Math 292A (Fall 1997, Instructor: M. Holst)

Homework #1 (Classical PDE and functional analysis)

Handed out: 10 October 1997
Due in class: 17 October 1997

- **Problem 1.** (Separating out algebraic structures from sets we commonly work with.)

  Recall that a vector space V is simply a set V with some additional algebraic structure involving an associated field K, namely the structure given by the ten rules presented in class.

  - Taking $V = \mathbb{R}^2$, $K = \mathbb{R}$, suppose that vector-addition in $\mathbb{R}^2$ adds an extra one to each component, so that $(3,1) + (5,0) = (9,2)$ rather than $(8,1)$. With scalar-vector multiplication the usual one, which vector space rules are broken?
  - Taking $V = \mathbb{R}$, $K = \mathbb{R}$, show that the set of all positive real numbers, with $x + y$ and $c \cdot x$ redefined to the usual xy and x^c, respectively, is a vector space. What is the "zero" vector in this new space?

- **Problem 2.** (In class we proved the Lax-Milgram Theorem by changing a bilinear-form equation into a linear operator equation, and then employing the Contraction Mapping Theorem. This argument was presented quite quickly in class; reproduce it more carefully here.)

  Let H be a Hilbert space. If $a(\cdot,\cdot)$ is a bounded and coercive bilinear form on H, and $f(\cdot)$ is a bounded linear functional on H, prove that the problem

  $$\text{Find } u \in H \text{ such that} \quad a(u,v) = f(v), \quad \forall v \in H,$$

  is equivalent to the problem

  $$\text{Find } u \in H \text{ such that } Au = F \in H$$

  where $F \in H$ is related to the linear form $f(\cdot)$ through the Riesz Representation Theorem, and the bounded linear operator A is related to the bilinear form $a(\cdot,\cdot)$ through the Bounded Operator Theorem (my name in class for the result analogous to the Riesz theorem).

- **Problem 3.** (This problem will help us handle systems of differential equations later.)

  Let $H_1, \ldots, H_J$ be inner-product spaces equipped with inner-products $(\cdot,\cdot)_{H_J}$. Define $H = H_1 \times \cdots \times H_J$, so that if $x \in H$, then $x = [x_1, \ldots, x_J]$, with $x_k \in H_k$. prove that

  $$(x,y)_H = (x_1,y_1)_{H_1} + \cdots + (x_J,y_J)_{H_J}$$

  defines an inner-product on H. (I.e., show that $(\cdot,\cdot)_H$ satisfies the inner-product properties.) If $H_k, 1 \le k \le J$ are Hilbert spaces, prove that H is also a Hilbert space. Show that norm on H induced by $(\cdot,\cdot)_H$ can be written as

  $$\|x\|_H = \left(\sum_{k=1}^{J} \|x_k\|_k^2 \right)^{1/2}$$

- **Extra Credit.** (This result is useful in finite element theory; the proof can be accomplished by showing inequality in both directions, using the Cauchy-Schwarz inequality in one direction, and the properties of the sup in the other.)

  Let H be a Hilbert space, equipped with the inner-product $(\cdot,\cdot)_H$, inducing the norm $\|\cdot\|_H$. Prove that for any $x \in H$, it holds that

  $$\|x\|_H = \sup_{\|y\|=1} |(x,y)_H|.$$