AMa104

Homework #5 (Final)

(Spectral Theory, Applications, and General Review)

Handed out: 6 December 1996
Due at noon: 13 December 1996

- **Problem 1.** (Spectral Theory and the Jordan Form) We learned in class that any square matrix A has a Jordan decomposition of the form

$$A = M^{-1}JM, \qquad \text{or} \qquad J = MAM^{-1},$$

where the matrix J consists of the Jordan blocks J_k:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}, \qquad J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}.$$

The relationship between the Jordan blocks and the geometric and algebraic multiplicity of the eigenvalues of A was discussed in class.

1. If the columns of M are denoted $\{v^1, \ldots, v^m\}$, show that

$$Av^i = \lambda v^i + \nu_i v^{i-1},$$

where λ is the eigenvalue in the Jordan block affecting v^i, and ν_i is either 0 or 1.

2. Use this result to construct the Jordan form (find J and M) of

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}.$$

I.e., first find the eigenvalues of A, and determine their algebraic and geometric multiplicities (are they repeated, and how many linearly independent eigenvectors correspond to each distinct eigenvalues). At this point you can write down J; you will find that there are two distinct eigenvalues, one of which is repeated to give the third eigenvalue. The single eigenvalue gives rise to a 1×1 block J_1, and the repeated eigenvalue gives rise to a 2×2 block J_2.

Now, determine the corresponding eigenvectors; you should find only two linearly independent ones. Therefore, for the two distinct eigenvalues λ_1 and λ_2, you have determined

$$Av^1 = \lambda_1 v^1, \qquad Av^2 = \lambda_2 v^2,$$

where we can think of λ_2 as being the eigenvalue with algebraic multiplicity 2. These two vectors v^1 and v^2 form the first two columns of M. To find the third column, use the previous result and solve

$$Av^3 = \lambda_2 v^3 + v^2$$

for v^3. Finally, form M^{-1}, and verify that $J = M^{-1}AM$.

3. For a typical $k \times k$ Jordan block J_p with λ on its diagonal, define the matrix $E_k = J_k - \lambda I$, which is the zero matrix except for ones on its first superdiagonal. Show that E_k^2 is the zero matrix except for ones on its second superdiagonal, that E_k^3 is the zero matrix except for ones on its third superdiagonal, and so on, so that $E_k^k = 0$.
4. Recall that the characteristic polynomial of A has the form:

$$P_n(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)^{m_1}(\lambda_2 - \lambda)^{m_2} \cdots (\lambda_s - \lambda)^{m_s},$$

where the $n \times n$ matrix A has s distinct eigenvalues. The powers m_k represent the algebraic multiplicity of each eigenvalue λ_k. Consider now $P_n(J)$, where J is the Jordan form of A. Use the previous result about powers of blocks of J to show that $P_n(J) = 0$.
5. Use the previous result to prove the Cayley-Hamilton Theorem: *If $P_n(\lambda)$ is the characteristic polynomial of the matrix A, then $P_n(A) = 0$.*

- **Problem 2.** (Applications of Matrix Theory and ODEs)

We have discussed in class the solution of homogeneous first-order linear systems of ODEs

$$\frac{du(t)}{dt} = Au(t),$$

where A is an $n \times n$ matrix, and $u(t)$ is an $n \times 1$ vector function of time t (each component of the vector u is a real-valued function of the single variable t). The solution to this problem was seen involve the matrix exponential:

$$u(t) = e^{At}u(0) = Me^{Jt}M^{-1}u(0),$$

where $A = MJM^{-1}$ is the Jordan decomposition. Consider now the inhomogeneous case:

$$\frac{du(t)}{dt} = Au(t) + f(t). \tag{1}$$

1. Pre-multiply (1) by e^{-tA} and integrate, justifying the steps, to conclude that

$$e^{-tA}u(t) - e^{0A}u(0) = \int_0^t e^{-\tau A} f(\tau) d\tau.$$

2. Use this result to prove: *The solution to the first-order inhomogeneous system of ODEs*

$$\frac{du(t)}{dt} = Au(t) + f(t), \qquad u(0) = u^0, \quad A \in \mathbb{R}^{n \times n},$$

is given by

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-\tau)A} f(\tau) d\tau = Me^{tJ}M^{-1}u(0) + \int_0^t Me^{(t-\tau)J}M^{-1} f(\tau) d\tau.$$

3. Let $M(t)$ denote the population of Micro$oft software (predator), and let $G(t)$ denote the population of GNU software (prey). A typical competing population "predator-prey" model might take the form ($\cdot$ denotes time derivatives):

$$\dot{M}(t) = -0.4M(t) + 0.5G(t) + m(t), \qquad M(t_0) = 100,$$
$$\dot{G}(t) = -kM(t) + 0.2G(t) + g(t), \quad G(t_0) = 1000.$$

Here, k is some parameter which represents in this case how defenseless GNU software is to the predatory tactics of the predator software. The functions $m(t)$ and $g(t)$ represent the production rates of both types of software by external sources ($g(t)$ might represent GNU software producers, and $m(t)$ might represent private software companies, for example). Taking $k = 0.1, m(t) = g(t) = 0.0$, solve the (homogeneous) system of ODEs using what you know now about matrix theory. How do things behave asymptotically? Now take $k = 0.1, m(t) = 1.0, g(t) = 1000.0$, and solve the (now inhomogenous) system. Again, how do things behave asymptotically?

- **Problem 3.** (Review question on Gaussian Elimination)

  Consider the linear system

$$Ax = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = b.$$

  1. Perform Gaussian elimination to produce an upper-triangular matrix U (and a modified vector b).
  2. Perform back-substitution to find the solution x.
  3. Form the three Gauss transformations which represent the three elementary row operations you had to perform to reduce A to U. In other words, start with the identity matrix I, and perform exactly the same elementary row operation on I that you did to A to zero out element a_{21} of A. Call this matrix B_1. Similarly, form the Gauss transformations B_2 and B_3 which correspond to zeroing out a_{31} and a_{32}. Verify that in fact
$$B_3 B_2 B_1 A = U.$$
  4. Each of these (lower-triangular) Gauss transformations has a particularly simple form; show that the inverse of these transformations is also particularly simple. Write down such a Gauss transformation which zeros element $a_{ij}, i > j$, of a general $n \times n$ matrix A. Write down the analogous simple inverse matrix. (Note that since you can write the inverse down, it must exist by construction for any Gauss transformation.)
  5. You showed in previous homeworks that the product of lower triangular matrices remains lower triangular, and the inverse of a lower triangular matrix remains lower triangular. Conclude (justifying the steps) that the composite matrix $B = B_3 B_2 B_1$ is lower triangular, that its inverse exists, and is also lower triangular.
  6. Form B^{-1}, and by defining $L = B^{-1}$ construct the decomposition: $A = LU$.
  7. Factor out the diagonal entries of the matrix U to form the decomposition: $A = LD\tilde{U}$, where D is a diagonal matrix, L is a lower-triangular matrix with ones on the diagonal, and U is an upper-triangular matrix with ones on the diagonal.