AMa104

Homework #1 (Matrices and Gaussian Elimination)

Handed out: 9 October 1996
Due in class: 17 October 1996

- **Problem 1.** (Strang 1.2.3) Describe the intersection of the three planes:

$$u + v + w + z = 6$$

$$u + w + z = 4$$

$$u + w = 2$$

all in a four-dimensional space. Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included?

- **Problem 2.** (Strang 1.4.13) The product of two lower triangular matrices is again lower triangular (similarly for the product of two upper triangular matrices). Confirm this with a 3×3 example, and then prove the result from the general definition of matrix-matrix multiplication.

- **Problem 3.** (Strang 1.5.14) Write down all six of the 3×3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices; they satisfy $PP^{-1} = I$ (they are in the same list).

- **Problem 4.** (Strang 1.6.4) If a matrix A is invertible, and $AB = AC$, prove that $B = C$. If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(i.e., A is singular), find an example such that $AB = AC$ but $B \neq C$.

- **Problem 5.** (Strang 1.6.5) If the inverse of A^2 is B, prove that the inverse of A is AB. (Thus, A is invertible when A^2 is invertible.)

- **Problem 6.** (Strang 1.6.12) Which properties of an invertible matrix A are preserved by it's inverse? (Prove your statements, possibly by giving counter examples.)
  - A is triangular
  - A is symmetric
  - A is tridiagonal
  - A has only whole number entries
  - A has only rational entries

- **Problem 7.** (Strang 1.6.14) Prove that even for the case of a rectangular matrix A, both AA^T and A^TA are always symmetric. Give an example that shows that they may not be equal, even for square matrices.

- **Problem 8.** (Strang 1.6.15) Prove that for any square matrix B, the matrix $A = B + B^T$ is always symmetric, the matrix $K = B - B^T$ is always skew-symmetric, and that B can always be written as some linear combination of A and K. Form A and K for the case when

$$B = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

- **Problem 9.** (Strang 1.6.23) If A and B are square matrices, prove that $I - AB$ is invertible if and only if $I - BA$ is invertible. (Hint: start with $B(I - AB) = (I - BA)B$.)

- **Problem 10.** (Strang 1.7.8) Solve $Ax = b = (1, 0, \ldots, 0)$ for the 10×10 Hilbert matrix A with $a_{ij} = 1/(i + j - 1)$, using any computer code (i.e., use MATLAB). Then make a small change in an entry of A or b; how do the two solutions compare?