)\right|$

$\leq \frac{1}{(n+1)!}$; since $|f^{(n+1)}(\xi)| \leq 1$

find n s.t $\frac{1}{(n+1)!} < 10^{-4}$

(Also accepted above interpretation of # of terms ... i.e. $(2n+1)$)

② see HW1 #32 for multiplicative property. see HW1 #36 for additive property. Combine these terms with these properties.

P3. From the Taylor series given, we have:

$$\frac{1}{2} e_n^2 f''(\xi_n) = e_n f'(x_n) - f(x_n) \quad \xi_n \text{ between } x_n \text{ \& } r$$

By Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad e_{n+1} = \frac{1}{2} \frac{e_n^2 f''(\xi_n)}{f'(x_n)}$$

$$\approx \frac{1}{2} e_n^2 \frac{f''(x_n)}{f'(x_n)} \approx \frac{1}{2} e_n^2 \frac{f''(r)}{f'(r)} = C e_n^2$$

P4 $\quad x^0 = (0, 1)^T$

$$F_1(x_1, x_2) = 4x_1^2 - x_2^2$$
$$F_2(x_1, x_2) = 4x_1 x_2^2 - x_1 - 1$$

$$F(\vec{x}) = \begin{bmatrix} 4x_1^2 - x_2^2 \\ 4x_1 x_2^2 - x_1 - 1 \end{bmatrix} \qquad J(\vec{x}) = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1 x_2 \end{bmatrix}$$

$$\vec{X}_{n+1} = \vec{X}_n - \left(J(\vec{x}_n)\right)^{-1} F(\vec{X}_n)$$

calculate x_1 and then x_2 using the above equation, yields

$$x_1 = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 13/24 \\ 5/4 \end{bmatrix}$$

$$f(x_1) = \begin{bmatrix} 7/36 \\ -1 \end{bmatrix}$$

P5

(a) Since f' is continuous, then f is also continuous, in $[a,b]$ and f' exists on $[a,b]$. We can therefore apply the MVT: (Note, could also use Taylor Exp)

$$f(x) - f(y) = f'(\xi)(x-y), \quad \xi \text{ between } y \text{ \& } x$$

$$\Rightarrow |f(x) - f(y)| \le |f'(\xi)|\,|x-y|$$

f' is continuous on a closed interval, and bounded by 1 in magnitude $\Rightarrow \sup |f'(x)| < 1 \ \forall x \in [a,b]$

$$\Rightarrow |f(x) - f(y)| \le \alpha\,|x-y| \qquad \alpha = \sup_{x \in [a,b]} |f'(x)| < 1$$

$\Rightarrow f$ is a contraction

(b) $[a,b]$ is a closed subset of real line. F is a contractive mapping of $[a,b]$ into $[a,b]$, then by CMT F has a unique fixed point.

*Notice, this is different than the HW problem that asked to prove F had a fixed pt.