AMa104

Homework #4

(Determinants and Spectral Theory)

Handed out: 19 November 1996
Due at noon: 5 December 1996

(Problems 1 through 6 cover determinant theory; Problem 6 is extra credit.)

- **Problem 1.** Compute the determinant of:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

  using:

  1. Cofactor expansion
  2. $PA = LDU$ decomposition

- **Problem 2.** Solve $Ax = b$ using Cramer's rule, where $Ax = b$ is given by

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

- **Problem 3.** Form the inverse of the matrix A of the previous problem by writing out all the cofactors, forming the cofactor matrix A_{cof} whose entries are the cofactors A_{ji}, and calculating the determinant of A.

- **Problem 4.** (Strang; review problem 4.4) Prove that the determinant is *invariant* under similarity transformations. I.e., let $A, B, M \in \mathbb{R}^{n\times n}$, and let M be nonsingular. You must show that if B is related to A through the similarity transformation $B = M^{-1}AM$, then it always holds that

$$\det(B) = \det(A).$$

- **Problem 5.** (Strang; review problem 4.5) Give a simple proof of the fact that for general $A, B \in \mathbb{R}^{n\times n}$ (even if $AB \neq BA$), it holds that

$$\det(AB) = \det(BA).$$

  The formula derived in class for the determinant of a product is the key to the proof; show that the analogous statement for sums is false, by giving a counter example which fails:

$$\det(A + B) = \det(A) + \det(B).$$

  Can you find a value of n for which this sum formula is actually a true for all $n \times n$ matrices?

- **Problem 6.** (Extra Credit) Using what we now know about determinants, give some simple proofs of the following two statements (the proofs were much harder in the case of operators...):

  1. If $\det(A) \neq 0$, then $(A^T)^{-1} = (A^{-1})^T$.
  2. If $\det(A) \neq 0$ and A is symmetric, then A^{-1} is also symmetric.

(Problems 7 through 12 cover spectral theory; Problem 12 is extra credit.)

- **Problem 7.** (Strang; problem 5.1.1) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}.$$

  Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

- **Problem 8.** (Strang; problem 5.1.10) Prove that A and A^T have the same eigenvalues. (Hint: compare their characteristic polynomials.)

- **Problem 9.** (Strang; problem 5.5.8) Write out the matrix A^H and compute $C = A^H A$ if

$$A = \begin{pmatrix} 1 & i & 0 \\ i & 0 & 1 \end{pmatrix}.$$

  How are C and C^H related? Does this hold for any $C = A^H A$?

- **Problem 10.** (Strang; problem 5.5.9) Using the matrix A from the previous problem,

  1. Find the null-space $\mathcal{N}(A)$ by solving $Ax = 0$.
  2. Show that $\mathcal{N}(A)$ is orthogonal to $\mathcal{R}(A^H)$, and *not* to $\mathcal{R}(A^T)$.

  I.e., the four fundamental subspaces in the complex setting are:

$$\mathcal{N}(A), \quad \mathcal{R}(A), \quad \mathcal{N}(A^H), \quad \mathcal{R}(A^H).$$

- **Problem 11.** (Important!) Commit to memory (and make sure you completely understand) the Table of Similarity Transformations on page 315 of Strang. To convince me you understand the material in the table, write down complete and thorough answers to the following questions:

  1. What is diagonalizability? Are all matrices diagonalizable? What is a defective matrix?
  2. What is the Jordan form, and do all matrices have one? What do the entries above the diagonal tell you?
  3. What is the Schur decomposition, and does it apply to all matrices, including defective ones? What sort of a similarity transformation is involved in the Schur decomposition? What is a unitary matrix, and how is it different from an orthogonal matrix?
  4. What is a normal matrix, and what happens to the matrices forming the Schur decomposition in this case?
  5. What is the spectral theorem? (I.e., how does the Schur decomposition simplify when the matrix is real symmetric or Hermitian?)
  6. How do the things above specialize for Skew-Hermitian matrices, and for orthogonal and unitary matrices?

- **Problem 12.** (Extra Credit) Prove that the determinant of a matrix $A \in \mathbb{R}^{n\times n}$ equals the product of the eigenvalues, and that the trace of a matrix $A \in \mathbb{R}^{n\times n}$ equals the sum of the eigenvalues. (Hint: For the determinant case, imagine you have the characteristic polynomial in factored form, and make a clever choice of the argument. For the trace case, there is a hint in problem 5.1.9 in Strang. Note that this problem is a combination of problems 5.1.8 and 5.1.9 in Strang; don't be tempted to peek in the back of the book!)