NON-CMC SOLUTIONS OF THE EINSTEIN CONSTRAINT EQUATIONS ON COMPACT MANIFOLDS WITH APPARENT HORIZON BOUNDARIES

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ABSTRACT. In this article we continue our effort to do a systematic development of the solution theory for conformal formulations of the Einstein constraint equations on compact manifolds with boundary. By building in a natural way on our recent work in Holst and Tsogtgerel (2013), and Holst, Nagy, and Tsogtgerel (2008, 2009), and also on the work of Maxwell (2004, 2005, 2009) and Dain (2004), under reasonable assumptions on the data we prove existence of both near- and far-from-constant mean curvature solutions for a class of Robin boundary conditions commonly used in the literature for modeling black holes, with a third existence result for constant mean curvature (CMC) appearing as a special case. Dain and Maxwell addressed initial data engineering for space-times that evolve to contain black holes, determining solutions to the conformal formulation on an asymptotically Euclidean manifold in the CMC setting, with interior boundary conditions representing excised interior black hole regions. Holst and Tsogtgerel compiled the interior boundary results covered by Dain and Maxwell, and then developed general interior conditions to model the apparent horizon boundary conditions of Dain and Maxwell for compact manifolds with boundary, and subsequently proved existence of solutions to the Lichnerowicz equation on compact manifolds with such boundary conditions. This paper picks up where Holst and Tsogtgerel left off, addressing the general non-CMC case for compact manifolds with boundary. As in our previous articles, our focus here is again on low regularity data and on the interaction between different types of boundary conditions. While our work here serves primarily to extend the solution theory for the compact with boundary case, we also develop several technical tools that have potential for use with the asymptotically Euclidean case.

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1. INTRODUCTION

This article represents the second installment in a systematic development of the solution theory for conformal formulations of the Einstein constraint equations on compact manifolds with boundary. Our development began in [13] by leveraging the technical tools we had developed in [12] for both the CMC (constant mean curvature) and non-CMC cases in the simpler setting of closed manifolds. The case of compact manifolds with boundary, while more complicated than the closed case, and often viewed as simply an approximation to the more physically realistic asymptotically Euclidean case, is itself an important problem in general relativity; it is particularly important in numerical relativity, where it arises in models of Cauchy surfaces containing asymptotically flat ends and/or trapped surfaces. Moreover, various technical obstacles that arise when extending the solution theory for closed manifolds developed in [12, 20] to the case of asymptotically Euclidean manifolds have analogues in the compact with boundary case.

Our results here build on the non-CMC analysis framework from [12], and leverage a number technical tools developed in [13] for the Lichnerowicz equation on compact manifolds with boundary. The framework in [12] is particularly effective for producing existence results for the non-CMC case without using the so-called near-CMC assumptions primarily because it isolates any assumptions about the strength of the nonlinear coupling between the two equations to the global barrier construction. The overall Schauder-type fixed-point argument in [12, 20] is based entirely on topological properties of the fixed-point map generated by the constraints, and on the properties of the spaces on which the map operates. Nearly all of the required properties can be established without resorting to any type of near-CMC condition that restricts the strength of the nonlinear coupling between the two constraints. This allows one to focus entirely on the problem of constructing global barriers free of the near-CMC condition (cf. [11, 20] for the first such constructions). It is useful to note that the "near-CMC" assumption allows for the mean curvature to be non-constant, but requires that the mean curvature not vanish and be bounded by some multiple of its gradient, while the "far-from-CMC" assumption simply means that the mean curvature is free of the near-CMC hypothesis.

We began a systematic study of the case of compact manifolds with boundary immediately after the work on the closed case in [12], which developed into [13]. The complexity of treating the boundary conditions carefully in [13] led us to focus that work on the Lichnerowicz equation alone, restricting the analysis of the boundary difficulties to that equation in isolation from the momentum constraint, as it is the main source of nonlinearity in the coupled constraint system. In [5] and [18], Dain and Maxwell had addressed initial data engineering for Einstein's equations that would evolve into spacetimes containing black holes. They determined solutions to the conformal formulation on asymptotically Euclidean manifolds with interior boundary conditions in the CMC case. The interior boundary results from [5, 18] were compiled in [13], and then general interior conditions were developed in order to model the apparent horizon boundary conditions of Dain and Maxwell for compact manifolds with boundary. In [13] we then proved existence of solutions to the Lichnerowicz equation on compact manifolds with the aforementioned boundary conditions.

The first difficulty encountered in completing the program in [13] was that even basic results such as Yamabe classification of nonsmooth metrics on compact manifolds with boundary were unavailable; only the smooth case had been previously examined (by Escobar [8, 9]). In order to develop a theory that mirrors that of the closed case, Yamabe classification was first generalized in [13] to nonsmooth metrics on compact manifolds with boundary. In particular, it was shown that two conformally equivalent rough metrics could not have scalar curvatures with distinct signs. (In the case of closed manifolds, Yamabe classification of rough metrics was also unavailable, and had to be established in [12].) Other results were also extended to compact manifolds with boundary, such as conformal invariance of the Hamiltonian constraint. The analysis framework from [12] was then used in [13] to establish several existence results for a large class of problems covering a broad parameter regime, which included most of the cases relevant in practice. As in the work on the non-CMC case for closed manifolds in [12], the focus in [13] (and in this article) is on low regularity data and on the interaction between different types of boundary conditions, which had not been carefully analyzed before.

We note that the Lichnerowicz equation was considered in isolation in [13], so that the CMC case with marginally trapped surface boundary conditions (cf. §2.3) was not examined. However, the results for the Lichnerowicz equation allowed for variable coefficients in the critical nonlinear term; this allows the results for the Lichnerowicz equation from [13] to be used to build the non-CMC results in this paper without modification. We should point out that the primary reason that the Lichnerowicz equation alone was considered in [13] instead of the CMC case, is that unlike the setting of closed manifolds, the constraint equations *do not decouple* in the CMC case; this is due to the boundary conditions remaining coupled when using models of asymptotically Euclidean manifolds with apparent horizon boundaries. Therefore, the treatment of the CMC case for manifolds with boundary requires essentially the same fully coupled topological fixed-point argument as the non-CMC case. In fact, in this article our result for the CMC case is simply a special case of our near-CMC result, and is not stated separately. An unfortunate impact is that the techniques used for closed manifolds in [12, 3, 19] to lower the regularity of the metric in the CMC case a full degree below the best known rough metric result in the non-CMC case (appearing in [12]) cannot be exploited for the CMC case on compact manifolds with boundary (see also Remark 3.4). We note that Dilts [7] recently has independently obtained solutions to a similar boundary value problem to ours in this article by also using the framework and supporting tools from [13, 12] in the special case of smooth ($W^{2,p}$) metrics; however he does not account for the coupling that occurs on the boundary, and therefore does not obtain solutions with apparent horizon boundaries. He also exploits the Green's function technique developed by Maxwell in [20] for smooth metrics in $W^{2,p}$ that avoids constructing a subsolution. We have avoided using the Green's function approach here in order to develop existence results for the roughest possible class of metrics ($h_{ab} \in W^{s,p}$, $p \in (1, \infty)$, $s > 1 + \frac{3}{p}$), which will necessitate the explicit construction of subsolutions.

In this article, we push our program further by considering the conformal formulation in the non-CMC case on a compact manifold with boundary, subject to the class of boundary conditions considered for the Lichnerowicz equation alone in [13]. Under reasonable assumptions on the data, we establish existence of both near- and far-from-CMC solutions to the conformal formulation on compact manifolds with interior (Robin-type) boundary conditions similar to those in [13, 5, 18], and exterior boundary conditions that are consistent with asymptotically Euclidean decay. As mentioned above, a third result, for the CMC case, now comes simply as a special case of the near-CMC result. While near-CMC and far-from-CMC existence results have been obtained for closed manifolds [14, 15, 1, 12, 20], and near-CMC existence results have been obtained for asymptotically Euclidean manifolds [4], results for compact manifolds with boundary have required the CMC assumption. We begin the discussion right where [13] left off, combining the technical tools and results from our prior work in [12] and [13]. By focusing on compact models of an asymptotically Euclidean manifold with truncated ends and excised black hole regions, in addition to extending the compact with boundary existence theory, we believe that the techniques developed in this article will prove useful for obtaining far-from-CMC solutions to the conformal formulation in the asymptotically Euclidean case.

Outline of the paper. In §2, we present some preliminary material on notation, the conformal method, and boundary conditions, briefly summarizing the more extensive presentations of these topics in [12, 13]. In particular, in §2.1 we give a brief overview of the notation we employ for spaces, norms, and related objects; in §2.2 we summarize the conformal method; and in §2.3 we give an overview of the boundary conditions of primary interest, following closely the presentation from [13]. In §3 we give an overview of the main results, summarized as two separate theorems for the near-CMC and far-from-CMC cases, analogous to two of the three main results for the closed case developed in [12]. In §4, we develop the necessary supporting results and estimate for the momentum constraint, and in §5 we similarly develop a number of supporting results needed for treating the Hamiltonian constraint in the overall fixed-point argument. In §6, we subsequently give several distinct global barrier constructions for the near-CMC and far-from-CMC cases. Finally, in §7 we give the proofs of the two main theorems. Although many of the technical tools we need have been established in [12, 13], some additional required results are included in Appendix A.

2. Preliminary material

The results in this article leverage and then build on the analysis framework and the supporting technical tools developed in our two previous articles [12, 13], including the material contained in the appendices of both works. We have made an effort to use completely consistent notation with these two prior works, and have also endeavored to avoid as much a possible any replication of the technical tools. However, in an effort to make the paper as self-contained as possible, we will give a brief summary below of the (quite standard) notation we use throughout all three articles for Sobolev classes, norms, and other objects.

2.1. Notation and conventions.

As in [12, 13], the function spaces employed throughout the article are fractional Sobolev classes; an overview of the construction of fractional order Sobolev spaces of sections of vector and tensor bundles can be found in the Appendix of [12], based on Besov spaces and partitions of unity. The case of the sections of the trivial bundle of scalars can also be found in [10], and the case of tensors can also be found in [21]. Throughout the article we will use standard notation for such function spaces; cf. the introduction to [12] for an extensive a summary. In particular, our notation for L^p and Sobolev spaces and norms of sections of vector bundles over compact manifolds is quite standard, which we briefly summarize below.

Let $\mathcal{M}$ be an n-dimensional smooth, compact manifold with non-empty boundary $\partial\mathcal{M}$. Let ∇_a be the Levi-Civita connection associated with the metric $h_{ab} \in C^\infty(T^0_2\mathcal{M})$, that is, the unique torsion-free connection satisfying $\nabla_a h_{bc} = 0$. Here, $X(T^r_s\mathcal{M})$ denotes a particular smoothness class of sections of the (r, s)-tensor bundle associated with the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$. Let $R_{abc}{}^d$ be the Riemann tensor of the connection ∇_a, where the sign convention used in this article is $(\nabla_a\nabla_b - \nabla_b\nabla_a)v_c = R_{abc}{}^d v_d$. Denote by $R_{ab} := R_{acb}{}^c$ the Ricci tensor and by $R := R_{ab}h^{ab}$ the Ricci scalar curvature of this connection. Integration on $\mathcal{M}$ can be defined with the volume form associated with the metric h_{ab}, allowing for the construction of L^p-type norms and spaces. Given an arbitrary tensor $u^{a_1\cdots a_r}{}_{b_1\cdots b_s}$ of type $m = r + s$, we define a real-valued function measuring its magnitude at any point $x \in \mathcal{M}$ as

$$|u| := (u^{a_1\cdots b_s} u_{a_1\cdots b_s})^{1/2}. \tag{2.1}$$

The L^p-norm of an arbitrary tensor field $u^{a_1\cdots a_r}{}_{b_1\cdots b_s}$ on $\mathcal{M}$ can then be defined for any $1 \leqslant p < \infty$ and for $p = \infty$ respectively using (2.1) as follows,

$$\|u\|_p := \left(\int_{\mathcal{M}} |u|^p \, dx \right)^{1/p}, \qquad \|u\|_\infty := \text{ess} \sup_{x\in\mathcal{M}} |u|. \tag{2.2}$$

The Lebesgue spaces $L^p(T^r_s\mathcal{M})$ of sections of the (r, s)-tensor bundle, for $1 \leqslant p \leqslant \infty$ can be construction through completion of $C^\infty(T^r_s\mathcal{M})$ with respect to the norm, with the case $p = 2$ giving Hilbert space structure. Denoting covariant derivatives of tensor fields as $\nabla^k u^{a_1\cdots a_m} := \nabla_{b_1} \cdots \nabla_{b_k} u^{a_1\cdots a_m}$, where k denotes the total number of derivatives represented by the tensor indices $(b_1, \ldots, b_k)$, for any non-negative integer k and for any $1 \leqslant p \leqslant \infty$, the Sobolev norm on $C^\infty(T^r_s\mathcal{M})$ is given as follows,

$$\|u\|^p_{k,p} := \sum_{l=0}^{k} \|\nabla^l u\|^p_p. \tag{2.3}$$

The Sobolev spaces $W^{k,p}(T^r_s\mathcal{M})$ of sections of the (r, s)-tensor bundle can be constructed through completion of $C^\infty(T^r_s\mathcal{M})$ with respect to this norm. For the remainder of this paper, we let $W^{k,p} = W^{k,p}(\mathcal{M})$ and $\mathbf{W}^{k,p} = W^{k,p}(T\mathcal{M})$. The Sobolev spaces $W^{k,p}(T^r_s\mathcal{M})$ are Banach spaces, and the case $p = 2$ is a Hilbert space. We have $L^p(T^r_s\mathcal{M}) = W^{0,p}(T^s_r\mathcal{M})$ and $\|s\|_p = \|s\|_{0,p}$. See the Appendix of [12] for a more careful construction that includes real order Sobolev spaces of sections of vector bundles.

We will also need to consider Sobolev spaces of functions and tensor bundles on the boundary components of $\mathcal{M}$. If $\partial\mathcal{M} = \Sigma_1 \cup \Sigma_2$, where each Σ_1 and Σ_2 are disjoint boundary components of $\mathcal{M}$, we explicitly denote the boundary component when referring to Sobolev spaces on that component. For example, the space of scalar valued Sobolev functions on Σ_i will be denoted by $W^{k,p}(\Sigma_i)$ and the space of (r, s)-tensors by $W^{k,p}(T^r_s\Sigma_i)$. We use the following notation for the norm that defines these spaces:

$$\|u\|^p_{k,p;\Sigma_i} := \sum_{l=0}^{k} \|\nabla^l u\|^p_{p;\Sigma_i}. \tag{2.4}$$

For the boundary value problem that we consider in this paper, we will need to form new Banach spaces from old Banach spaces using the direct product. Given two Banach spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$,

$$X \times Y \text{ is a Banach space with norm } \left(\|\cdot\|^q_X + \|\cdot\|^q_Y\right)^{1/q}, \quad q \geqslant 1. \tag{2.5}$$

In particular, if Σ_i represents a boundary component of $\mathcal{M}$ for $i \in \{1, 2\}$, we will have need to consider spaces of the form

$$W^{s,p} \times W^{t,q}(\Sigma_1) \times W^{t,q}(\Sigma_2),$$

with a norm given by the sum of appropriate powers of the norms of the respective spaces.

Let C^∞_+ be the set of nonnegative smooth (scalar) functions on $\mathcal{M}$. Then we can define order cone

$$W^{s,p}_+ := \left\{ \phi \in W^{s,p} : \langle \phi, \varphi \rangle \geqslant 0 \quad \forall\, \varphi \in C^\infty_+ \right\}, \tag{2.6}$$

with respect to which the Sobolev spaces $W^{s,p} = W^{s,p}(\mathcal{M})$ are ordered Banach spaces. Here $\langle \cdot, \cdot \rangle$ represents the (unique) extension of the L^2-inner product to a bilinear form

$W^{s,p} \otimes W^{-s,p'} \to \mathbb{R}$, with $\frac{1}{p'} + \frac{1}{p} = 1$. The order relation is then $\phi \geqslant \psi$ iff $\phi - \psi \in W^{s,p}_+$. We note that this order cone is normal only for $s = 0$.

Given two ordered Banach spaces X and Y with order cones X_+, Y_+, we will have a need to define an order cone on the Banach space $X \times Y$. We define the order cone

$$(X \times Y)_+ = \{(x, y) \in X \times Y \ : \ x \in X_+, \ y \in Y_+\}. \tag{2.7}$$

In particular, if $\mathcal{M}$ is a manifold with boundary $\partial\mathcal{M} = \Sigma_1 \cup \Sigma_2$, then we may define an order cone on $W^{s,p}(\mathcal{M}) \times W^{t,q}(\Sigma_1) \times W^{t,q}(\Sigma_2)$ using definition (2.6). See Appendix of [12], where the key ideas of ordered Banach spaces are reviewed.

2.2. **The Einstein Constraint Equations and the Conformal Formulation.** We give a quick overview of the Einstein constraint equations in general relativity, and then define weak formulations that are fundamental to both solution theory and the development of approximation theory, following closely [12, 13].

Let $(M, g_{\mu\nu})$ be a 4-dimensional spacetime, that is, M is a 4-dimensional, smooth manifold, and $g_{\mu\nu}$ is a smooth, Lorentzian metric on M with signature $(-, +, +, +)$. Let ∇_μ be the Levi-Civita connection associated with the metric $g_{\mu\nu}$. The Einstein equation is

$$G_{\mu\nu} = \kappa T_{\mu\nu},$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor, and $\kappa = 8\pi G/c^4$, with G the gravitation constant and c the speed of light. The Ricci tensor is $R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma}$ and $R = R_{\mu\nu} g^{\mu\nu}$ is the Ricci scalar, where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, that is $g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu{}^\nu$. The Riemann tensor is defined by $R_{\mu\nu\sigma}{}^{\rho} w_\rho = \big(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\big) w_\sigma$, where w_μ is any 1-form on M. The stress energy tensor $T_{\mu\nu}$ is assumed to be symmetric and to satisfy the condition $\nabla_\mu T^{\mu\nu} = 0$ and the **dominant energy condition**, that is, the vector $-T^{\mu\nu} v_\nu$ is timelike and future-directed, where v^μ is any timelike and future-directed vector field. In this section Greek indices μ, ν, σ, ρ denote abstract spacetime indices, that is, tensorial character on the 4-dimensional manifold M. They are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively. Latin indices a, b, c, d will denote tensorial character on a 3-dimensional manifold.

The map $t : M \to \mathbb{R}$ is a **time function** iff the function t is differentiable and the vector field $-\nabla^\mu t$ is a timelike, future-directed vector field on M. Introduce the hypersurface $\mathcal{M} := \{x \in M \, : \, t(x) = 0\}$, and denote by n_μ the unit 1-form orthogonal to $\mathcal{M}$. By definition of $\mathcal{M}$ the form n_μ can be expressed as $n_\mu = -\alpha \, \nabla_\mu t$, where α, called the lapse function, is the positive function such that $n_\mu n_\nu \, g^{\mu\nu} = -1$. Let $\hat{h}_{\mu\nu}$ and $\hat{k}_{\mu\nu}$ be the first and second fundamental forms of $\mathcal{M}$, that is,

$$\hat{h}_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu, \qquad \hat{k}_{\mu\nu} := -\hat{h}_\mu{}^\sigma \nabla_\sigma n_\nu.$$

The Einstein constraint equations on $\mathcal{M}$ are given by

$$\big(G_{\mu\nu} - \kappa T_{\mu\nu}\big) \, n^\nu = 0.$$

A well known calculation allows us to express these equations involving tensors on M as equations involving *intrinsic* tensors on $\mathcal{M}$. The result is the following equations,

$$^3\hat{R} + \hat{k}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \tag{2.8}$$

$$\hat{D}^a \hat{k} - \hat{D}_b \hat{k}^{ab} + \kappa \hat{\jmath}^a = 0, \tag{2.9}$$

where tensors $\hat{h}_{ab}$, $\hat{k}_{ab}$, $\hat{\jmath}_a$ and $\hat{\rho}$ on a 3-dimensional manifold are the pull-backs on $\mathcal{M}$ of the tensors $\hat{h}_{\mu\nu}$, $\hat{k}_{\mu\nu}$, $\hat{\jmath}_\mu$ and $\hat{\rho}$ on the 4-dimensional manifold M. We have introduced the energy density $\hat{\rho} := n_\mu n_\mu T^{\mu\nu}$ and the momentum current density $\hat{\jmath}_\mu := -\hat{h}_{\mu\nu} n_\sigma T^{\nu\sigma}$.

We have denoted by $\hat{D}_a$ the Levi-Civita connection associated to $\hat{h}_{ab}$, so $(\mathcal{M}, \hat{h}_{ab})$ is a 3-dimensional Riemannian manifold, with $\hat{h}_{ab}$ having signature $(+,+,+)$, and we use the notation $\hat{h}^{ab}$ for the inverse of the metric $\hat{h}_{ab}$. Indices have been raised and lowered with $\hat{h}^{ab}$ and $\hat{h}_{ab}$, respectively. We have also denoted by ${}^3\hat{R}$ the Ricci scalar curvature of the metric $\hat{h}_{ab}$. Finally, recall that the constraint equations (2.8)-(2.9) are indeed equations on $\hat{h}_{ab}$ and $\hat{k}_{ab}$ due to the matter fields satisfying the energy condition $-\hat{\rho}^2 + \hat{\jmath}_a\hat{\jmath}^a \leqslant 0$ (with strict inequality holding at points on $\mathcal{M}$ where $\hat{\rho} \neq 0$; see [22]), which is implied by the dominant energy condition on the stress-energy tensor $T^{\mu\nu}$ in spacetime.

The Conformal Formulation. Let ϕ denote a positive scalar field on $\mathcal{M}$, and decompose the extrinsic curvature tensor $\hat{k}_{ab} = \hat{l}_{ab} + \frac{1}{3}\hat{h}_{ab}\hat{\tau}$, where $\hat{\tau} := \hat{k}_{ab}\hat{h}^{ab}$ is the trace and then $\hat{l}_{ab}$ is the traceless part of the extrinsic curvature tensor. Then, introduce the following conformal re-scaling:

$$\begin{aligned} \hat{h}_{ab} &=: \phi^4\, h_{ab}, \quad \hat{l}^{ab} =: \phi^{-10}\, l^{ab}, \quad \hat{\tau} =: \tau, \\ \hat{\jmath}^a &=: \phi^{-10}\, j^a, \quad \hat{\rho} =: \phi^{-8}\, \rho. \end{aligned} \tag{2.10}$$

We have introduced the Riemannian metric h_{ab} on the 3-dimensional manifold $\mathcal{M}$, which determines the Levi-Civita connection D_a, and so we have that $D_a h_{bc} = 0$. We have also introduced the symmetric, traceless tensor l_{ab}, and the non-physical matter sources j^a and ρ. The different powers of the conformal re-scaling above are carefully chosen so that the constraint equations (2.8)-(2.9) transform into the following equations

$$-8\Delta\phi + {}^3R\phi + \tfrac{2}{3}\tau^2\phi^5 - l_{ab}l^{ab}\phi^{-7} - 2\kappa\rho\phi^{-3} = 0, \tag{2.11}$$

$$-D_b l^{ab} + \tfrac{2}{3}\phi^6 D^a\tau + \kappa j^a = 0, \tag{2.12}$$

where in equation above, and from now on, indices of unhatted fields are raised and lowered with h^{ab} and h_{ab} respectively. We have also introduced the **Laplace-Beltrami operator** with respect to the metric h_{ab}, acting on smooth scalar fields; it is defined as follows

$$\Delta\phi := h^{ab}D_aD_b\phi. \tag{2.13}$$

Equations (2.11)-(2.12) can be obtained by a straightforward albeit long computation. In order to perform this calculation it is useful to recall that both $\hat{D}_a$ and D_a are connections on the manifold $\mathcal{M}$, and so they differ on a tensor field $C_{ab}{}^c$, which can be computed explicitly in terms of ϕ, and has the form

$$C_{ab}{}^c = 4\delta_{(a}{}^c D_{b)}\ln(\phi) - 2h_{ab}h^{cd}D_d\ln(\phi).$$

We remark that the power four on the re-scaling of the metric $\hat{h}_{ab}$ and $\mathcal{M}$ being 3-dimensional imply that ${}^3\hat{R} = \phi^{-5}({}^3R\phi - 8\Delta\phi)$, or in other words, that ϕ satisfies the **Yamabe-type problem**:

$$-8\Delta\phi + {}^3R\phi - {}^3\hat{R}\phi^5 = 0, \quad \phi > 0, \tag{2.14}$$

where ${}^3\hat{R}$ denotes the scalar curvature corresponding to the physical metric $\hat{h}_{ab} = \phi^4 h_{ab}$. Note that for any other power in the re-scaling, terms proportional to $h^{ab}(D_a\phi)(D_b\phi)/\phi^2$ appear in the transformation. The set of all metrics on a compact manifold can be classified into the three disjoint Yamabe classes $\mathcal{Y}^+(\mathcal{M})$, $\mathcal{Y}^0(\mathcal{M})$, and $\mathcal{Y}^-(\mathcal{M})$, corresponding to whether one can conformally transform the metric into a metric with strictly positive, zero, or strictly negative scalar curvature, respectively, cf. [16] (See also the Appendix of [13]). We note that the **Yamabe problem** is to determine, for a given metric h_{ab}, whether there exists a conformal transformation ϕ solving (2.14) such that ${}^3\hat{R} = \text{const}$. Arguments similar to those above for ϕ force the power negative ten on the re-scaling of the tensor $\hat{l}^{ab}$ and $\hat{\jmath}^a$, so terms proportional to $(D_a\phi)/\phi$ cancel out in (2.12). Finally, the ratio between the conformal re-scaling powers of $\hat{\rho}$ and $\hat{\jmath}^a$ is chosen such that the inequality $-\rho^2 + h_{ab}j^aj^b \leqslant 0$ implies the inequality $-\hat{\rho}^2 + \hat{h}_{ab}\hat{\jmath}^a\hat{\jmath}^b \leqslant 0$. For a complete discussion of all possible choices of re-scaling powers, see the Appendix of [12].

There is one more step to convert the original constraint equation (2.8)-(2.9) into a determined elliptic system of equations. This step is the following: Decompose the symmetric, traceless tensor l_{ab} into a divergence-free part σ_{ab}, and the symmetrized and traceless gradient of a vector, that is, $l^{ab} =: \sigma^{ab} + (\mathcal{L}w)^{ab}$, where $D_a\sigma^{ab} = 0$ and we have introduced the **conformal Killing operator** $\mathcal{L}$ acting on smooth vector fields and defined as follows

$$(\mathcal{L}w)^{ab} := D^aw^b + D^bw^a - \tfrac{2}{3}(D_cw^c)h^{ab}. \tag{2.15}$$

Therefore, the constraint equations (2.8)-(2.9) are transformed by the conformal re-scaling into the following equations

$$-8\Delta\phi + {}^3R\phi + \tfrac{2}{3}\tau^2\phi^5 - [\sigma_{ab} + (\mathcal{L}w)_{ab}][\sigma^{ab} + (\mathcal{L}w)^{ab}]\phi^{-7} - 2\kappa\rho\phi^{-3} = 0, \tag{2.16}$$

$$-D_b(\mathcal{L}w)^{ab} + \tfrac{2}{3}\phi^6D^a\tau + \kappa j^a = 0. \tag{2.17}$$

In the next section we interpret these equations above as partial differential equations for the scalar field ϕ and the vector field w^a, while the rest of the fields are considered as given fields. Given a solution ϕ and w^a of equations (2.16)-(2.17), the physical metric $\hat{h}_{ab}$ and extrinsic curvature $\hat{k}^{ab}$ of the hypersurface $\mathcal{M}$ are given by

$$\hat{h}_{ab} = \phi^4h_{ab}, \qquad \hat{k}^{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab}] + \tfrac{1}{3}\,\phi^{-4}\tau h^{ab},$$

while the matter fields are given by Eq (2.10).

From this point forward, for simplicity we will denote the Levi-Civita connection of the metric h_{ab} on the 3-dimensional manifold $\mathcal{M}$ as ∇_a rather than D_a, and the Ricci scalar of h_{ab} will be denoted by R instead of ${}^3R$. Let $(\mathcal{M}, h)$ be a 3-dimensional Riemannian manifold, where $\mathcal{M}$ is a smooth, compact manifold with non-empty boundary $\partial\mathcal{M}$, and $h \in C^\infty(T^0_2\mathcal{M})$ is a positive definite metric. With the shorthand notation $C^\infty = C^\infty(\mathcal{M}\times\mathbb{R})$ and $\boldsymbol{C}^\infty = C^\infty(T\mathcal{M})$, let $L : C^\infty \to C^\infty$ and $I\!\!L : \boldsymbol{C}^\infty \to \boldsymbol{C}^\infty$ be the operators with actions on $\phi \in C^\infty$ and $\boldsymbol{w} \in \boldsymbol{C}^\infty$ given by

$$L\phi := -\Delta\phi, \tag{2.18}$$

$$(I\!\!L\boldsymbol{w})^a := -\nabla_b(\mathcal{L}\boldsymbol{w})^{ab}, \tag{2.19}$$

where Δ denotes the Laplace-Beltrami operator defined in (2.13), and where $\mathcal{L}$ denotes the conformal Killing operator defined in (2.15). We will also use the index-free notation $I\!\!L\boldsymbol{w}$ and $\mathcal{L}\boldsymbol{w}$.

The freely specifiable functions of the problem are a scalar function τ, interpreted as the trace of the physical extrinsic curvature; a symmetric, traceless, and divergence-free, contravariant, two index tensor σ; the non-physical energy density ρ and the non-physical momentum current density vector $\boldsymbol{j}$ subject to the requirement $-\rho^2 + \boldsymbol{j}\cdot\boldsymbol{j} \leqslant 0$. The term non-physical refers here to a conformal rescaled field, while physical refers to a conformally non-rescaled term. The requirement on ρ and $\boldsymbol{j}$ mentioned above and the particular conformal rescaling used in the semi-decoupled decomposition imply that the same inequality is satisfied by the physical energy and momentum current densities. This is a necessary condition (although not sufficient) in order that the matter sources in spacetime satisfy the dominant energy condition. The definition of various energy conditions can be found in [22, page 219]. Introduce the non-linear operators $f : C^\infty \times \boldsymbol{C}^\infty \to C^\infty$ and

$\mathbb{F} : C^\infty \to \boldsymbol{C}^\infty$ given by

$$f(\phi, \boldsymbol{w}) = a_\tau \phi^5 + a_R \phi - a_\rho \phi^{-3} - a_w \phi^{-7}, \quad \text{and} \quad \mathbb{F}(\phi) = \boldsymbol{b}_\tau \, \phi^6 + \boldsymbol{b}_j,$$

where the coefficient functions are defined as follows

$$\begin{aligned} a_\tau &:= \tfrac{1}{12}\tau^2, & a_R &:= \tfrac{1}{8}R, & a_\rho &:= \tfrac{\kappa}{4}\rho, \\ a_{\boldsymbol{w}} &:= \tfrac{1}{8}(\sigma + \mathcal{L}\boldsymbol{w})_{ab}(\sigma + \mathcal{L}\boldsymbol{w})^{ab}, & \boldsymbol{b}_\tau^a &:= \tfrac{2}{3}\nabla^a \tau, & \boldsymbol{b}_j^a &:= \kappa j^a. \end{aligned} \tag{2.20}$$

Notice that the scalar coefficients a_τ, a_w, and a_ρ are non-negative, while there is no sign restriction on a_R.

With these notations, the **classical formulation** (or the strong formulation) of the coupled Einstein constraint equations reads as: Given the freely specifiable smooth functions τ, σ, ρ, and $\boldsymbol{j}$ in $\mathcal{M}$, find a scalar field ϕ and a vector field $\boldsymbol{w}$ in $\mathcal{M}$ solution of the system

$$L\phi + f(\phi, \boldsymbol{w}) = 0 \qquad \text{and} \qquad \mathbb{L}\boldsymbol{w} + \mathbb{F}(\phi) = 0 \qquad \text{in } \mathcal{M}. \tag{2.21}$$

2.3. **Boundary Conditions.** Following [13], the two main types of boundary conditions that we consider in this paper are exterior boundary conditions and interior boundary conditions. Exterior boundary conditions occur when the asymptotic ends of a manifold are removed and one needs to impose the correct decay conditions. The interior boundary conditions arise when singularities are excised from the manifold and then conditions are imposed on the boundary so that the region is either a trapped or marginally trapped surface (cf. [13, 5, 18]). We let $\partial\mathcal{M} = \Sigma_I \cup \Sigma_E$, where Σ_I and Σ_E denote the interior and exterior boundary, respectively. Moreover, we assume that the interior and exterior boundaries are the union of finitely many disjoint components:

$$\Sigma_I = \bigcup_{i=1}^{M} \Sigma_i \quad \text{and} \quad \Sigma_E = \bigcup_{i=M+1}^{N} \Sigma_i.$$

On a 3-dimensional manifold, the exterior boundary condition for the conformal factor ϕ is that it must satisfy

$$\partial_r \phi + \frac{1}{r}(\phi - 1) = \mathcal{O}(r^{-3}), \tag{2.22}$$

where r is the flat-space radial coordinate. This condition is chosen to ensure that the conformal data accurately models initial data for the asymptotically Euclidean case. More specifically, this condition is chosen to ensure the correct decay estimates for ϕ and to give accurate values for the total energy [13, 23].

The solution $\boldsymbol{w}$ to the momentum constraint must also satisfy certain Robin type conditions to accurately model asymptotically Euclidean data. In [23], the vector Robin condition

$$(\mathcal{L}\boldsymbol{w})^{bc}\nu_c \left(\delta^a_b - \frac{1}{2}\nu^a \nu_b\right) + \frac{6}{7r}\boldsymbol{w}^b \left(\delta^a_b - \frac{1}{8}\nu^a \nu_b\right) = \mathcal{O}(r^{-3}) \tag{2.23}$$

is given for a 3-dimensional asymptotically Euclidean manifold. Here ν is the outward pointing normal vector field to Σ_E with respect to the non physical metric g and r is the radius of a large spherical domain. Taking the right hand side in the above expression to be zero, and using the fact that $(\delta^a_b + \nu^a \nu_b)$ is the inverse of $(\delta^a_b - \frac{1}{2}\nu^a \nu_b)$, we can rewrite (2.23) as

$$(\mathcal{L}\boldsymbol{w})^{ab}\nu_b + \frac{6}{7r}\boldsymbol{w}^b \left(\delta^a_b + \frac{3}{4}\nu^a \nu_b\right) = 0. \tag{2.24}$$

Therefore, we impose the general vector Robin condition on the momentum constraint for the exterior boundary:

$$(\mathcal{L}\boldsymbol{w})^{ab}\nu_b + C^a_b \boldsymbol{w}^b = 0, \tag{2.25}$$

which is general enough to include (2.24).

There are many different interior boundary conditions that have been imposed in the literature. For the sake of completeness, we will give a brief review of the boundary conditions mentioned in [13], where Holst and Tsogtgerel compile a complete list of interior conditions modeling marginally trapped surfaces. While the following interior boundary conditions are presented for n-dimensional manifolds, we will focus on the boundary condition given in (2.33) in the 3-dimensional case.

Let Σ_i denote an interior boundary component and let $\hat{\nu}$ be the outward pointing normal vector with respect to the physical metric $\hat{g}$. The expansion scalars corresponding to the outgoing and ingoing future directed geodesics to Σ_i are then given by

$$\hat{\theta}_{\pm} = \mp(n-1)\hat{H} + \mathrm{tr}_{\hat{g}}\hat{K} - \hat{K}(\hat{\nu},\hat{\nu}), \tag{2.26}$$

where $(n-1)\hat{H} = \mathrm{div}_{\hat{g}}\hat{\nu}$ is the mean extrinsic curvature of Σ_i. The surface Σ_i is called a trapped surface if $\hat{\theta}_{\pm} < 0$ and a marginally trapped surface if $\hat{\theta}_{\pm} \leqslant 0$. See [5, 18, 22] for details.

Writing the expansion scalars in terms of the conformal quantities and setting $\overline{q} = \frac{n}{n-2}$ as in [13], we have that

$$\hat{\theta}_{\pm} = \mp(n-1)\phi^{-\overline{q}}\left(\frac{2}{n-2}\partial_\nu\phi + H\phi\right) + (n-1)\tau - \phi^{-2\overline{q}}S(\nu,\nu), \tag{2.27}$$

where $\nu = \phi^{\overline{q}-1}\hat{\nu}$ is the unit normal with respect to g, and $\partial_\nu\phi$ is the derivative of ϕ along ν. In (2.27), we have also used that fact that the mean curvature $\hat{H}$ satisfies

$$\hat{H} = \phi^{-\overline{q}}\left(\frac{2}{n-2}\partial_\nu\phi + H\phi\right), \tag{2.28}$$

where H is the mean curvature with respect to g.

As in [13], we let $\theta_+ = \phi^{\overline{q}-e}\hat{\theta_+}$ be the specified, scaled expansion factor for some $e \in \mathbb{R}$, and obtain

$$\frac{2(n-1)}{n-2}\partial_\nu\phi + (n-1)H\phi - (n-1)\tau\phi^{\overline{q}} + S(\nu,\nu)\phi^{-\overline{q}} + \theta_+\phi^e = 0. \tag{2.29}$$

Similarly, by specifying $\theta_- = \phi^{\overline{q}-e}\hat{\theta_-}$ for some $e \in \mathbb{R}$, we obtain

$$\frac{2(n-1)}{n-2}\partial_\nu\phi + (n-1)H\phi + (n-1)\tau\phi^{\overline{q}} - S(\nu,\nu)\phi^{-\overline{q}} - \theta_-\phi^e = 0. \tag{2.30}$$

In (2.29), θ_- remains unspecified, and in (2.30), θ_+ is unspecified. So in either case, to ensure that $\theta_{\pm} \leqslant 0$, conditions have to be imposed on either τ or S in order to ensure that the unspecified expansion factor satisfies the marginally trapped surface condition. In [13], Holst and Tsogtgerel developed general conditions on the initial data to ensure that the trapped surface conditions are satisfied. In the case of (2.29) with specified θ_+, one assumes that ϕ_- satisfies $\phi_- \leqslant \phi$, $\tau \leqslant 0$ on Σ_I, $e = -\overline{q}$, and requires either that

$$S(\nu,\nu) \leqslant 0, \tag{2.31}$$

$$2|S(\nu,\nu)| + |\theta_+| \leqslant 2(n-1)|\tau|\phi_-^{2\overline{q}},$$

or that

$$S(\nu,\nu) \geqslant 0, \qquad (2.32)$$
$$|\theta_+| \leqslant 2S(\nu,\nu) + 2(n-1)|\tau|\phi_-^{2\overline{q}}.$$

In the case of (2.30), where θ_- is specified, one assumes that $e = \overline{q}$, $S(\nu,\nu) \geqslant 0$, and

$$2(n-1)\tau + |\theta_-| \leqslant 2S(\nu,\nu)\phi_+^{-2\overline{q}} \quad \text{on } \Sigma_I, \qquad (2.33)$$

where ϕ_+ satisfies $\phi_+ \geqslant \phi$. In [13], the authors assume that $\tau \geqslant 0$ on Σ_I. However, in Theorem 3.2 we assume that $\tau \leqslant 0$ on Σ_I and in Theorem 3.3 we only assume that τ satisfies (2.33).

Conditions (2.29)-(2.30) are nonlinear, Robin conditions on the inner boundary components for the conformal factor ϕ. Equations (2.31)-(2.32) constitute Robin boundary conditions on the inner boundary components for the momentum constraint, which we will discuss in more detail in the next section when we formulate the our boundary value problem. See [13, 5, 18] for a complete discussion of the boundary conditions stated above.

3. Overview of the Main Results

The main results for this paper concern the existence of far-from-CMC and near-CMC solutions to the conformal formulation of the Einstein constraint equations on a compact, 3-dimensional manifold $\mathcal{M}$ with boundary Σ. We assume that

$$\partial\mathcal{M} = \Sigma_I \cup \Sigma_E, \qquad (3.1)$$

where the boundary segments Σ_I and Σ_E are decomposed further into finite segments as

$$\Sigma_I = \bigcup_{i=1}^{M} \Sigma_i, \quad \text{and} \quad \Sigma_E = \bigcup_{i=M+1}^{N} \Sigma_i, \ (M<N), \quad \text{with} \quad \Sigma_i \cap \Sigma_j = \emptyset \quad \text{if } i \neq j.$$

We show that under certain conditions, the following system

$$-\Delta\phi + a_R\phi + a_\tau\phi^5 - a_{\mathbf{w}}\phi^7 - a_\rho\phi^{-3} = 0, \qquad (3.2)$$
$$I\!L\boldsymbol{w} + b_\tau\phi^6 + \mathbf{b}_j = 0, \qquad (3.3)$$

subject to the boundary conditions

$$\partial_\nu\phi + \frac{1}{2}H\phi + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)\phi^3 - \frac{1}{4}S(\nu,\nu)\phi^{-3} = 0, \text{ on } \Sigma_I, \qquad (3.4)$$
$$(\mathcal{L}\boldsymbol{w})^{ab}\nu_b = V^a, \text{ on } \Sigma_I, \qquad (3.5)$$
$$\partial_\nu\phi + c\phi = g, \text{ on } \Sigma_E, \qquad (3.6)$$
$$(\mathcal{L}\boldsymbol{w})^{ab}\nu_b + C^a_b\boldsymbol{w}^b = 0, \text{ on } \Sigma_E, \qquad (3.7)$$

has a solution. In (3.4)-(3.7) we assume that

$$c > 0, \quad g > 0 \quad \text{and} \quad g = \delta(c + \mathcal{O}(R^{-3})), \quad \delta > 0, \qquad (3.8)$$
$$\exists\alpha > 0 \quad \text{such that} \quad \int_{\partial\mathcal{M}} C_{ab}V^aV^b \geqslant \alpha|V|_{L^2(\partial\mathcal{M})}, \quad \forall V \in \mathbf{L}^2.$$

The coefficients $a_R, a_\tau, a_{\boldsymbol{w}}$ and a_ρ are defined in (2.20), and H and θ_- are the mean extrinsic curvature for the boundary and expansion factor for the incoming null geodesics. The operators $\mathcal{L}$ and $I\!L$ are the conformal Killing operator and its divergence, defined in (2.18).

Remark 3.1. *In* (3.5)*, the vector* $\mathbf{V}$ *will be chosen so that*

$$V^a\nu_a = (2\tau + |\theta_-|/2)B^6 - \sigma(\nu,\nu), \tag{3.9}$$

where B *is a positive function. The condition implies that* $S(\nu,\nu) = (2\tau + |\theta_-|/2)B^6$*, which is similar to the marginally trapped surface condition* (2.33)*. The general approach is to solve* (3.1) *with boundary conditions* (3.4)-(3.7)*, and then argue that we can choose* $B > \|\phi\|_\infty$ *sufficiently large so that the marginally trapped surface condition is satisfied.*

Now that we have clarified the statement of our problem, we can formally state our main results as the following two theorems. Our first main result (Theorem 3.2 below) covers both the Near-CMC and CMC cases.

Theorem 3.2. (Near-CMC and CMC $W^{s,p}$ solutions, $p \in (1,\infty)$, $s \in (1+\frac{3}{p},\infty)$) *Let* $(\mathcal{M}, h_{ab})$ *be a* 3*-dimensional, compact Riemannian manifold with boundary satisfying the conditions* (3.1)*. Let* $h_{ab} \in W^{s,p}(T^0_2\mathcal{M})$*, where* $p \in (1,\infty)$ *and* $s \in (1+\frac{3}{p},\infty)$ *are given. With* $d = s - \frac{3}{p}$*, select* q *and* e *to satisfy:*

- $\frac{1}{q} \in (0,1) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}] \cap [\frac{1-d}{3}, \frac{3+sp}{6p})$,
- $e \in [1,\infty) \cap [s-1, s] \cap [\frac{3}{q} + d - 1, \frac{3}{q} + d]$.

Let Eq (3.8) *hold and assume the data satisfies:*

- $\theta_- \in W^{s-1-\frac{1}{p},p}(\Sigma_I)$,
- $c, g \in W^{s-1-\frac{1}{p},p}(\Sigma_E)$,
- $C^a_b \in W^{e-1-\frac{1}{q},q}(T^1_1\Sigma_E)$,
- $\mathbf{V} \in \mathbf{W}^{e-1,q}$, $V^a\nu_a = (2\tau + |\theta_-|/2)B^6 - \sigma(\nu,\nu)$,
- $\tau \in W^{s-1,p}$ *if* $e \geqslant 2$*, and* $\tau \in W^{1,z} \cap L^\infty$ *otherwise, with* $z = \frac{3p}{3+\max\{0,2-s\}p}$,
- $(4\tau^\vee + |\theta|^\vee) > 0$ *on* Σ_I,
- $\sigma \in W^{e-1,q}$,
- $\rho \in W^{s-2,p}_+$,
- $\mathbf{j} \in \mathbf{W}^{e-2,q}$.

In addition, assume that $a_\tau^\vee > \mathtt{k}_1$*, where* a_τ *is defined in* (2.20)*, and where*

$$\mathtt{k}_1 = 2C^2(\|\mathbf{b}_\tau\|_z)^2, \tag{3.10}$$

with C *is a positive constant. If at least one of the following additional conditions hold:*

(a) $\rho^\vee > 0$*,*
(b) $a_\sigma^\vee$ *is sufficiently large,*

where a_σ *is defined in* (2.20)*, then there exists a solution* $\phi \in W^{s,p}$ *with* $\phi > 0$ *and* $\mathbf{w} \in \mathbf{W}^{e,q}$ *to equations* (3.2)-(3.3) *with boundary conditions* (3.4)-(3.8)*. Moreover, with an additional smallness assumption on* τ *on* Σ_I*, the marginally trapped surface boundary condition in* (2.33) *is satisfied.*

Our second main result (Theorem 3.3 below) covers three distinct non-CMC cases for which both the Near-CMC and CMC assumptions are violated. Case (a) in Theorem 3.3 puts no restrictions on the size of τ or $D\tau$, but requires a smallness condition on the exterior boundary data that leads to a departure of the model from faithfully approximating asymptotically Euclidean boundaries, while preserving the trapped surface conditions. Cases (b) and (c) remove the smallness condition on the exterior boundary to faithfully preserve the asymptotically Euclidean model by introducing smallness conditions on τ and/or $D\tau$ to satisfy the trapped surface conditions, yet still allow for violation of the near-CMC condition.

Theorem 3.3. (Non-CMC $W^{s,p}$ solutions, $p \in (1,\infty)$, $s \in (1+\frac{3}{p},\infty)$)
Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact Riemannian manifold with boundary satisfying the conditions (3.1). *Let $h_{ab} \in W^{s,p}(T^0_2\mathcal{M})$ and be in $\mathcal{Y}^+$, where $p \in (1,\infty)$ and $s \in (1+\frac{3}{p},\infty)$ are given. With $d = s - \frac{3}{p}$, select q and e to satisfy:*

- $\frac{1}{q} \in (0,1) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}] \cap [\frac{1-d}{3}, \frac{3+sp}{6p})$,
- $e \in [1,\infty) \cap [s-1, s] \cap [\frac{3}{q} + d - 1, \frac{3}{q} + d]$.

Let Eq (3.8) *hold and assume the data satisfies:*

- $\theta_- \in W^{s-1-\frac{1}{p},p}(\Sigma_I) \cap L^\infty(\Sigma_I)$,
- $c, g \in W^{s-1-\frac{1}{p},p}(\Sigma_E)$,
- $C^a_b \in W^{e-1-\frac{1}{q},q}(T^1_1\Sigma_E)$,
- $\mathbf{V} \in \mathbf{W}^{e-1,q}$, $V^a\nu_a = (2\tau + |\theta_-|/2)B^6 - \sigma(\nu,\nu)$,
- $\tau \in W^{s-1,p}$ *if* $s \geqslant 2$, *and* $\tau \in W^{1,z} \cap L^\infty$ *otherwise, with* $z = \frac{3p}{3+\max\{0,2-s\}p}$,
- $(4\tau^\vee + |\theta|^\vee) > 0$ *on* Σ_I,
- $\sigma \in W^{e-1,q}$ *with* $\|\sigma\|_\infty$ *sufficiently small,*
- $\rho \in W^{s-2,p}_+ \cap L^\infty \backslash \{0\}$, *with* $\|\phi\|_\infty$ *sufficiently small,*
- $\mathbf{j} \in \mathbf{W}^{e-2,q}$ *with* $\|\mathbf{j}\|_{e-2,q}$ *sufficiently small.*

Additionally assume that at least one of the following hold:

(a) $\delta > 0$ *is sufficiently small in* (3.8)*;*
(b) $a^\vee_R > 0$ *is sufficiently large;*
(c) $\|\theta_-\|_\infty$ *is sufficiently small, and* $D\tau$ *is sufficiently small.*

Then:

Case (a): The function B in (3.9) *can be chosen so that the marginally trapped surface boundary condition in* (2.33) *is satisfied, and subsequently there exists a solution $\phi \in W^{s,p}$ with $\phi > 0$ and $\mathbf{w} \in \mathbf{W}^{e,q}$ to equations* (3.2)-(3.3) *with boundary conditions* (3.4)-(3.8).

Cases (b) and (c): There exists a solution $\phi \in W^{s,p}$ with $\phi > 0$ and $\mathbf{w} \in \mathbf{W}^{e,q}$ to equations (3.2)-(3.3) *with boundary conditions* (3.4)-(3.8)*. With an additional smallness assumption on τ on Σ_I, the marginally trapped surface boundary condition in* (2.33) *may be satisfied.*

Remark 3.4. *We pointed out earlier that while Holst and Tsogtgerel in* [13] *proved existence (and when possible, uniqueness) of solutions to the Lichnerowicz equation for a rather extensive collection of boundary conditions similar to those discussed in Section 2.3, they did not attempt to prove existence of CMC solutions to the coupled system* (3.2)-(3.3) *satisfying the marginally trapped surface conditions* (2.31)-(2.33)*. The dependence of the coefficient $S(\nu,\nu)$ on the size of the conformal factor ϕ as required by the marginally trapped surface conditions leaves the equations coupled even in the CMC case; hence any results for the CMC case would require non-CMC techniques, and this was left for this second paper. In the case of closed manifolds, the CMC condition decouples the equations so that obtaining existence results for the Lichnerowicz equation for rough metrics is essentially sufficient (modulo some well-known estimates for the conformal Killing operator) to obtain analogous rough metric results for the (decoupled) system, allowing for lower regularity solutions than in the non-CMC case. This is due to the fact that the CMC decoupling in the case of closed manifolds frees one from estimating $\mathbf{w}$ in terms of ϕ; such estimates require additional regularity assumptions on h. Holst and Tsogtgerel in* [13] *proved existence of Lichnerowicz solutions for metrics $h \in W^{s,p}$, with $s > \frac{3}{p}$; these results are a direct analogue of the CMC results in* [12], *made possible by ignoring this boundary coupling that we now account for here. In the case of compact manifolds with boundary, where the marginally trapped surface conditions produce a boundary coupling between the momentum and Hamiltonian constraints that remains even in the CMC setting, there appears to be little hope of obtaining a lower regularity CMC result using these same techniques, along the lines of what was possible in* [12, 13]*. Therefore, Theorem 3.2 above will be sufficient for obtaining the roughest possible CMC solutions using our approach here, and we will not state explicitly a separate CMC existence result.*

For the above problem, one views each $\Sigma_I = \bigcup_{i=1}^{M} \Sigma_i$ as the interior black hole regions contained within a compact subset of an asymptotically Euclidean manifold. The exterior boundary conditions (3.6)-(3.7) come from the decay conditions (2.22)-(2.23). We note that the condition $g = \delta(c + \mathcal{O}(R^{-3}))$ in (3.8) is consistent with asymptotically Euclidean decay when the conformal factor ϕ tends to $\delta > 0$ at infinity. The interior boundary conditions (3.4)-(3.5) on ϕ and $\boldsymbol{w}$ are derived from the marginally trapped surface condition (2.33) discussed in Section 2.3 in the event that the expansion factor θ_- is specified. The components $\Sigma_E = \bigcup_{i=M+1}^{N} \Sigma_i$ represent the asymptotic ends of this manifold and the exterior boundary conditions (3.6)-(3.7) are imposed so that the solutions on the compact region $\mathcal{M}$ exhibit the correct asymptotic behavior.

In order to solve the above problem, we will require the coupled fixed point solution framework used in [12], which is based on Theorem 3.5 below. The are two main difficulties we encounter in attempting to apply the solution framework of [12] to this particular problem. The first difficulty lies in reformulating the conformal equations (3.2), with the boundary conditions (3.4)-(3.7), in a manner that allows us to utilize this framework. This requires adapting many of the supporting results in [12] to incorporate our boundary problem and reformulating the boundary problem itself. Following the approach taken in [13], we must formulate the conformal equations with boundary conditions (3.4)-(3.7) as a nonlinear, fixed point problem on a certain closed, convex and bounded subset of a Banach space. We must then show that the operator defining our fixed point problem is continuous, compact, and invariant on this subspace. In order to show that this operator is invariant, one requires what are known as global sub- and super-solutions for the above system. While this was done for the case on the Lichnerowicz equation on compact manifolds with boundary in [13], determining global sub- and super-solutions in the non-CMC setting for our boundary value problem is the other primary difficulty in applying the fixed point framework from [12].

In the following section, we restate the fixed point theorems used in [12] for convenience. Then the rest of the paper is dedicated to reformulating our problem in this framework, adapting results from [12] and [13], and then determining global sub-and super-solutions.

3.1. **Coupled Fixed Point Theorems and Outline of Proofs.** In Theorem 3.5 below we give some abstract fixed-point results which form the basic framework for our analysis of the coupled constraints. These topological fixed-point theorems will be the main tool by which we shall establish Theorems 3.3-3.2 above. They have the important feature that the required properties of the abstract fixed-point operators S and T appearing in Theorem 3.5 below can be established in the case of the Einstein constraints without using the near-CMC condition; this is not the case for fixed-point arguments for the constraints based on k-contractions (cf. [15, 1]) and the Implicit Function Theorem (cf. [4]) which require near-CMC conditions. The bulk of the paper then involves establishing the required properties of S and T without using the near-CMC condition, and finding suitable global barriers ϕ_- and ϕ_+ for defining the required set U that are similarly free of the near-CMC condition (when possible).

We now set up the basic abstract framework we will use. Let X, Y, $\mathbb{X}$, and $\mathbb{Y}$ be Banach spaces, and let $F : X \times Y \to \mathbb{X}$ and $G : X \to \mathbb{Y}$ be (generally nonlinear) operators. Let $A_{I\!L} : Y \to \mathbb{Y}$ be a linear invertible operator, and let $A_L : X \to \mathbb{X}$ be a linear invertible operator satisfying the maximum principle, meaning that $A_L u \leqslant A_L v \Rightarrow u \leqslant v$. The order structures on X and $\mathbb{X}$ (and hence on their duals, which we denote respectively as X^* and $\mathbb{X}^*$) for interpreting the maximum principle will be inherited from ordered Banach spaces Z and $\mathbb{Z}$ (see the Appendix of [12]) through the compact embeddings $X \hookrightarrow Z$ and $\mathbb{X} \hookrightarrow \mathbb{Z}$, which will also make available compactness arguments. To formulate our problem in this abstract setting, let γ_I be the trace operator onto Σ_I and γ_E be the trace operator onto Σ_E. As in [13] we define the following linear and nonlinear operators:

$$A_L(\phi) = \begin{pmatrix} -\Delta\phi + a_R\phi \\ \gamma_I(\partial_\nu\phi) + \frac{1}{2}H(\gamma_I\phi) \\ \gamma_E(\partial_\nu\phi) + c(\gamma_E\phi) \end{pmatrix} \tag{3.11}$$

$$A_{I\!L}(\boldsymbol{w}) = \begin{pmatrix} I\!L\boldsymbol{w} \\ \gamma_I((\mathcal{L}\boldsymbol{w})^{ab}\nu_b) \\ \gamma_E((\mathcal{L}\boldsymbol{w})^{ab}\nu_b) + C^a_b(\gamma_E\boldsymbol{w}^b) \end{pmatrix} \tag{3.12}$$

$$F(\phi, \boldsymbol{w}) = \begin{pmatrix} a_\tau\phi^5 - a_{\boldsymbol{w}}\phi^{-7} - a_\rho\phi^{-3} \\ \left(\frac{1}{2}\gamma_I(\tau) - \frac{1}{4}\theta_-\right)(\gamma_I(\phi))^3 - \frac{1}{4}S(\nu,\nu)(\gamma_I(\phi))^{-3} \\ -g \end{pmatrix} \tag{3.13}$$

$$G(\phi) = \begin{pmatrix} b_\tau\phi^6 + \mathbf{b}_j \\ \mathbf{V} \\ \mathbf{0} \end{pmatrix}. \tag{3.14}$$

For $\phi \in W^{s,p}$ and $\boldsymbol{w} \in W^{e,q}$ satisfying the exponent conditions of Theorems 3.3-3.2, we have that

$$A_L : X = W^{s,p} \to W^{s-2,p} \times W^{s-1-\frac{1}{p},p}(\Sigma_I) \times W^{s-1-\frac{1}{p},p}(\Sigma_E) = \mathbb{X}, \tag{3.15}$$
$$A_{I\!L} : Y = W^{e,q} \to \mathbf{W}^{e-2,q} \times W^{e-1-\frac{1}{q},q}(T\Sigma_I) \times W^{e-1-\frac{1}{q},q}(T\Sigma_E) = \mathbb{Y},$$
$$F : X \times Y = W^{s,p} \times \mathbf{W}^{e,q} \to W^{s-2,p} \times W^{s-1-\frac{1}{p},p}(\Sigma_I) \times W^{s-1-\frac{1}{p},p}(\Sigma_E) = \mathbb{X},$$
$$G : X = W^{s,p} \to \mathbf{W}^{e-2,q} \times W^{e-1-\frac{1}{q},q}(T\Sigma_I) \times W^{e-1-\frac{1}{q},q}(T\Sigma_E) = \mathbb{Y}.$$

Then following the discussion in [12], the coupled Hamiltonian and momentum constraints with boundary conditions (3.4) can be viewed abstractly as coupled operator equations of the form:

$$A_L(\phi) + F(\phi, w) = 0, \tag{3.16}$$
$$A_{I\!L}(\mathbf{w}) + G(\phi) = 0, \tag{3.17}$$

or equivalently as the coupled fixed-point equations

$$\phi = T(\phi, w), \tag{3.18}$$
$$w = S(\phi), \tag{3.19}$$

for appropriately defined fixed-point maps $T : X \times Y \to X$ and $S : X \to Y$. The obvious choice for S is the *Picard map* for (3.17)

$$S(\phi) = -A_{I\!L}^{-1}G(\phi), \tag{3.20}$$

which also happens to be the solution map for (3.17). On the other hand, there are a number of distinct possibilities for T, ranging from the solution map for (3.16), to the *Picard map* for (3.16), which inverts only the linear part of the operator in (3.16):

$$T(\phi, w) = -A_L^{-1} F(\phi, w). \tag{3.21}$$

Assume now that T is as in (3.21), and (for fixed $w \in Y$) that ϕ_- and ϕ_+ are sub- and super-solutions of the semi-linear operator equation (3.16) in the sense that

$$A_L(\phi_-) + F(\phi_-, w) \leqslant 0, \qquad A_L(\phi_+) + F(\phi_+, w) \geqslant 0.$$

The linear operator A_L is invertible and satisfies the maximum principle, which we will show in Section 5. These conditions imply (see [12]) that for fixed $w \in Y$, ϕ_- and ϕ_+ are also sub- and super-solutions of the equivalent fixed-point equation:

$$\phi_- \leqslant T(\phi_-, w), \qquad \phi_+ \geqslant T(\phi_+, w).$$

For developing results on fixed-point iterations in ordered Banach spaces, it is convenient to work with maps which are monotone increasing in ϕ, for fixed $w \in Y$:

$$\phi_1 \leqslant \phi_2 \quad \Longrightarrow \quad T(\phi_1, w) \leqslant T(\phi_2, w).$$

The map T that arises as the Picard map for a semi-linear problem will generally not be monotone increasing; however, if there exists a continuous, linear, monotone increasing map $J : X \to \mathbb{X}$, then one can always introduce a positive shift s into the operator equation

$$A_L^s(\phi) + F^s(\phi, w) = 0,$$

with $A_L^s = A_L + sJ$ and $F^s(\phi, w) = F(\phi, w) - sJ\phi$. Since $s > 0$ the shifted operator A_L^s retains the maximum principle property of A_L, and if s is chosen sufficiently large, then F^s is monotone decreasing in ϕ. Under the additional condition on J and s that A_L^s is invertible, the shifted Picard map

$$T^s(\phi, w) = -(A_L^s)^{-1} F^s(\phi, w)$$

is now monotone increasing in ϕ. See Section 5 for verification of these properties of T^s.

We now give the main abstract existence result from [12] for systems of the form (3.18)–(3.19).

Theorem 3.5. (Coupled Fixed-Point Principle [12]**)** *Let X and Y be Banach spaces, and let Z be a real ordered Banach space having the compact embedding $X \hookrightarrow Z$. Let $[\phi_-, \phi_+] \subset Z$ be a nonempty interval which is closed in the topology of Z, and set $U = [\phi_-, \phi_+] \cap \overline{B}_M \subset Z$ where $\overline{B}_M$ is the closed ball of finite radius $M > 0$ in Z about the origin. Assume U is nonempty, and let the maps*

$$S : U \to \mathcal{R}(S) \subset Y, \qquad T : U \times \mathcal{R}(S) \to U \cap X,$$

be continuous maps. Then there exist $\phi \in U \cap X$ and $w \in \mathcal{R}(S)$ such that

$$\phi = T(\phi, w) \quad \textit{and} \quad w = S(\phi).$$

Proof. See [12]. □

Remark 3.6. *We make some brief remarks about Theorem 3.5 (see also the discussion following this results in* [12]*). Theorem 3.5 was specifically engineered for the analysis of the fully coupled Einstein constraint equations; it allows one to establish simple sufficient conditions on the map T to yield the core invariance property by using barriers in an ordered Banach space (for a review of ordered Banach spaces, see the Appendix* *of* [12]*). If the ordered Banach space Z in Theorem 3.5 had a* normal *order cone, then the closed interval $[\phi_-, \phi_+]$ would automatically be bounded in the norm of Z (see the Appendix of* [12] *for this result). The interval by itself is also non-empty and closed by assumption, and trivially convex (see the Appendix of* [12]*), so that Theorem 3.5 would follow immediately from a variation of the Schauder Theorem by simply taking $U = [\phi_-, \phi_+]$. Note that the closed ball $\overline{B}_M$ in Theorem 3.5 can be replaced with any non-empty, convex, closed, and bounded subset of Z having non-trivial intersection with the interval $[\phi_-, \phi_+]$.*

Following our approach in [12], the overall argument used here to prove the non-CMC results in Theorems 3.2 and 3.3 using Theorem 3.5 involves the following steps:

Step 1: *The choice of function spaces*. We will choose the spaces for use of Theorem 3.5 as follows:
- $X = W^{s,p}$, with $p \in (3, \frac{\alpha+1}{3}),\ \alpha > 8$, and $s(p) \in (1 + \frac{3}{p}, 2)$.
- $Y = \boldsymbol{W}^{e,q}$, with e and q as given in the theorem statements.
- $Z = W^{\tilde{s},p}$, $\tilde{s} \in (1 + \frac{3}{p} - \frac{4}{\alpha}, 1 + \frac{3}{p})$, so that $X = W^{s,p} \hookrightarrow W^{\tilde{s},p} = Z$ is compact.
- $U = [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M \subset W^{\tilde{s},p} = Z$, with ϕ_- and ϕ_+ global barriers (sub- and super-solutions, respectively) for the Hamiltonian constraint equation which satisfy the compatibility condition: $0 < \phi_- \leqslant \phi_+ < \infty$.

Step 2: *Construction of the mapping S*. Assuming the existence of "global" weak sub- and super-solutions ϕ_- and ϕ_+, and assuming the fixed function $\phi \in U = [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M \subset W^{\tilde{s},p} = Z$ is taken as data in the momentum constraint, we establish continuity and related properties of the momentum constraint solution map $S : U \to \mathcal{R}(S) \subset \boldsymbol{W}^{e,q} = Y$. (§4)

Step 3: *Construction of the mapping T*. Again existence of "global" weak sub- and super-solutions ϕ_- and ϕ_+, with fixed $w \in \mathcal{R}(S) \subset \boldsymbol{W}^{e,q} = Y$ taken as data in the Hamiltonian constraint, we establish continuity and related properties of the Picard map $T : U \times \mathcal{R}(S) \to U \cap W^{s,p}$. Invariance of T on $U = [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M \subset W^{\tilde{s},p}$ is established using a combination of *a priori* order cone bounds and norm bounds. (§5)

Step 4: *Barrier construction*. Global weak sub- and super-solutions ϕ_- and ϕ_+ for the Hamiltonian constraint are explicitly constructed to build a nonempty, convex, closed, and bounded subset $U = [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M \subset W^{\tilde{s},p}$, which is a strictly positive interval. These include variations of known barrier constructions which require the near-CMC condition, and also some new barrier constructions which are free of the near-CMC condition. (§6) ***Note: This is the only place in the argument where near-CMC conditions may potentially arise.***

Step 5: *Application of fixed-point theorem*. The global barriers and continuity properties are used together with the abstract topological fixed-point result (Theorem 3.5) to establish existence of solutions $\phi \in U \cap W^{s,p}$ and $w \in \boldsymbol{W}^{e,q}$ to the coupled system: $w = S(\phi), \phi = T(\phi, w)$. (§7)

Step 6: *Bootstrap*. The above application of a fixed-point theorem is actually performed for some low regularity spaces, i.e., for $s \leqslant 2$ and $e \leqslant 2$, and a bootstrap argument is then given to extend the results to the range of s and p given in the statement of the Theorem. (§7)

As was the case in [12, 13], the ordered Banach space Z plays a central role in Theorem 3.5 and its application here. We will use $Z = W^{t,q}$, $t \geqslant 0$, $1 \leqslant q \leqslant \infty$, with order cone defined as in (2.6). Given such an order cone, one can define the closed interval

$$[\phi_-, \phi_+]_{t,q} = \{\phi \in W^{t,q} : \phi_- \leqslant \phi \leqslant \phi_+\} \subset W^{t,q},$$

which as noted earlier is denoted more simply as $[\phi_-, \phi_+]_q$ when $t = 0$, and as simply $[\phi_-, \phi_+]$ when $t = 0$, $q = \infty$. If we consider the interval $U = [\phi_-, \phi_+]_{t,q} \subset W^{t,q} = Z$ defined using this order structure, for use with Theorem 3.5 it is important that U be convex (with respect to the vector space structure of Z), closed (in the topology of Z), and (when possible) bounded (in the metric given by the norm on Z). It will also be important that U be nonempty as a subset of Z; this will involve choosing compatible ϕ_- and ϕ_+. Regarding convexity, closure, and boundedness, we have the following lemma from [12].

Lemma 3.7. (Order cone intervals in $W^{t,q}$ [12]) *For $t \geqslant 0$, $1 \leqslant q \leqslant \infty$, the set*

$$U = [\phi_-, \phi_+]_{t,q} = \{\phi \in W^{t,q} : \phi_- \leqslant \phi \leqslant \phi_+\} \subset W^{t,q}$$

is convex with respect to the vector space structure of $W^{t,q}$ and closed in the topology of $W^{t,q}$. For $t = 0$, $1 \leqslant q \leqslant \infty$, the set U is also bounded with respect to the metric space structure of $L^q = W^{0,q}$.

Proof. See [12]. □

4. MOMENTUM CONSTRAINT

In this section we fix a particular scalar function $\phi \in W^{s,p}$ with $sp > 3$, and consider separately the momentum constraint equation (3.3) with boundary conditions (3.5)-(3.7) to be solved for the vector valued function $\boldsymbol{w}$. The result is a linear elliptic system of equations for this variable $\boldsymbol{w} = \boldsymbol{w}_\phi$. Our goal is not only to develop some existence results for the momentum constraint, but also to derive the estimates for the momentum constraint solution map S that we will need later in our analysis of the coupled system.

Let $(\mathcal{M}, h)$ be a 3-dimensional Riemannian manifold, where $\mathcal{M}$ is a smooth, compact manifold with boundary satisfying (3.1) with $p \in (1, \infty)$, $s \in (1 + \frac{3}{p}, \infty)$, and $h \in W^{s,p}$ is a positive definite metric. With

$$q \in (1, \infty), \qquad \text{and} \qquad e \in (2 - s, s] \cap (-s + \tfrac{3}{p} - 1 + \tfrac{3}{q}, s - \tfrac{3}{p} + \tfrac{3}{q}], \tag{4.1}$$

fix the source terms

$$\boldsymbol{b}_\tau, \boldsymbol{b}_j \in \boldsymbol{W}^{e-2,q}, \mathbf{V} \in \mathbf{W}^{e-1,q}, \text{ and } C^a_b \in W^{e-1-\frac{1}{q},q}(T^1_1\Sigma_E) \cap L^\infty(T^1_1\Sigma_E), \tag{4.2}$$

where C^a_b satisfies (3.8). Fix a function $\phi \in W^{s,p}$, and let

$$A_{I\!L} : W^{e,q} \to \mathbf{W}^{e-2,q} \times W^{e-1-\frac{1}{q},q}(T\Sigma_I) \times W^{e-1-\frac{1}{q},q}(T\Sigma_E),$$
$$G : W^{s,p} \to \mathbf{W}^{e-2,q} \times W^{e-1-\frac{1}{q},q}(T\Sigma_I) \times W^{e-1-\frac{1}{q},q}(T\Sigma_E).$$

be as in (3.12) and (3.14).

The momentum constraint equation with Robin boundary conditions is the following: find an element $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ that is a solution of

$$A_{I\!L}\boldsymbol{w} + G(\phi) = 0. \tag{4.3}$$

4.1. **Weak Formulation.** In order to show that (4.3) has a solution, we employ the Lax-Milgram Theorem in the case when $p \in (1,\infty)$, $s > 1+\frac{3}{p}$, $e = 1$ and $q = 2$ to show that the weak formulation has a solution. We will then utilize *a priori* estimates to show that solutions exist for the exponent ranges specified above.

Using the volume form given by h and integration by parts, the weak formulation of (4.3) is then to find $\boldsymbol{w} \in \mathbf{W}^{1,2}$ such that for all $\boldsymbol{v} \in \mathbf{W}^{1,2}$,

$$\begin{aligned}\int_{\mathcal{M}} (\mathcal{L}\boldsymbol{w})_{ab}(\mathcal{L}\boldsymbol{v})^{ab}\,dx + \int_{\Sigma_E} C^a_b \gamma_E \boldsymbol{w}^b \gamma_E \boldsymbol{v}_a\,ds = -\int_{\mathcal{M}} (b^a_\tau \phi^6 + b^a_j)\boldsymbol{v}_a\,dx \\ + \int_{\Sigma_I} \gamma_I \mathbf{V}^a \gamma_I \boldsymbol{v}_a\,ds, \qquad (4.4)\end{aligned}$$

where dx is the measure induced by h and ds is the measure induced by the metric on $\partial\mathcal{M}$ that is inherited from h.

Remark 4.1. *We observe that the bilinear form* (4.4) *is well defined for* $\mathbf{v} \in \mathbf{W}^{1,2}$*, given that* $\gamma_i \mathbf{v} \in W^{\frac{1}{2},2}(T\Sigma_i)$*,* $i \in \{I, E\}$*, and* $\mathbf{V}, (\mathbf{b}_\tau \phi^6 + \mathbf{b}_j) \in \mathbf{L}^2$ *and* $C^a_b \in L^\infty(T^1_1 \Sigma_I)$*.*

Letting

$$a_{\mathcal{L}}(\boldsymbol{w},\boldsymbol{v}) = \int_{\mathcal{M}} (\mathcal{L}\boldsymbol{w})_{ab}(\mathcal{L}\boldsymbol{v})^{ab}\,dx + \int_{\Sigma_E} C^a_b \gamma_E \boldsymbol{w}^b \gamma_E \boldsymbol{v}_a\,ds, \qquad (4.5)$$

and

$$\boldsymbol{f}(\boldsymbol{v}) = -\int_{\mathcal{M}} (b^a_\tau \phi^6 + b^a_j)\boldsymbol{v}_a\,dx + \int_{\Sigma_I} \gamma_I \mathbf{V}^a \gamma_I \boldsymbol{v}_a\,ds, \qquad (4.6)$$

we say that $A_{\mathbb{L}}(\boldsymbol{w}) + G(\phi) = 0$ weakly if $a_{\mathcal{L}}(\boldsymbol{w},\boldsymbol{v}) = f(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbf{W}^{1,2}$.

Our approach to proving that (4.3) is weakly solvable will be to verify that the shifted, bounded linear operator

$$a^s_{\mathcal{L}}(\boldsymbol{w},\boldsymbol{v}) = a_{\mathcal{L}}(\boldsymbol{u},\boldsymbol{v}) + s(\boldsymbol{w},\boldsymbol{v}), \qquad (4.7)$$

is coercive for some $s > 0$. We will then apply the Lax-Milgram Theorem and Riesz-Schauder Theory to conclude that (4.3) has a unique, weak solution in $\mathbf{W}^{1,2}$.

4.1.1. *Gårding's Inequality.* The primary inequality that we will need to establish in order to show that (4.7) is coercive is the Gårding inequality. We just mention here that the Gårding type inequality for the particular case of the space $W^{1,2}_0$ can be proven for a general class of bilinear forms called strongly elliptic. See [24], exercise 22.7b, page 396. A bilinear form $a : W^{1,2}_0 \times W^{1,2}_0 \to \mathbb{R}$ with action

$$\begin{aligned} a(u,v) = \int_{\mathcal{M}} a_{ac_1\cdots c_n bd_1\cdots d_n} \nabla^a u^{c_1\cdots c_n} \nabla^b v^{d_1\cdots d_n}\,dx \\ + \int_{\mathcal{M}} b_{c_1\cdots c_n d_1\cdots d_n} u^{c_1\cdots c_n} v^{d_1\cdots d_n}\,dx \end{aligned}$$

is **strongly elliptic** iff there exists a positive constant α_0 such that

$$a_{ac_1\cdots c_n bd_1\cdots d_n} \zeta^a \zeta^b u^{c_1\cdots c_n} u^{d_1\cdots d_n} \geqslant \alpha_0\, \zeta_a \zeta^a\, u_{c_1\cdots c_n} u^{c_1\cdots c_n}$$

for all vectors $\zeta \in \mathbb{R}^3$ and all tensors $u_{c_1\cdots c_n} \in \mathbb{R}^{3n}$. Notice that the bilinear form $a_{\mathcal{L}} : \boldsymbol{W}^{1,2}_0 \times \boldsymbol{W}^{1,2}_0 \to \mathbb{R}$ given by $a_{\mathcal{L}}(\boldsymbol{u},\boldsymbol{v}) = (\mathcal{L}\boldsymbol{u}, \mathcal{L}\boldsymbol{v})$ is strongly elliptic, as the following calculation shows:

$$\left[\zeta^a u^c + \zeta^c u^a - \frac{2}{3}h^{ac}(\zeta_d u^d)\right]\left[\zeta_a u_c + \zeta_c u_a - \frac{2}{3}h_{ac}(\zeta_e u^e)\right] = 2(\zeta_a\zeta^a)(u_b u^b) + \frac{2}{3}\left(\zeta_a u^a\right)^2 \geqslant 2(\zeta_a\zeta^a)(u_b u^b).$$

Hence, a Gårding type inequality is satisfied by the bilinear form $a_{\mathcal{L}}$ on the Hilbert space $\boldsymbol{W}_0^{1,2}$. However, this space is too small in our case where we need the same inequality on the space $\boldsymbol{W}^{1,2}$.

We extend the Gårding inequality to the space $\boldsymbol{W}^{1,2}$ in the following two results.

Lemma 4.2. (Gårding's inequality for $\mathcal{L}$) *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact, Riemannian manifold, with Lipschitz boundary, and with a metric $h \in W^{s,p}, p \in (1,\infty), s \in (1+\frac{3}{p},\infty)$. Then, there exists a positive constant k_0 such that the following inequality holds*

$$k_0\, \|\boldsymbol{u}\|_{1,2}^2 \leqslant \|\boldsymbol{u}\|_2^2 + \|\mathcal{L}\boldsymbol{u}\|_2^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}. \tag{4.8}$$

Proof. *(Lemma 4.2.)* See [6] for the proof. □

Using Lemma 4.2, we can immediately establish the same type of inequality for the bilinear form $a_{\mathcal{L}}(\boldsymbol{u},\boldsymbol{u})$ in (4.5) provided that C^a_b is positive definite in the sense of (3.8).

Corollary 4.3. (Gårding's inequality for $a_{\mathcal{L}}(\boldsymbol{w},\boldsymbol{v})$) *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact, Riemannian manifold, with Lipschitz boundary and with a metric $h \in W^{s,p}$, $p \in (1,\infty), s \in (1+\frac{3}{p},\infty)$. Let $a_{\mathcal{L}}(\mathbf{u},\mathbf{u})$ be the bilinear form defined in* (4.5) *for a positive definite tensor $C^a_b \in \mathbf{L}^\infty(T^1_1(\Sigma_E))$ in the sense of* (3.8)*. Then, there exists a positive constant k_1 such that the following inequality holds*

$$k_1\, \|\mathbf{u}\|_{1,2}^2 \leqslant \|\mathbf{u}\|_2^2 + a_{\mathcal{L}}(\mathbf{u},\mathbf{u}) \qquad \forall \mathbf{w} \in \mathbf{W}^{1,2}. \tag{4.9}$$

Proof. *(Corollary 4.3.)* The definition of the bilinear form in (4.5) implies that

$$a_{\mathcal{L}}(\boldsymbol{u},\boldsymbol{u}) = \|\mathcal{L}\boldsymbol{u}\|_2^2 + \langle C\gamma_E\boldsymbol{u}, \gamma_E\boldsymbol{u}\rangle_{\Sigma_I}, \quad \forall \boldsymbol{u} \in \boldsymbol{W}^{1,2}.$$

Lemma 4.2 and the fact that C^a_b is positive definite imply the result. □

The above results combined with Riesz-Schauder theory allow us to conclude that (4.5) is weakly solvable in Theorem 4.4 below. We note that while the positivity assumption (3.8) used in Corollary 4.3 can be removed by using a more complex proof involving a trace inequality, the positivity assumption (3.8) is essential to showing injectivity in Theorem 4.4 below. However, the positivity property is available in the practical situations of interest such as (2.25).

Theorem 4.4. (Momentum constraint) *Suppose $(\mathcal{M}, h)$ is a connected, 3-dimensional manifold with boundary satisfying* (3.1) *and with $h \in W^{s,p}$, $p \in (1,\infty)$, $s > 1+\frac{3}{p}$. Assume that the data $\mathbf{b}_\tau, \mathbf{b}_j \in \mathbf{W}^{-1,2}$, $\mathbf{V} \in \mathbf{L}^2$, $\sigma \in L^2(T^0_2\mathcal{M})$, and let the tensor $C^a_b \in L^\infty(T^1_1(\Sigma_i))$ be positive definite in the sense of* (3.8)*. Then there exists a unique solution to the weak formulation of the momentum constraint* (4.5)*, and there exists a constant $C > 0$ independent $\tau, \mathbf{j}, \mathbf{V}$ and σ such that the following estimate holds:*

$$\|\mathbf{w}\|_{1,2} \leqslant C\left(\|\mathbf{b}_\tau\phi^6\|_{-1,2} + \|\mathbf{b}_j\|_{-1,2} + \|\gamma_I\mathbf{V}\|_{-\frac{1}{2},2;\Sigma_I}\right). \tag{4.10}$$

Proof. Setting $s > 0$, Corollary 4.3 implies that the bilinear form (4.7) is coercive. By the Lax-Milgram Theorem, for any $\boldsymbol{h} \in \mathbf{W}^{-1,2}$, we have that there exists a unique element $\boldsymbol{w} \in \mathbf{W}^{1,2}$ which satisfies

$$a^s_{\mathcal{L}}(\boldsymbol{w},\boldsymbol{v}) = a_{\mathcal{L}}(\boldsymbol{w},\boldsymbol{v}) + (\boldsymbol{w},\boldsymbol{v}) = \boldsymbol{h}(\boldsymbol{v}).$$

This defines a bounded, invertible operator $L^s : \mathbf{W}^{1,2} \to \mathbf{W}^{-1,2}$, where

$$L^s \boldsymbol{w} = \boldsymbol{h} \Longleftrightarrow a^s_{\mathcal{L}}(\boldsymbol{w}, \boldsymbol{v}) = h(\boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \mathbf{W}^{1,2}. \tag{4.11}$$

If we let L be a similar operator defined by

$$L\boldsymbol{w} = \boldsymbol{h} \Longleftrightarrow a_{\mathcal{L}}(\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{h}(\boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \mathbf{W}^{1,2}, \tag{4.12}$$

we have that

$$\begin{gathered} L + sI = L^s, \text{ and} \\ L\boldsymbol{w} = \boldsymbol{h} \Longleftrightarrow L^s\boldsymbol{w} = \boldsymbol{h} + s\boldsymbol{w}. \end{gathered} \tag{4.13}$$

Therefore, rewriting (4.13), we have that

$$L\boldsymbol{w} = \boldsymbol{h} \Longleftrightarrow \boldsymbol{w} - s(L^s)^{-1}\boldsymbol{w} = (L^s)^{-1}\boldsymbol{h}. \tag{4.14}$$

Standard elliptic PDE theory tells us that the operator

$$K\boldsymbol{w} = s(L^s)^{-1}\boldsymbol{w},$$

is compact, and we can therefore apply the Fredholm alternative to conclude that the operator L is Fredholm with index zero. Thus, $dim(ker(L)) = codim(R(L))$ and to conclude that the operator L is invertible (which implies the existence and uniqueness of solutions to the weak formulation), we need only show that its kernel is trivial.

Assume that L has a nontrivial kernel. This implies that there exists some $\boldsymbol{w} \in \mathbf{W}^{1,2}$ such that

$$a_{\mathcal{L}}(\boldsymbol{w}, \boldsymbol{v}) = \int_{\mathcal{M}} (\mathcal{L}\boldsymbol{w})_{ab}(\mathcal{L}\boldsymbol{v})^{ab}\, dx + \int_{\Sigma_E} C^a_b \gamma_E \boldsymbol{w}^b \gamma_E \boldsymbol{v}_a\, ds = 0, \tag{4.15}$$

for all $\boldsymbol{v} \in \mathbf{W}^{1,2}$. Therefore,

$$\begin{aligned} 0 \leqslant C\|\nabla\boldsymbol{w}\|_2^2 \leqslant \int_{\mathcal{M}} (\mathcal{L}\boldsymbol{w})_{ab}(\mathcal{L}\boldsymbol{v})^{ab}\, dx = -\int_{\Sigma_E} C^a_b \gamma_E \boldsymbol{w}^b \gamma_E \boldsymbol{v}_a\, ds \\ \leqslant -\alpha\|\gamma_E\boldsymbol{w}\|^2_{2;\Sigma_E} \leqslant 0, \end{aligned} \tag{4.16}$$

where $\alpha > 0$ by the positive definite assumption on C^a_b. If $\boldsymbol{w} \neq 0$, then one of the above two inequalities must be strict. In particular, if $\|\nabla\boldsymbol{w}\|_2 = 0$, then $\boldsymbol{w}$ is constant and the assumption that $\boldsymbol{w} \neq 0$ implies that $\|\boldsymbol{w}\|_{L^2;\Sigma_E} > 0$. On the other hand, if $\|\boldsymbol{w}\|_{2;\Sigma_E} = 0$ and $\boldsymbol{w} \neq 0$, then $\boldsymbol{w}$ is non-constant and $\|\nabla\boldsymbol{w}\|_2 > 0$. In either case, we have a contradiction which allows us to conclude that L has a trivial kernel and is invertible. Given that $\boldsymbol{f} \in \mathbf{W}^{-1,2}$, where $\boldsymbol{f}$ is defined in (4.6), the weak formulation momentum constraint (4.3) has a solution.

In order to establish the *a priori* estimate (4.10), we apply the open mapping theorem to conclude that L is open. Given that L is invertible, we can then conclude that $L^{-1} : \mathbf{W}^{-1,2} \to \mathbf{W}^{1,2}$ is a bounded linear operator. So there exists some $C > 0$ such that for any $\boldsymbol{h} \in \mathbf{W}^{-1,2}$,

$$\|L^{-1}\boldsymbol{h}\|_{1,2} \leqslant C\|\boldsymbol{h}\|_{-1,2}. \tag{4.17}$$

This implies that if $\boldsymbol{w}$ is our unique, weak solution to the momentum constraint for $\boldsymbol{f} \in \mathbf{W}^{-1,2}$ given by (4.6), then

$$\|\boldsymbol{w}\|_{1,2} \leqslant C\|\boldsymbol{f}\|_{-1,2}. \tag{4.18}$$

The above bound (4.18) implies that

$$\|\boldsymbol{w}\|_{1,2} \leqslant C\left(\|\mathbf{b}_\tau\phi^6 + \mathbf{b}_j\|_{-1,2} + \|\gamma_I\mathbf{V}\|_{-\frac{1}{2},2;\Sigma_I}\right), \tag{4.19}$$

which is the desired estimate in (4.10). □

Now that we have shown that weak solutions $\boldsymbol{w} \in \mathbf{W}^{1,2}$ exist, we utilize the following regularity theorem to show that the Momentum constraint 4.3 has a solution in $\mathbf{W}^{e,q}$, with e, q satisfying (4.1), provided that the Robin data and coefficients functions satisfy (4.2).

Theorem 4.5. (Regularity $\mathbf{W}^{e,q}$) *Let $(\mathcal{M}, h)$ be a connected, 3-dimensional compact manifold with boundary satisfying* (3.1)*, and with $h \in W^{s,p}$, $p \in (1,\infty)$, $s \in (1+\frac{3}{p},\infty)$. Let $\partial\mathcal{M}$ be C^k, where $k \geqslant e > 1$, and suppose that the Robin data and coefficients satisfy the regularity assumptions* (4.1)*. Then there exists a solution* $\mathbf{w}$ *to the momentum constraint* (4.3) *in $\boldsymbol{W}^{e,q}$, and there exist positive constants C_1 and C_2 such that the following estimate holds,*

$$\|\mathbf{w}\|_{e,q} \leqslant C \left(\|\mathbf{b}_\tau \phi^6\|_{e-2,q} + \|\mathbf{b}_j\|_{e-2,q} + \|\gamma_I \mathbf{V}\|_{e-1-\frac{1}{q},q;\Sigma_I} \right). \tag{4.20}$$

Proof. *(Theorem 4.5.)* We give just an outline of a proof following a standard approach. Viewing the metric in local coordinates and applying interior and boundary estimates, and then applying a partition of unity argument, one obtains the above result. In particular, one decomposes both the vector Laplacian $\mathbb{L}$ and the boundary operators into a sum of constant coefficient and slightly perturbed non-constant coefficient operators as in Proposition 5.1 in [18] and Lemma B.3 in [13]. One then applies results for constant coefficient elliptic operators, interpolation inequalities, and (4.10) to obtain the result. □

Remark 4.6. *If the data $\mathbf{V}, \boldsymbol{b}_\tau, \boldsymbol{b}_j$ satisfies the hypotheses of Theorem 4.5, and $q \in (3,\infty)$, $1+\frac{3}{q} < e \leqslant 2$, then $\mathbf{W}^{e,q} \hookrightarrow \mathbf{W}^{1,\infty}$ and we have that*

$$\|\mathcal{L}\mathbf{w}\|_\infty \leqslant \|\mathbf{w}\|_{1,\infty} \leqslant C\|\mathbf{w}\|_{e,q}.$$

Combining this and the a priori estimate (4.20) *and using the embedding $L^z \hookrightarrow W^{e-2,q}$ for appropriate z, we have that*

$$\|\mathcal{L}\mathbf{w}\|_\infty \leqslant C \left(\|\phi\|_\infty^6 \|\mathbf{b}_\tau\|_z + \|\mathbf{b}_j\|_{e-2,q} + \|\gamma_I \mathbf{V}\|_{e-1-\frac{1}{q},q;\Sigma_I} \right). \tag{4.21}$$

Furthermore, if $\mathbf{X} \in W^{e-1-\frac{1}{q},q}(T\mathcal{M})$ and $\mathbf{V}$ is a vector field such that

$$\gamma_I \mathbf{V} = \left((2\tau + |\theta_-|/2)B^6 - \sigma(\nu,\nu) \right) \nu + \mathbf{X},$$

we can utilize (4.21) *to obtain*

$$\begin{aligned} \|\mathcal{L}\mathbf{w}\|_\infty \leqslant C \Big(& \|\phi_+\|_\infty^6 \|\mathbf{b}_\tau\|_z + \|\mathbf{b}_j\|_{e-2,q} + \|\tau B^6\|_{e-1-\frac{1}{q},q;\Sigma_I} \\ & + \|\theta_- B^6\|_{e-1-\frac{1}{q},q;\Sigma_I} + \|\sigma(\nu,\nu)\|_{e-1-\frac{1}{q},q;\Sigma_I} + \|\mathbf{X}\|_{e-1-\frac{1}{q},q;\Sigma_I} \Big). \end{aligned} \tag{4.22}$$

Note that we have replaced ϕ with ϕ_+ in (4.21) *given that we are assuming ϕ_+ is an* a priori *upper bound on ϕ.*

The bounds in equation (4.22) *will be essential to control $a_{\mathbf{w}}$ in the Hamiltonian constraint in terms of the global super-solution ϕ_+. This will be necessary in order to obtain our global sub-and super-solutions later.*

Theorems 4.4 and 4.5 imply that the Picard map (in this case the solution map),

$$S : W^{s,p} \to \mathbf{W}^{e,q},$$
$$S(\phi) = -A_{\mathbb{L}}^{-1} G(\phi),$$

is well-defined. In order to apply the Coupled Fixed Point Theorem 3.5, we will additionally require that S be continuous, which we show in the following Lemma.

Lemma 4.7. (Properties of the map S) *In addition to the conditions* (4.1) *and* (4.2) *imposed in the beginning of this section, let $e \in [0,2]$ and $\mathbf{b}_\tau \in L^z$ with $z = \frac{3q}{\max\{0,(2-e)\}q+3}$. Let the assumptions for Theorems 4.4 and 4.5, so that in particular the momentum constraint* (4.3) *is uniquely solvable in $\boldsymbol{W}^{e,q}$. With some $\phi_+ \in W^{s,p}$ satisfying $\phi_+ > 0$, let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be the solutions to the momentum constraint with the source functions ϕ_1 and ϕ_2 from the set $[0,\phi_+] \cap W^{s,p}$, respectively. Then,*

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{e,q} \leqslant C\, \|\phi_+\|_\infty^5 \|\boldsymbol{b}_\tau\|_z\, \|\phi_1 - \phi_2\|_{s,p}. \tag{4.23}$$

Proof. The functions ϕ_1 and ϕ_2 pointwise satisfy the following inequalities

$$\begin{aligned} \phi_2^n - \phi_1^n &= \Big(\sum_{j=0}^{n-1} \phi_2^j \phi_1^{n-1-j}\Big)(\phi_2 - \phi_1) \leqslant n\,(\phi_+)^{n-1}\,|\phi_2 - \phi_1|, \\ -\big[\phi_2^{-n} - \phi_1^{-n}\big] &= \frac{\phi_2^n - \phi_1^n}{(\phi_2\phi_1)^n} \leqslant n\,\frac{(\phi_+)^{n-1}}{(\phi_-)^{2n}}\,|\phi_2 - \phi_1|, \end{aligned} \tag{4.24}$$

for any integer $n > 0$.

By Theorems 4.4 and 4.5, for a fixed $\phi \in (0,\phi^+)$, S^{-1} is an invertible operator between

$$\mathbb{Y} = W^{e-2,q}(T\mathcal{M}) \times W^{e-1-\frac{1}{q},q}(T\Sigma_I) \times W^{e-1-\frac{1}{q},q}(T\Sigma_E)$$

and $Y = W^{e,q}(T\mathcal{M})$. Hence, by the Bounded Inverse Theorem

$$\|\boldsymbol{w}\|_{e,q} \leqslant C\|G(\phi)\|_{\mathbb{Y}}.$$

Therefore

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{e,q} \leqslant C\|G(\phi_1) - G(\phi_2)\|_{\mathbb{Y}} = \|b_\tau(\phi_1^6 - \phi_2^6)\|_{e-2,q}, \tag{4.25}$$

given that the boundary terms on $W^{e-1-\frac{1}{q},q}(\Sigma_I)$ and $W^{e-1-\frac{1}{q},q}(\Sigma_E)$ do not depend on ϕ, and so the norms corresponding to these terms in the $\mathbb{Y}$ norm vanish.

Using (4.25), the inequalities (4.24), and the embeddings $W^{s,p} \hookrightarrow L^\infty$, $L^z \hookrightarrow W^{e-2,q}$, we obtain

$$\begin{aligned} \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{e,q} &\leqslant C\|\boldsymbol{b}_\tau(\phi_1^6 - \phi_2^6)\|_{e-2,q} \leqslant C\|\boldsymbol{b}_\tau(\phi_1^6 - \phi_2^6)\|_z \leqslant C\|\boldsymbol{b}_\tau\|_z\|\phi_1^6 - \phi_2^6\|_\infty \\ &\leqslant 6C\|\phi_+\|_\infty^5\|\boldsymbol{b}_\tau\|_z\,\|\phi_1 - \phi_2\|_{s,p}. \end{aligned}$$

□

5. THE HAMILTONIAN CONSTRAINT AND THE PICARD MAP T

In this section we fix a particular function $a_{\boldsymbol{w}}$ in an appropriate space and we then separately look for weak solutions of the Hamiltonian constraint (3.2) with Robin boundary conditions (3.4)-(3.6). For convenience, we reformulate the problem here in a self-contained manner. Our goal here is primarily to establish some properties and derive some estimates for a Hamiltonian constraint fixed-point map T that we will need later in our analysis of the coupled system.

Let $(\mathcal{M}, h)$ be a 3-dimensional Riemannian compact manifold with boundary satisfying (3.1) and with $p \in (1,\infty)$ and $s \in (\frac{3}{p},\infty) \cap [1,\infty)$, $h \in W^{s,p}$ is a positive definite metric. Recall the operators

$$A_L(\phi) = \begin{pmatrix} -\Delta\phi + a_R\phi \\ \gamma_I(\partial_\nu\phi) + \frac{1}{2}H(\gamma_I\phi) \\ \gamma_E(\partial_\nu\phi) + c(\gamma_E\phi) \end{pmatrix}, \tag{5.1}$$

$$F(\phi, \boldsymbol{w}) = \begin{pmatrix} a_\tau\phi^5 - a_{\boldsymbol{w}}\phi^{-7} - a_\rho\phi^{-3} \\ \left(\frac{1}{2}\gamma_I(\tau) - \frac{1}{4}\theta_-\right)(\gamma_I(\phi))^3 - \frac{1}{4}S(\nu,\nu)(\gamma_I(\phi))^{-3} \\ -g \end{pmatrix}, \tag{5.2}$$

introduced in Section 3.1. The dependence of $F(\phi, \boldsymbol{w})$ on $\boldsymbol{w}$ is hidden in the fact that the coefficient $a_{\boldsymbol{w}}$ depends on $\boldsymbol{w}$ and $S(\nu,\nu) = \mathcal{L}\mathbf{w}(\nu,\nu) + \sigma(\nu,\nu)$, cf. (2.20).

Fix the source functions

$$a_\tau, a_\rho, a_{\boldsymbol{w}} \in W^{s-2,p}_+,\ a_R = \frac{1}{8}R \in W^{s-2,p}, \quad \text{and}$$
$$\theta_-, H \in W^{s-1-\frac{1}{p},p}(\Sigma_I), \quad c, g \in W^{s-1-\frac{1}{p},p}(\Sigma_E),$$

where R is the scalar curvature of the metric h and H is the mean extrinsic curvature on Σ_I induced by h. (By Corollary A.5(b) in [13], we know $h_{ab} \in W^{s,p}$ implies $R \in W^{s-2,p}$ and $H \in W^{s-1-\frac{1}{p},p}(\Sigma_I)$. Here the pointwise multiplication by an element of $W^{s,p}$ defines a bounded linear map in $W^{s-2,p}$ since $s-2 \geqslant -s$ and $2(s-\frac{3}{p}) > 0 > 2-3$, cf. Corollary A.5(a) in [13]. Therefore we have that

$$A_L : W^{s,p} \to W^{s-2,p} \times W^{s-1-\frac{1}{p},p}(\Sigma_I) \times W^{s-1-\frac{1}{p},p}(\Sigma_E), \tag{5.3}$$

$$F : W^{s,p} \times \mathbf{W}^{e,q} \to W^{s-2,p} \times W^{s-1-\frac{1}{p},p}(\Sigma_I) \times W^{s-1-\frac{1}{p},p}(\Sigma_E). \tag{5.4}$$

We then formulate the Hamiltonian constraint equation with Robin boundary conditions as follows: find an element that is a solution of

$$A_L(\phi) + F(\phi, \boldsymbol{w}) = 0. \tag{5.5}$$

Recall from Section 3.1 that our approach for finding weak solutions to (5.5) is to reformulate the problem as a fixed point problem of the form

$$\phi = (A^s_L)^{-1}F^s(\phi, \boldsymbol{w}) = T^s(\phi, \boldsymbol{w}), \tag{5.6}$$

where we assume that $\boldsymbol{w}$ is fixed and A^s_L and F^s are the shifted operators defined in Section 3.1. In order for this map to be well-defined, we obviously require A^s_L to be an invertible map. Furthermore, we also will require the map T^s to be monotonically increasing in ϕ, which will require A^s_L to satisfy the maximum principle. These two properties of A^s_L are verified in Lemmas B.7 and B.8 in [13].

Now that we are sure that the map T^s is well-defined, we discuss some key properties of this map that are essential in applying the coupled fixed point Theorem.

5.1. **Invariance of T^s given Global Sub-and Super-Solutions.** To establish existence results for weak solutions to the Hamiltonian constraint equation using fixed-point arguments, we must show that the fixed point operator T^s in (5.6) is invariant on a certain subspace. This will require the existence of generalized (weak) sub- and super-solutions (sometimes called barriers) which will be derived later in §6. Let us recall the definition of sub- and super-solutions in the following, in a slightly generalized form that will be necessary in our study of the coupled system.

A function $\phi_- \in (0,\infty) \cap W^{s,p}$ is called a **sub-solution** of (5.1) iff the function ϕ_- satisfies the inequality

$$A_L\phi_- + F(\phi_-, \boldsymbol{w}) \leqslant 0, \tag{5.7}$$

for some $a_{\boldsymbol{w}} \in W^{s-2,p}$. A function $\phi_+ \in (0,\infty) \cap W^{s,p}$ is called a **super-solution** of (5.1) iff the function ϕ_+ satisfies the inequality

$$A_L\phi_+ + F(\phi_+, \boldsymbol{w}) \geqslant 0, \tag{5.8}$$

for some $a_{\boldsymbol{w}} \in W^{s-2,p}$. We say a pair of sub- and super-solutions is *compatible* if they satisfy

$$0 < \phi_- \leqslant \phi_+ < \infty, \tag{5.9}$$

so that the interval $[\phi_-, \phi_+] \cap W^{s,p}$ is both nonempty and bounded. In the following discussion, we will assume that ϕ_- and ϕ_+ are a compatible pair of barriers.

Now the we have discussed the basic properties of the linear mapping A_L, we turn to the properties of the fixed-point mapping $T^s : U \times \mathcal{R}(S) \to X$ for the Hamiltonian constraint, where we define T^s as in (5.6). In the following, we analyze the behavior of $T^s(\phi)$ for $\phi_- \leqslant \phi \leqslant \phi_+$. For ease of notation, we let

$$\mathbb{X} = W^{s-2,p} \times W^{s-1-\frac{1}{p},p}(\Sigma_I) \times W^{s-1-\frac{1}{p},p}(\Sigma_E). \tag{5.10}$$

Lemma 5.1. (Properties of the map T) *In the above described setting, assume that $p \in (\frac{\alpha+1}{\alpha-1}, \frac{\alpha+1}{3})$ for $\alpha > 4$ and $s \in (\frac{3}{p}, \infty) \cap [1, 3 - \frac{1}{p'}]$. With $\mathbf{a} = (a, a_I, a_E) \in \mathbb{X}_+$ satisfying $a_i \neq 0$ and $\psi \in W^{s,p}_+$, let*

$$\Psi = (\psi, 0, 0) \quad \textit{and} \quad a_{\boldsymbol{w}}\Psi = (a_{\boldsymbol{w}}\psi, 0, 0) \in \mathbb{X}.$$

Then let $\mathbf{a}_s = \mathbf{a} + a_{\boldsymbol{w}}\Psi \in \mathbb{X}$. Fix the functions $\phi_-, \phi_+ \in W^{s,p}$ such that $0 < \phi_- \leqslant \phi_+$, and define the shifted operators

$$A^s_L : W^{s,p} \to \mathbb{X}, \qquad A^s_L\phi := A_L\phi + \mathbf{a}_s\phi, \tag{5.11}$$

$$F^s_{\boldsymbol{w}} : [\phi_-, \phi_+]_{s,p} \to \mathbb{X}, \qquad F^s_{\boldsymbol{w}}(\phi) := F_{\boldsymbol{w}}(\phi) - \mathbf{a}_s\phi. \tag{5.12}$$

For $\phi \in [\phi_-, \phi_+]_{s,p}$ and $a_{\boldsymbol{w}} \in W^{s-2,p}$, let

$$T^s(\phi, a_{\boldsymbol{w}}) := -(A^s_L)^{-1}F^s_{\boldsymbol{w}}(\phi). \tag{5.13}$$

Then, the map $T^s : [\phi_-, \phi_+]_{s,p} \times W^{s-2,p} \to W^{s,p}$ is continuous in both arguments. Moreover, there exists $\tilde{s} \in (1 + \frac{3}{p} - \frac{4}{\alpha}, 1 + \frac{3}{p})$ and constants C_1, C_2 such that

$$\|T(\phi, a_{\boldsymbol{w}})\|_{s,p} \leqslant C_1(1 + \|a_{\boldsymbol{w}}\|_{s-2,p})\|\phi\|_{\tilde{s},p} + C_2, \tag{5.14}$$

for all $\phi \in [\phi_-, \phi_+]_{s,p}$ and $a_{\boldsymbol{w}} \in W^{s-2,p}$.

Proof. We first bound

$$\begin{aligned}
\|F^s_{\boldsymbol{w}}(\phi)\|_{\mathbb{X}} \leqslant{}& \|a_\tau\phi^5 - a_{\mathbf{w}}\phi^{-7} - a_\rho\phi^{-3} - (a + a_{\mathbf{w}}\psi)\phi\|_{s-2,p} \\
&+ \left\|\left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)(\gamma_I(\phi))^3 - \frac{1}{4}S(\nu,\nu)(\gamma_I(\phi))^{-3} - a_I\gamma_I(\phi)\right\|_{s-1-\frac{1}{p},p;\Sigma_I} \\
&+ \|g - a_E\gamma_E(\phi)\|_{s-1-\frac{1}{p},p;\Sigma_E}, \\
={}& \|f^s_{\boldsymbol{w}}(\phi)\|_{s-2,p} + \|h^s(\phi)\|_{s-1-\frac{1}{p},p;\Sigma_I} + \|g^s\|_{s-1-\frac{1}{p},p;\Sigma_E}.
\end{aligned} \tag{5.15}$$

By applying Lemma 29 from [12] (recalled as Lemma A.6 in [13]), for any $\tilde{s} \in (\frac{3}{p}, s]$, $s - 2 \in [-1, 1]$ and $\frac{1}{p} \in (\frac{s-1}{2}\delta, 1 - \frac{3-s}{2}\delta)$ with $\delta = \frac{1}{p} - \frac{\tilde{s}-1}{3}$, we have

$$\begin{aligned}
\|f^s_{\boldsymbol{w}}(\phi)\|_{s-2,p} \leqslant C\Big(&\|a_\tau\|_{s-2,p}\,\|\phi_+^4\|_\infty + \|a_\rho\|_{s-2,p}\,\|\phi_-^{-4}\|_\infty \\
&+ \|a_{\boldsymbol{w}}\|_{s-2,p}\,(\|\phi_-^{-8}\|_\infty + \|\psi\|_{\tilde{s},p}) + \|a\|_{s-2,p}\Big)\|\phi\|_{\tilde{s},p}.
\end{aligned} \tag{5.16}$$

Let us verify that $\frac{1}{p}$ is indeed in the prescribed range. First, given the assumptions on $\tilde{s}$ we have $\delta = \frac{1}{3} + \frac{1}{p} - \frac{\tilde{s}}{3} < \frac{4}{3\alpha}$. By subsequently taking into account $s \geqslant 1$, we infer that $1 - \frac{3-s}{2}\delta \geqslant 1 - \frac{4}{3\alpha} = \frac{3\alpha-4}{3\alpha}$. This shows $\frac{1}{p} < 1 - \frac{3-s}{2}\delta$ for $p > \frac{3\alpha}{3\alpha-4}$, which is not sharp, but will be sufficient for our analysis. For the other bound, we need $\frac{1}{p} > \frac{s-1}{2}\delta$. Given that $\delta < \frac{4}{3\alpha}$, we have that $\frac{s-1}{2}\delta < \frac{2(s-1)}{3\alpha}$ and because $1 \leqslant s \leqslant 3 - \frac{1}{p'}$, we have $\frac{2(s-1)}{3\alpha} < \frac{4}{3\alpha}$. The assumption that $p < \frac{\alpha+1}{3}$ implies that $p < \frac{3\alpha}{4}$, and therefore that $\frac{s-1}{2}\delta < \frac{4}{3\alpha} < \frac{1}{p}$. So $\frac{1}{p}$ is in the prescribed range.

Applying Lemma 29 from [12] again we have that

$$\begin{aligned} \|h(\phi)\|_{s-1-\frac{1}{p},p;\Sigma_I} \leqslant C_1 \Big(&\|\tau\|_{s-1-\frac{1}{p},p;\Sigma_I} \|(\gamma_I(\phi))^2\|_{\infty;\Sigma_I} \\ &+ \|\theta_-\|_{s-1-\frac{1}{p},p;\Sigma_I} \|(\gamma_I(\phi))^2\|_{\infty;\Sigma_I} \\ &+ \|a_I\|_{s-1-\frac{1}{p},p;\Sigma_I} \Big) \|\gamma_I(\phi)\|_{\tilde{s}-\frac{1}{p},p;\Sigma_I} \\ &+ C_2 \Big(\|\tau\|_{s-1-\frac{1}{p},p;\Sigma_I} + \|\theta_-\|_{s-1-\frac{1}{p},p;\Sigma_I} \Big) \|B\|_\infty^5 \|B\|_{\tilde{s}-\frac{1}{p},p;\Sigma_I} \end{aligned} \tag{5.17}$$

where we have used the fact that $S(\nu,\nu) = (2\gamma_I(\tau) + |\theta_-|/2)B^6$.

We again verify the conditions of Lemma 29 from [12]. Given that $s \in [1, 3 - \frac{1}{p'}]$, we observe that $s - 1 - \frac{1}{p} \in [-1, 1]$. We also require $\frac{1}{p} \in (\frac{s-\frac{1}{p}}{2}\delta, 1 - \frac{2+\frac{1}{p}-s}{2}\delta)$. We observe that $0 < \delta = \frac{1}{p} - (\frac{\tilde{s}-\frac{1}{p}-1}{2}) = \frac{1}{2} - (\frac{\tilde{s}-\frac{3}{p}}{2}) < \frac{2}{\alpha}$ since $\tilde{s} \in (1 + \frac{3}{p} - \frac{4}{\alpha}, 1 + \frac{3}{p})$. Since $s \geqslant 1$, we have that $1 - \frac{2+\frac{1}{p}-s}{2}\delta \geqslant 1 - \frac{1+\frac{1}{p}}{\alpha} = 1 - \frac{1}{\alpha} - \frac{1}{\alpha p}$. Requiring $\frac{1}{p} < 1 - \frac{1}{\alpha} - \frac{1}{\alpha p}$ implies that $p > \frac{\alpha+1}{\alpha-1}$, which holds by assumption. Finally, we observe that $\frac{s-\frac{1}{p}}{2}\delta < \frac{s-\frac{1}{p}}{\alpha} \leqslant \frac{3-\frac{1}{p}}{\alpha} < \frac{1}{p}$ if $p < \frac{\alpha+1}{3}$, which holds by assumption.

We must now bound the terms in (5.17) involving the $L^\infty(\Sigma_I)$ norm of $\gamma_I(\phi)$ in terms of our sub-and-super solutions. Let $u \in W^{s,p}$. Then there exists a sequence $\{u_m\} \subset C^\infty(\overline{\mathcal{M}})$ such that $u_m \to u$ in $W^{s,p}$, and because $s > \frac{3}{p}$, $u_m \to u$ in L^∞. Moreover, by the continuity of $\{u_m\}$ we clearly have that

$$\|\gamma_I(u_m)\|_{\infty;\Sigma_I} = \sup_{x\in\Sigma_I} |u_m(x)| \leqslant \|u_m\|_\infty \leqslant \|u\|_\infty + \epsilon(m),$$

where $\epsilon(m) \to 0$ as $m \to \infty$. We therefore have that $\|u\|_{\infty;\Sigma_I} \leqslant \|u\|_\infty$. Using this fact, (5.17) becomes

$$\begin{aligned} \|h(\phi)\|_{s-1-\frac{1}{p},p;\Sigma_I} \leqslant C_1 \Big(&\|\tau\|_{s-1-\frac{1}{p},p;\Sigma_I} \|\phi_+^2\|_\infty \\ &+ \|\theta_-\|_{s-1-\frac{1}{p},p;\Sigma_I} \|\phi_+^2\|_\infty \\ &+ \|a_I\|_{s-1-\frac{1}{p},p;\Sigma_I} \Big) \|\gamma_I(\phi)\|_{\tilde{s}-\frac{1}{p},p;\Sigma_I} \\ &+ C_2 \Big(\|\tau\|_{s-1-\frac{1}{p},p;\Sigma_I} + \|\theta_-\|_{s-1-\frac{1}{p},p;\Sigma_I} \Big) \|B\|_\infty^5 \|B\|_{\tilde{s}-\frac{1}{p},p;\Sigma_I}. \end{aligned} \tag{5.18}$$

Similarly, we have that

$$\|g^s\|_{s-1-\frac{1}{p},p;\Sigma_E} \leqslant \|g\|_{s-1-\frac{1}{p},p;\Sigma_I} + C\|a_E\|_{s-1-\frac{1}{p},p;\Sigma_E} \|\gamma_E(\phi)\|_{\tilde{s}-\frac{1}{p},p;\Sigma_E}. \tag{5.19}$$

Combining Eqs (5.15), (5.16), (5.18),(5.19) and utilizing the Trace Theorem to obtain the bound

$$\|\phi\|_{\tilde{s}-\frac{1}{p},p;\Sigma_i} \leqslant \|\phi\|_{\tilde{s},p}, \quad \text{for} \quad i \in \{I, E\},$$

we have that

$$\|F^s_{\mathbf{w}}(\phi)\|_{\mathbb{X}} \leqslant C_1(1+\|a_{\mathbf{w}}\|_{s-2,p})\|\phi\|_{\tilde{s},p} + C_2. \tag{5.20}$$

To finalize the proof of (5.14), note that the operator A^s_L is invertible by Lemma B.8 in [13], since the function $\mathbf{a}_s$ is positive. The inverse $(A^s_L)^{-1}:\mathbb{X}\to W^{s,p}$ is bounded by the Bounded Inverse Theorem; this gives (5.14), with possibly different constants than in (5.20).

The continuity of the mapping $F^s_{\mathbf{w}}:[\phi_-,\phi_+]_{s,p}\to\mathbb{X}$ for any $a_{\mathbf{w}}\in W^{s-2,p}$ follows because $F^s_{\mathbf{w}}$ is a composition of continuous maps, and the continuity of $a_{\mathbf{w}}\mapsto F^s_{\mathbf{w}}(\phi)$ for fixed $\phi\in[\phi_-,\phi_+]_{s,p}$ is obvious. Being the composition of continuous maps, $(\phi,a_{\mathbf{w}})\mapsto T^s_{\mathbf{w}}(\phi)$ is also continuous. □

The following lemma shows that by choosing the shift sufficiently large, we can make the map T^s monotone increasing. This result is important for ensuring that the Picard map T for the Hamiltonian constraint is invariant on the interval $[\phi_-,\phi_+]$ defined by sub- and super-solutions. There is an obstruction that the scalar curvature and should be continuous, which is a sufficient condition to guarantee that pointwise multiplication in the space $W^{s,p}\otimes W^{s-2,p}\to W^{s-2,p}$ be continuous operation. This assumption can be handled in the general case by conformally transforming the metric to a metric with continuous scalar curvature and using the conformal covariance of the Hamiltonian constraint, cf. Section 7.1 and Lemma A.2. (We omit explicitly writing the trace maps γ_I and γ_E for quantities evaluated on the boundaries Σ_I and Σ_E in the statement of Lemma 5.2 below without danger of confusion.)

Lemma 5.2. (Monotone increasing property of T) *In addition to the conditions of Lemma 5.1, let a_R, H and c be continuous and define the shift function $\mathbf{a}_s=\mathbf{a}+a_{\mathbf{w}}\Psi$, where $\mathbf{a}$ and $a_{\mathbf{w}}\Psi$ are as in Lemma 5.1 and*

$$a=\max\{1,a_R\}+5a_\tau\phi_+^4+3a_\rho\frac{\phi_+^2}{\phi_-^6},\quad \psi=7\frac{\phi_+^6}{\phi_-^{14}}, \tag{5.21}$$

$$a_I=\max\{1,H\}+\frac{3}{2}|\tau|\phi_+^2+\frac{3}{4}|\theta_-|\phi_+^2+\frac{3B^6}{4}(2|\tau|+|\theta_-|/2)\frac{\phi_+^2}{\phi_-^6},\quad \textit{on } \Sigma_I$$

$$a_E=\max\{1,c\},\quad \textit{on } \Sigma_E.$$

Then, for any fixed $a_{\mathbf{w}}\in W^{s-2,p}$, the map $\phi\mapsto T^s(\phi,a_{\mathbf{w}}):[\phi_-,\phi_+]_{s,p}\to W^{s,p}$ is monotone increasing.

Proof. By Lemma B.7 in [13] the shifted operator A^s_L satisfies the maximum principle, hence the inverse $(A^s_L)^{-1}:\mathbb{X}\to W^{s,p}$ is monotone increasing.

Now we will show that the operator $F^s_{\mathbf{w}}$ is monotone decreasing in ϕ. Given any two functions ϕ_2, $\phi_1\in[\phi_-,\phi_+]_{s,p}$ with $\phi_2\geqslant\phi_1$, we have

$$F^s_{\mathbf{w}}(\phi_2)-F^s_{\mathbf{w}}(\phi_1)= \tag{5.22}$$

$$\begin{pmatrix} a_\tau\big[\phi_2^5-\phi_1^5\big]-(a+a_{\mathbf{w}}\psi)[\phi_2-\phi_1]-a_\rho\big[\phi_2^{-3}-\phi_1^{-3}\big]-a_{\mathbf{w}}\big[\phi_2^{-7}-\phi_1^{-7}\big],\\ (\frac{1}{2}\tau-\theta_-/4)[(\gamma_I(\phi_2))^3-(\gamma_I(\phi_1))^3]\\ -((\frac{1}{2}\tau+\frac{1}{8}|\theta_-|)B^6)[(\gamma_I(\phi_2))^{-3}-(\gamma_I(\phi_1))^{-3}]-a_I[\gamma_I(\phi_2)-\gamma_I(\phi_1)],\\ -a_E[\gamma_E(\phi_2)-\gamma_E(\phi_1)]\end{pmatrix}.$$

The inequalities (4.24), the condition $0<\phi_1\leqslant\phi_2$, and the choice of $\mathbf{a}_s$ imply that

$$F^s_{\mathbf{w}}(\phi_2)-F^s_{\mathbf{w}}(\phi_1)\leqslant 0,$$

which establishes that $F^s_{\mathbf{w}}$ is monotone decreasing.

Both the operator $(A_L^s)^{-1}$ and the map $-F_{\boldsymbol{w}}^s$ are monotone increasing, therefore the composition map $T^s(\cdot, a_{\boldsymbol{w}}) = -(A_L^s)^{-1}F_{\boldsymbol{w}}^s(\cdot)$ is also monotone increasing. □

Lemma 5.3. (Barriers for T and the Hamiltonian constraint) *Let the conditions of Lemma 5.2 hold, with ϕ_- and ϕ_+ sub- and super-solutions of the Hamiltonian constraint equation* (5.5)*, respectively. Then, we have $T^s(\phi_+, a_{\boldsymbol{w}}) \leqslant \phi_+$ and $T^s(\phi_-, a_{\boldsymbol{w}}) \geqslant \phi_-$.*

Proof. We have

$$\phi_+ - T^s(\phi_+, a_{\boldsymbol{w}}) = (A_L^s)^{-1}\big[A_L^s(\phi_+) + F_{\boldsymbol{w}}^s(\phi_+)\big],$$

which is nonnegative since ϕ_+ is a super-solution and $(A_L^s)^{-1}$ is linear and monotone increasing. The proof of the other inequality is completely analogous. □

Since we are no longer using normal order cones, our non-empty, convex, closed interval $[\phi_-, \phi_+]_{s,p}$ is not necessarily bounded as a subset of $W^{s,p}$. Therefore, we also need *a priori* bounds in the norm on $W^{s,p}$ to ensure the Picard iterates stay inside the intersection of the interval with the closed ball $\overline{B}_M$ in $W^{s,p}$ of radius M, centered at the origin. We first establish a lemma to this effect that will be useful for both the non-CMC and CMC cases.

Lemma 5.4. (Invariance of T on the ball $\overline{B}_M$) *Let the conditions of Lemma 5.1 hold, and let $a_{\boldsymbol{w}} \in W^{s-2,p}$. Additionally assume that $s \in (1 + \frac{3}{p} - \frac{4}{\alpha}, \infty)$. Then for any $\tilde{s} \in (1 + \frac{3}{p} - \frac{4}{\alpha}, \min\{s, 1 + \frac{3}{p}\})$ with $(\alpha > 4)$, and for some $t \in (1 + \frac{3}{p} - \frac{4}{\alpha}, \tilde{s})$, there exists a closed ball $\overline{B}_M \subset W^{\tilde{s},p}$ of radius $M = \mathcal{O}\left([1 + \|a_{\boldsymbol{w}}\|_{s-2,p}]^{\tilde{s}/(\tilde{s}-t)}\right)$ such that*

$$\phi \in [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M \quad \Rightarrow \quad T^s(\phi, a_{\boldsymbol{w}}) \in \overline{B}_M.$$

Proof. From Lemma 5.1, there exist $t \in (1 + \frac{3}{p} - \frac{4}{\alpha}, \tilde{s})$ and $C_1, C_2 > 0$ such that

$$\|T^s(\phi, a_{\boldsymbol{w}})\|_{\tilde{s},p} \leqslant C_1(1 + \|a_{\boldsymbol{w}}\|_{s-2,p})\|\phi\|_{t,p} + C_2, \qquad \forall \phi \in [\phi_-, \phi_+]_{\tilde{s},p}.$$

For any $\varepsilon > 0$, the norm $\|\phi\|_{t,p}$ can be bounded by the interpolation estimate

$$\|\phi\|_{t,p} \leqslant \varepsilon\|\phi\|_{\tilde{s},p} + C\varepsilon^{-t/(\tilde{s}-t)}\|\phi\|_p,$$

where C is a constant independent of ε. Since ϕ is bounded from above by ϕ_+, $\|\phi\|_p$ is bounded uniformly, and now demanding that $\phi \in \overline{B}_M$, we get

$$\|T^s(\phi, a_{\boldsymbol{w}})\|_{\tilde{s},p} \leqslant C_1[1 + \|a_{\boldsymbol{w}}\|_{s-2,p}]\left(M\varepsilon + C\varepsilon^{-t/(\tilde{s}-t)}\right) + C_2, \tag{5.23}$$

with a possibly different constant C_1. Choosing ε such that $2\varepsilon C_1[1 + \|a_{\boldsymbol{w}}\|_{s-2,p}] = 1$ and setting $M = 2(CC_1[1 + \|a_{\boldsymbol{w}}\|_{s-2,p}]\varepsilon^{-t/(\tilde{s}-t)} + C_2)$, we can ensure that the right hand side of (5.23) is bounded by M. □

6. Barriers for the Hamiltonian constraint

The results developed in §5 for a particular fixed-point map T for analyzing the Hamiltonian constraint equation and the coupled system rely on the existence of generalized (weak) sub- and super-solutions, or barriers. There, the Hamiltonian constraint with Robin boundary conditions was studied in isolation from the momentum constraint with Robin type boundary conditions, and these generalized barriers only needed to satisfy the conditions given at the beginning of §5 for a given fixed function $\boldsymbol{w}$ appearing as a source term in the nonlinearity of the Hamiltonian constraint. Therefore, these types of barriers are sometimes referred to as *local barriers*, in that the coupling to the momentum constraint is ignored. In order to establish existence results for the coupled system in the non-CMC case, it will be critical that the sub- and super-solutions satisfy one additional property that now reflects the coupling, giving rise to the term *global barriers*. It will be useful now to define this global property precisely.

Definition 6.1. *A sub-solution ϕ_- is called* **global** *iff it is a sub-solution of* (5.5) *for all vector fields $\boldsymbol{w}_\phi$ solution of* (4.3) *with source function $\phi \in [\phi_-, \infty) \cap W^{s,p}$. A super-solution ϕ_+ is called* **global** *iff it is a super-solution of* (5.5) *for all vector fields $\boldsymbol{w}_\phi$ solution of* (4.3) *with source function $\phi \in (0, \phi_+] \cap W^{s,p}$. A pair $\phi_- \leqslant \phi_+$ of sub- and super-solutions is called an* **admissible pair** *if ϕ_- and ϕ_+ are sub- and super-solutions of* (5.5) *for all vector fields $\boldsymbol{w}_\phi$ of* (4.3) *with source function $\phi \in [\phi_-, \phi_+] \cap W^{s,p}$.*

It is obvious that if ϕ_- and ϕ_+ are respectively global sub- and super-solutions, then the pair ϕ_-, ϕ_+ is admissible in the sense above, provided they satisfy the compatibility condition (5.9).

Here our primary interests is in developing existence results for weak (and strong) non-CMC solutions to the coupled system with Robin boundary conditions which are free of the near-CMC assumption. This assumption had appeared in two distinct places in all prior literature in the case of closed manifolds [15, 1]; the first assumption appears in the construction of a fixed-point argument based on strict k-contractions, and the second assumption appears in the construction of global super-solutions. In the case of compact manifolds with boundary, the only existence results to date require the mean curvature to be constant [13, 5, 18]. In this section, we construct global super-solutions for the coupled constraint equations with Robin boundary conditions that are free of the near-CMC assumption, along with some compatible sub-solutions. These sub- and super-solution constructions are needed for the general fixed-point result for the coupled system, leading to our main non-CMC results (Theorem 3.3).

Throughout this section, we will assume that the background metric h belongs to $W^{s,p}$ with $p \in (1,\infty)$ and $s \in (\frac{3}{p},\infty) \cap (1,2]$. Recall that $r = \frac{3p}{3+(2-s)p}$, so that the continuous embedding $L^r \hookrightarrow W^{s-2,p}$ holds. Given a symmetric two-index tensor $\sigma \in L^{2r}$ and a vector field $\boldsymbol{w} \in \boldsymbol{W}^{1,2r}$, introduce the functions $a_\sigma = \frac{1}{8}\sigma^2 \in L^r$ and $a_{\mathcal{L}\boldsymbol{w}} = \frac{1}{8}(\mathcal{L}\boldsymbol{w})^2 \in L^r$. Note that under these conditions $a_{\boldsymbol{w}}$ belongs to $L^r \hookrightarrow W^{s-2,p}$, and that if $a_\sigma, a_{\mathcal{L}\boldsymbol{w}} \in L^\infty$, we have the pointwise estimate

$$a_{\boldsymbol{w}}^\wedge \leqslant 2a_\sigma^\wedge + 2a_{\mathcal{L}\boldsymbol{w}}^\wedge. \tag{6.1}$$

Here and in what follows, given any scalar function $u \in L^\infty$, we use the notation

$$u^\wedge := \text{ess sup}\, u, \qquad u^\vee := \text{ess inf}\, u.$$

In the event that a given function f is defined on some portion of the boundary (i.e. on Σ_I or Σ_E), we slightly abuse notation and let $f^\vee$ and $f^\wedge$ denote

$$f^\vee = \min_{\Sigma_i}(\text{ess inf}|_{\Sigma_i} f), \quad \text{and} \quad f^\wedge = \max_{\Sigma_i}(\text{ess sup}|_{\Sigma_i} f).$$

In some places we will assume that when the vector field $\boldsymbol{w} \in \boldsymbol{W}^{1,2r}$ is given by the solution of the momentum constraint equation (4.3) with the source term $\phi \in W^{s,p}$,

$$a_{\mathcal{L}\boldsymbol{w}}^\wedge \leqslant \mathtt{k}(\phi_+) := \mathtt{k}_1 \|\phi_+\|_\infty^{12} + \mathtt{k}_2, \tag{6.2}$$

with some positive constants $\mathtt{k}_1$ and $\mathtt{k}_2$ and ϕ_+ an *a priori* upper bound on ϕ. We can verify this assumption e.g. when the conditions of Theorem 4.5 and Remark 4.6 are satisfied, since from Remark 4.6 we get

$$\begin{aligned} a^{\wedge}_{\mathcal{L}\boldsymbol{w}} =& \|\mathcal{L}\boldsymbol{w}\|^2_{\infty} \\ \leqslant& C^2 \Big(\|\phi_+\|^6_{\infty} \|\mathbf{b}_\tau\|_z + \|\mathbf{b}_j\|_{e-2,q} + \|\tau B^6\|_{e-1-\frac{1}{q},q;\Sigma_I} \\ &\quad + \|\theta_- B^6\|_{e-1-\frac{1}{q},q;\Sigma_I} + \|\sigma(\nu,\nu)\|_{e-1-\frac{1}{q},q;\Sigma_I} + \|\mathbf{X}\|_{e-1-\frac{1}{q},q;\Sigma_I} \Big)^2, \end{aligned} \tag{6.3}$$

given that $S(\nu,\nu) = (2\gamma_I(\tau) + |\theta_-|/2)B^6$. If we apply Lemma 29 in [13] to the term

$$\|\tau B^6\|_{e-1-\frac{1}{q},q;\Sigma_I},$$

appearing in (6.3), we obtain

$$\|\tau B^6\|_{e-1-\frac{1}{q},q;\Sigma_I} \leqslant C \left(\|\tau\|_{e-1-\frac{1}{q},q;\Sigma_I} \|B\|^5_{\infty} \|B\|_{\tilde{s},p} \right), \tag{6.4}$$

where $q \in (3, \frac{\alpha+1}{3}),\ \alpha > 8,\ e \in (1+\frac{3}{q}, 2],\ p \in (3,\infty)$ and $\tilde{s} \in (1+\frac{3}{p}-\frac{4}{\alpha}, 1+\frac{3}{p})$, which is similar to conditions in Lemma 5.1. A similar bound can obtained for the term $\|\theta_- B^6\|_{e-1-\frac{1}{q},q;\Sigma_I}$, and combining these estimates with (6.3) we obtain bound (6.2) with the constants

$$\begin{aligned} \mathtt{k}_1 &= 2C^2(\|\boldsymbol{b}_\tau\|_z)^2, \\ \mathtt{k}_2 &= 2C^2 \Big(\|\boldsymbol{b}_j\|_{e-2,q} + \|\sigma(\nu,\nu)\|_{e-1-\frac{1}{q},q;\Sigma_I} + \|\mathbf{X}\|_{e-1-\frac{1}{q},q;\Sigma_I} \\ &\qquad + \|B\|^5_{\infty}\|B\|_{\tilde{s},p} \Big(\|\tau\|_{e-1-\frac{1}{q},q;\Sigma_I} + \|\theta_-\|_{e-1-\frac{1}{q},q;\Sigma_I} \Big) \Big)^2. \end{aligned} \tag{6.5}$$

6.1. **Constant barriers.** Now we will present some global sub- and super-solutions for the Hamiltonian constraint equation (5.5) which are constant functions. The proofs are based on the arguments in [12] for the case of closed manifolds. To simplify notation, we will omit the trace operators γ_I and γ_E from the boundary operators.

Lemma 6.2. (Global super-solution) *Let $(\mathcal{M}, h)$ be a 3-dimensional, smooth, compact Riemannian manifold with metric $h \in W^{s,p}$, $s > \frac{3}{p}$ and non-empty boundary satisfying the conditions* (3.1)*. Suppose that the estimate* (6.2) *holds for the solution of the momentum constraint equation, and assume that a_R is uniformly bounded from below, $a_\rho, a_\sigma \in L^\infty$, $c^\vee > 0$, $(2\tau^\vee + |\theta_-|^\vee) > 0$ on Σ_I and g is uniformly bounded from above. With the parameter $\varepsilon > 0$ to be chosen later, define the following rational polynomials*

$$q_{1,\varepsilon}(\chi) = (a^\vee_\tau - \mathtt{K}_{1\varepsilon})\,\chi^5 + a^\vee_R\,\chi - a^\wedge_\rho\,\chi^{-3} - \mathtt{K}_{2\varepsilon}\chi^{-7}, \tag{6.6}$$

$$q_2(\chi) = \frac{1}{2}H^\vee\chi + \left(\frac{1}{2}\tau^\vee + \frac{|\theta_-|^\vee}{4}\right)\chi^3 - \left(\frac{1}{2}\tau^\wedge + \frac{1}{8}|\theta_-|^\wedge\right)B^6\chi^{-3} \tag{6.7}$$

$$q_3(\chi) = c^\vee\chi - g^\wedge, \tag{6.8}$$

where $\mathtt{K}_{1\varepsilon} := (1+\frac{1}{\varepsilon})\mathtt{k}_1$ and $\mathtt{K}_{2\varepsilon} := (1+\varepsilon)a^\wedge_\sigma + (1+\frac{1}{\varepsilon})\mathtt{k}_2$.

We distinguish the following two cases:

(a) In case $\mathtt{k}_1 < a^\vee_\tau$, choose $\varepsilon > \dfrac{\mathtt{k}_1}{a^\vee_\tau - \mathtt{k}_1}$. If $q_{1\varepsilon}$ has a root, let $\phi_1 = \phi_1(a^\vee_\tau - \mathtt{K}_{1\varepsilon}, a^\vee_R, a^\wedge_\rho, \mathtt{K}_{2\varepsilon})$ be the largest positive root of $q_{1\varepsilon}$, and if $q_{1\varepsilon}$ has no positive roots, let $\phi_1 = 1$. Similarly, let ϕ_2 be the largest positive root of q_2 if its exists, otherwise let $\phi_2 = 1$. Now, the constant $\phi_+ = \max\{\phi_1, \phi_2, g^\wedge/c^\vee\}$ is a global super-solution of the Hamiltonian constraint equation (5.5).

(b) In case $\mathtt{k}_1 \geqslant a^\vee_\tau$, choose $\varepsilon > 0$. In addition, assume that $a^\vee_R > 0$ is sufficiently large and that both $a^\wedge_\rho$ and $\mathtt{K}_{2\varepsilon}$ are sufficiently small, so that $q_{1\varepsilon}$ has two positive roots,

with the largest being as large as $\max\{\phi_3, g^\wedge/c^\vee\}$*, where* ϕ_3 *is the largest positive root of* q_3*. Then, the largest root* $\phi_+ = \phi_2(a_\tau^\vee - \mathtt{K}_{1\varepsilon}, a_R^\vee, a_\rho^\wedge, \mathtt{K}_{2\varepsilon})$ *of* $q_{1\varepsilon}$ *is a super-solution of the Hamiltonian constraint equation (5.5).*

Proof. We look for a super-solution among the constant functions. Let χ be any positive constant. Then we have

$$A(\chi) + F(\chi, \boldsymbol{w}) = F(\chi, \boldsymbol{w}) = \begin{pmatrix} a_\tau\chi^5 + a_R\chi - a_\rho\chi^{-3} - a_{\boldsymbol{w}}\chi^{-7} \\ \frac{1}{2}H\chi + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)\chi^3 - \frac{1}{4}S(\nu,\nu)\chi^{-3} \\ c\chi - g \end{pmatrix}.$$

In order for χ to be a super-solution of (5.5), we require that $F(\chi, \boldsymbol{w}) \geqslant \boldsymbol{0}$, which implies that each of the components of $F(\chi, \boldsymbol{w})$ must be nonnegative. Given any $\varepsilon > 0$, the inequality $2|\sigma_{ab}(\mathcal{L}\boldsymbol{w})^{ab}| \leqslant \varepsilon\sigma^2 + \frac{1}{\varepsilon}(\mathcal{L}\boldsymbol{w})^2$ implies that

$$8a_{\boldsymbol{w}} = \sigma^2 + (\mathcal{L}\boldsymbol{w})^2 + 2\sigma_{ab}(\mathcal{L}\boldsymbol{w})^{ab} \leqslant (1+\varepsilon)\,\sigma^2 + (1+\tfrac{1}{\varepsilon})\,(\mathcal{L}\boldsymbol{w})^2,$$

hence, taking into account (6.2), for any $\boldsymbol{w} \in \boldsymbol{W}^{1,2r}$ that is a solution of the momentum constraint equation (4.3) with any source term $\phi \in (0, \chi]$, the constant $a_{\boldsymbol{w}}^\wedge$ must fulfill the inequality

$$a_{\boldsymbol{w}}^\wedge \leqslant (1+\varepsilon)a_\sigma^\wedge + (1+\tfrac{1}{\varepsilon})a_{\mathcal{L}\boldsymbol{w}}^\wedge \leqslant \mathtt{K}_{1\varepsilon}\|\phi_+\|_\infty^{12} + \mathtt{K}_{2\varepsilon}. \tag{6.9}$$

Using the fact that $S(\nu,\nu) = (2\tau + |\theta_-|/2)B^6$ and letting $\phi_+ = \chi$, we have that for any constant $\chi > 0$ and all $\phi \in (0, \chi]$, it holds that

$$\begin{aligned} F(\chi, \boldsymbol{w}_\phi) &\geqslant \begin{pmatrix} a_\tau^\vee\chi^5 + a_R^\vee\chi - a_\rho^\wedge\chi^{-3} - \left(\mathtt{K}_{1\varepsilon}\,\|\phi_+\|_\infty^{12} + \mathtt{K}_{2\varepsilon}\right)\chi^{-7} \\ \frac{1}{2}H^\vee\chi + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)\chi^3 - \frac{1}{4}((2\tau + |\theta_-|/2)B^6)\chi^{-3} \\ c^\vee\chi - g^\wedge \end{pmatrix} \\ &\geqslant \begin{pmatrix} B_\varepsilon\chi^5 + a_R^\vee\chi - a_\rho^\wedge\chi^{-3} - \mathtt{K}_{2\varepsilon}\chi^{-7} \\ \frac{1}{2}H^\vee\chi + \left(\frac{1}{2}\tau^\vee + \frac{|\theta_-|^\vee}{4}\right)\chi^3 - \left(\frac{1}{2}\tau^\wedge + \frac{1}{8}|\theta_-|^\wedge\right)(B^\wedge)^6\chi^{-3} \\ c^\vee\chi - g^\wedge \end{pmatrix} \\ &= \begin{pmatrix} q_{1\varepsilon}(\chi) \\ q_2(\chi) \\ q_3(\chi) \end{pmatrix}, \end{aligned} \tag{6.10}$$

where $B_\varepsilon := a_\tau^\vee - \mathtt{K}_{1\varepsilon}$.

Clearly we have that $q_2(\chi) \geqslant 0$ for χ sufficiently large if $2\tau^\vee + |\theta_-|^\vee > 0$. Similarly, $q_3(\chi) \geqslant 0$ for $\chi \geqslant g^\wedge/c^\vee$. We calculate the first and second derivative of $q_{1\varepsilon}$ as

$$\begin{aligned} q_\varepsilon'(\chi) &= 5B_\varepsilon\chi^4 + a_R^\vee + 3a_\rho^\wedge\chi^{-4} + 7\mathtt{K}_{2\varepsilon}\chi^{-8}, \\ q_\varepsilon''(\chi) &= 20B_\varepsilon\chi^3 - 12a_\rho^\wedge\chi^{-5} - 56\mathtt{K}_{2\varepsilon}\chi^{-9}. \end{aligned} \tag{6.11}$$

Consider the case (a). In this case, because of the choice $\varepsilon > \frac{\mathtt{k}_1}{a_\tau^\vee - \mathtt{k}_1}$, we have $B_\varepsilon > 0$, and so $q_{1\varepsilon}(\chi) > 0$ for sufficiently large χ, and $q_{1\varepsilon}$ is increasing. The function $q_{1\varepsilon}$ has no positive root only if $a_\rho^\wedge = \mathtt{K}_{2\varepsilon} = 0$. So if $q_{1\varepsilon}$ has no positive root, let $\phi_1 = 1$ and $q_{1\varepsilon}(\chi) \geqslant 0$ for all $\chi \geqslant 0$. If $q_{1\varepsilon}$ has at least one positive root, let ϕ_1 be the largest positive root and $q_{1\varepsilon}(\chi) \geqslant 0$ for all $\chi \geqslant \phi_1$. Similarly, let ϕ_2 be the largest positive root of q_2 if it exists, otherwise let $\phi_2 = 1$. Then $q_2(\chi) \geqslant 0$ for all $\chi \geqslant \phi_2$ given that $2\tau^\vee + |\theta_-|^\vee > 0$. Recalling now that any constant χ satisfies $A(\chi) = 0$, we conclude that

$$A(\chi) + F(\chi, \boldsymbol{w}_\phi) \geqslant 0 \qquad \forall\,\chi \geqslant\, \max\{\phi_1, \phi_2, g^\wedge/c^\vee\}\,,\ \forall\,\phi \in (0, \chi], \tag{6.12}$$

implying that $\phi_+ = \max\{\phi_1, \phi_2, g^\wedge/c^\vee\}$ is a global super-solution of the Hamiltonian constraint (5.5).

For the case (b), since $B_\varepsilon < 0$ and $a_\rho^\wedge$ and $\mathtt{K}_{2\varepsilon}$ are nonnegative, the first derivative $q'_{1\varepsilon}(\chi)$ is strictly decreasing for $\chi > 0$, and since $q'_{1\varepsilon}(\chi) > 0$ for sufficiently small $\chi > 0$ and $q'_\varepsilon(\chi) < 0$ for sufficiently large $\chi > 0$, the derivative $q'_{1\varepsilon}$ has a unique positive root, at which the polynomial $q_{1\varepsilon}$ attains its maximum over $(0, \infty)$. This maximum is positive if both $a_\rho^\wedge$ and $\mathtt{K}_{2\varepsilon}$ are sufficiently small, and hence the polynomial $q_{1\varepsilon}$ has two positive roots $\phi_{1\epsilon} \leqslant \phi_{2\epsilon}$. Moreover, if $a_R^\vee$ is sufficiently large, $\phi_{2\epsilon} \geqslant \max\{\phi_3, g^\wedge/c^\vee\}$, where ϕ_3 is the largest positive root of q_3 if it exists, and $\phi_3 = 1$ otherwise. Similar to the above we conclude that

$$A(\chi) + F(\chi, \boldsymbol{w}_\phi) \geqslant 0, \text{ for } \chi = \phi_{2\epsilon}, \forall\, \phi \in (0, \chi],$$

implying that $\phi_+ = \phi_{2\epsilon}$ is a global super-solution of the Hamiltonian constraint (5.5). $\square$

Remark 6.3. *In order for the condition $S(\nu, \nu) = (2\tau + |\theta_-|)B^6$ to imply the marginally trapped surface condition* (2.33)*, if suffices to construct a global super-solution ϕ_+ and choose B to be a constant such that $B > \|\phi_+\|_\infty$. In light of* (6.5)*, we observe that for both cases (a) and (b) above we can choose B large and $\|\tau\|_{e-1-\frac{1}{q},q;\Sigma_I}$ and $\|\theta_-\|_{e-1-\frac{1}{q},q;\Sigma_I}$ sufficiently small so that the coefficient $K_{2\epsilon}$ in* (6.10) *remains unchanged and $(2\tau^\wedge + |\theta_-|^\wedge/2)B^6$ decreases in size. This ensures that we can choose $B > \|\phi_+\|_\infty$ in the above construction.*

Case (a) of the above lemma has the condition $\mathtt{k}_1 < a_\tau^\vee$, which is analogous to the near-CMC condition. The above condition also requires that the extrinsic mean curvature τ is nowhere zero. Noting that there are solutions even for $\tau \equiv 0$ in some cases (cf. [14]), the condition $\inf \tau \neq 0$ appears as a rather strong restriction. We see that case (b) of the above lemma removes this restriction, in exchange for the sign condition on R and size conditions on R, ρ, j, and σ.

In the next Lemma we construct a global sub-solution using a pre-existing global super-solution.

Lemma 6.4. (Global sub-solution) *Let $(\mathcal{M}, h)$ be a 3-dimensional, smooth, compact Riemannian manifold with metric $h \in W^{s,p}$, $s \geqslant \frac{3}{p}$ and non-empty boundary satisfying the conditions* (3.1)*. Assume that c, H, a_R and τ are uniformly bounded from above, and g is uniformly bounded from below. Let $\phi_+ > 0$ be a global super-solution of the Hamiltonian constraint and suppose that $(4\tau^\vee + |\theta_-|^\vee) > 0$ on Σ_I. Let ϕ_1 be the unique positive root of the polynomial*

$$q(\chi) = \frac{1}{2}\max\{1, H^\wedge\}\chi + \left(\frac{1}{2}\tau^\wedge + \frac{|\theta_-|^\wedge}{4}\right)\chi^3 - \left(\frac{1}{2}\tau^\vee + \frac{1}{8}|\theta_-|^\vee\right)B^6\chi^{-3} \qquad (6.13)$$

We distinguish between the following two cases:

(a) If $a_\rho^\vee > 0$, let ϕ_2 be the unique positive root of the polynomial

$$q_\rho(\chi) = a_\tau^\wedge \chi^8 + \max\{1, a_R^\wedge\}\chi^4 - a_\rho^\vee. \qquad (6.14)$$

Then

$$\phi_- = \min\{\phi_1, \phi_2, g^\vee/H^\wedge\}$$

is a global sub-solution.

(b) Let $a_\sigma^\vee > \mathtt{k}(\phi_+)$, where $\mathtt{k}$ is as in (6.2)*. Then, for some $\varepsilon \in (\mathtt{k}(\phi_+)/a_\sigma^\vee, 1)$, if ϕ_3 is the unique positive root of the polynomial*

$$q_\sigma(\chi) = a_\tau^\wedge\, \chi^{12} + \max\{1, a_R^\wedge\}\, \chi^8 - \mathtt{K}_\varepsilon,$$

where $\mathtt{K}_\varepsilon := (1-\varepsilon)a_\sigma^\vee - \left(\frac{1}{\varepsilon}-1\right)\mathtt{k}(\phi_+)$, *then*

$$\phi_- = \min\{\phi_1, \phi_3, g^\vee/H^\wedge\}$$

is a global sub-solution of (5.5).

Proof. If χ is constant, then we have

$$A(\chi) + F(\chi, \boldsymbol{w}_\phi) = F(\chi, \boldsymbol{w}_\phi) = \begin{pmatrix} a_\tau\chi^5 + a_R\chi - a_\rho\chi^{-3} - a_{\boldsymbol{w}}\chi^{-7} \\ \frac{1}{2}H\chi + \frac{1}{2}\tau\chi^3 - \frac{1}{4}S(\nu,\nu)\chi^{-3} - \frac{1}{4}\theta_-\chi^3 \\ c\chi - g \end{pmatrix}.$$

In order for χ to be a sub-solution of (5.5), we require that $F(\chi, \boldsymbol{w}_\phi) \leqslant \mathbf{0}$, which implies that each of the components of $F(\chi, \boldsymbol{w}_\phi)$ must be non-positive.

In case (a), we have that

$$\begin{aligned} &F(\chi, \boldsymbol{w}_\phi) \\ &\leqslant \begin{pmatrix} a_\tau^\wedge\chi^5 + a_R^\wedge\chi - a_\rho^\vee\chi^{-3} \\ \frac{1}{2}H^\wedge\chi + (\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge)\chi^3 - \frac{1}{4}((2\tau^\vee + |\theta_-|^\vee/2)B^6)\chi^{-3} \\ c^\wedge\chi - g^\vee \end{pmatrix} \qquad (6.15) \\ &\leqslant \begin{pmatrix} a_\tau^\wedge\chi^5 + \max\{1, a_R^\wedge\}\chi - a_\rho^\vee\chi^{-3} \\ \frac{1}{2}\max\{1, H^\wedge\}\chi + (\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge)\chi^3 - \frac{1}{4}((2\tau^\vee + |\theta_-|^\vee/2)(B^\vee)^6)\chi^{-3} \\ c^\wedge\chi - g^\vee \end{pmatrix}. \end{aligned}$$

We observe that each component in (6.15) will be non-positive provided that we have $\phi_- = \chi = \min\{\phi_1, \phi_2, g^\vee/c^\wedge\}$, where $\phi_- > 0$ given that $a_\rho > 0$, $(4\tau^\vee + |\theta_-|^\vee) > 0$ on Σ_I and $g^\vee/c^\wedge > 0$ by (3.8).

In case (b), we observe that if $\chi > 0$ is any constant function and $\boldsymbol{w} \in \boldsymbol{W}^{1,2r}$, then we have

$$F(\chi, \boldsymbol{w}_\phi) \leqslant \begin{pmatrix} a_\tau^\wedge\chi^5 + C\chi - a_{\boldsymbol{w}}^\vee\chi^{-7} \\ \frac{1}{2}H^\wedge\chi + (\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge)\chi^3 - ((\frac{1}{2}\tau^\vee + \frac{1}{8}|\theta_-|^\vee)(B^\vee)^6)\chi^{-3} \\ c^\wedge\chi - g^\vee \end{pmatrix}, \quad (6.16)$$

where we have used that a_ρ is nonnegative, and introduced the constant $C = \max\{1, a_R^\wedge\}$.

Given any $\varepsilon > 0$, the inequality $2|\sigma_{ab}(\mathcal{L}\boldsymbol{w})^{ab}| \leqslant \varepsilon\sigma^2 + \frac{1}{\varepsilon}(\mathcal{L}\boldsymbol{w})^2$ implies that

$$8a_{\boldsymbol{w}} = \sigma^2 + (\mathcal{L}\boldsymbol{w})^2 + 2\sigma_{ab}(\mathcal{L}\boldsymbol{w})^{ab} \geqslant (1-\varepsilon)\,\sigma^2 - (\tfrac{1}{\varepsilon}-1)\,(\mathcal{L}\boldsymbol{w})^2,$$

hence, taking into account (6.2), for any $\boldsymbol{w} \in \boldsymbol{W}^{1,2r}$ that is a solution of the momentum constraint equation (4.3) with any source term $\phi \in (0, \phi_+]$, the constant $a_{\boldsymbol{w}}^\vee$ must fulfill the inequality

$$a_{\boldsymbol{w}}^\vee \geqslant (1-\varepsilon)a_\sigma^\vee - (\tfrac{1}{\varepsilon}-1)a_{\mathcal{L}\boldsymbol{w}}^\wedge \geqslant (1-\varepsilon)a_\sigma^\vee - (\tfrac{1}{\varepsilon}-1)\mathtt{k}(\phi_+) =: \mathtt{K}_\varepsilon.$$

We use the above estimate in (6.16) to get, for any $\boldsymbol{w} \in \boldsymbol{W}^{1,2r}$ that is a solution of the momentum constraint equation (4.3) with any source term $\phi \in (0, \phi_+]$,

$$F(\chi, \boldsymbol{w}_\phi) \leqslant \begin{pmatrix} a_\tau^\wedge\chi^5 + C\chi - \mathtt{K}_\varepsilon\chi^{-7}. \\ \frac{1}{2}H^\wedge\chi + (\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge)\chi^3 - ((\frac{1}{2}\tau^\vee + \frac{1}{8}|\theta_-|^\vee)(B^\vee)^6)\chi^{-3} \\ c^\wedge\chi - g^\vee \end{pmatrix}. \quad (6.17)$$

Because of the choice $\mathtt{k}(\phi_+)/a_\sigma^\vee < \varepsilon < 1$, we have $\mathtt{K}_\varepsilon > 0$. So with the unique positive root ϕ_3 of

$$q_\sigma(\chi) := a_\tau^\wedge\,\chi^5 + C\,\chi - \mathtt{K}_\varepsilon\,\chi^{-7},$$

we have $q_\sigma(\chi) \leqslant 0$ for any constant $\chi \in (0, \phi_3]$. Taking $\phi_- = \min\{\phi_1, \phi_3, g^\vee/c^\wedge\}$, we have that $F(\phi, \boldsymbol{w}_\phi) \leqslant 0$, which completes the proof. □

6.2. **Non-constant Barriers.** The barriers constructed in the previous section require that the scalar curvature be strictly positive and sufficiently large or that the near-CMC condition be satisfied. Few restrictions are placed on the size of the data $\theta_-, \rho, \mathbf{j}, \sigma(\nu,\nu)$ and $\mathbf{X}$. In this section we develop non-constant global sub-and super-solutions using an auxiliary problem similar to the one considered in Lemma A.1 in Appendix A. The advantage of this construction is that we only require the metric $h \in \mathcal{Y}^+$. However, the tradeoff is that we will require the data $|\theta_-|, \rho, \mathbf{j}, \sigma(\nu,\nu)$ and $\mathbf{X}$ to be sufficiently small. Additionally, we will either have to require that b_τ is sufficiently small or that $\delta > 0$ is sufficiently small, where we recall that $g = \delta(c + \mathcal{O}(R^{-3}))$. This assumption is analogous to the smallness condition on the Dirichlet data ϕ_D in [7].

Lemma 6.5. (Non-Constant Global super-solution for small $D\tau$) *Let $(\mathcal{M}, h)$ be a 3-dimensional, smooth, compact Riemannian manifold with metric $h \in W^{s,p}$, $s > \frac{3}{p}$ in the positive Yamabe class and non-empty boundary satisfying the conditions* (3.1)*. Assume that the estimate* (6.2) *holds for the solution of the momentum constraint equation for two positive constants k_1 and k_2, which can be chosen sufficiently small. Additionally assume that $a_\rho, a_\sigma \in L^\infty$, $a_\rho^\wedge, a_\sigma^\wedge$ are sufficiently small and that $(2\tau^\vee + |\theta_-|^\vee) > 0$ on Σ_I. Then if $u \in W^{s,p}$ satisfies*

$$
\begin{aligned}
-\Delta u + a_R u &= \Lambda_1 > 0, \\
\gamma_I \partial_\nu u + \frac{1}{2} H \gamma_I u &= \Lambda_2 > 0, \\
\gamma_E \partial_\nu u + c\gamma_E u &= \Lambda_3 > 0,
\end{aligned}
\tag{6.18}
$$

for positive functions $\Lambda_1, \Lambda_2, \Lambda_3$, then there exists a constant $\beta > 0$ such that $\phi_+ = \beta u$ is a positive global super-solution of the Hamiltonian constraint equation (5.5).

Proof. We first observe that by Lemmas B.8 and B.7 in [13], the solution u exists and is positive. Evaluating (5.5) at ϕ_+ we have

$$
A(\phi_+) + F(\phi_+, \boldsymbol{w}) = \begin{pmatrix} \beta\Lambda_1 + a_\tau \phi_+^5 - a_\rho \phi_+^{-3} - a_{\boldsymbol{w}} \phi_+^{-7} \\ \beta\Lambda_2 + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)(\gamma_I \phi_+)^3 - \frac{1}{4} S(\nu,\nu)(\gamma_I\phi_+)^{-3} \\ \beta\Lambda_3 - g \end{pmatrix}.
$$

If we use the point-wise bound (6.1) and the estimate (6.2) with k_1 and k_2, we have that $a_{\boldsymbol{w}} \leqslant K_1(\phi_+^\wedge)^{12} + K_2$, where $K_1 = 2k_1$ and $K_2 = 2a_\sigma^\wedge + 2k_2$. Letting $k_3 = u^\wedge/u^\vee$ and recalling that $\theta_- \leqslant 0$, we then have

$$
\begin{aligned}
&A(\phi_+) + F(\phi_+, \boldsymbol{w}) \\
&\geqslant \begin{pmatrix} \beta\Lambda_1 + a_\tau^\vee \phi_+^5 - a_\rho^\wedge \phi_+^{-3} - [K_1(\phi_+^\wedge)^{12} + K_2]\phi_+^{-7} \\ \beta\Lambda_2 + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)(\gamma_I\phi_+)^3 - \left(\left(\frac{1}{2}\tau + \frac{1}{8}|\theta_-|\right)B^6\right)(\gamma_I\phi_+)^{-3} \\ \beta\Lambda_3 - g^\wedge \end{pmatrix} \\
&\geqslant \begin{pmatrix} \beta\Lambda_1 - \beta^5 K_1 k_3^{12} u^5 - \beta^{-3} a_\rho^\wedge u^{-3} - \beta^{-7} K_2 u^{-7} \\ \beta\Lambda_2 + \beta^3\left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)(\gamma_I u)^3 - \beta^{-3}\left(\frac{1}{2}\tau + \frac{1}{8}|\theta_-|\right)B^6(\gamma_I u)^{-3} \\ \beta\Lambda_3 - g^\wedge \end{pmatrix} \\
&\geqslant \begin{pmatrix} \beta\Lambda_1^\vee - \beta^5 K_1 k_3^{12}(u^\wedge)^5 - \beta^{-3} a_\rho^\wedge (u^\vee)^{-3} - \beta^{-7} K_2 (u^\vee)^{-7} \\ \beta\Lambda_2^\vee + \beta^3\left(\frac{1}{2}\tau^\vee + \frac{1}{4}|\theta_-|^\vee\right)(\gamma_I u^\vee)^3 - \beta^{-3}\left(\frac{1}{2}\tau^\wedge + \frac{1}{8}|\theta_-|^\wedge\right)(B^\wedge)^6(\gamma_I u^\vee)^{-3} \\ \beta\Lambda_3^\vee - g^\wedge \end{pmatrix}
\end{aligned}
$$

Therefore, $A(\phi_+) + F(\phi_+, \boldsymbol{w}) \geqslant 0$ provided that we can choose β, k_1, k_2, a_ρ and a_σ so that

$$\beta\Lambda_1^\vee - \beta^5 K_1 k_3^{12}(u^\wedge)^5 - \beta^{-3}a_\rho^\wedge(u^\vee)^{-3} - \beta^{-7}K_2(u^\vee)^{-7} \geqslant 0, \tag{6.19}$$

$$\beta\Lambda_2^\vee + \beta^3\left(\frac{1}{2}\tau^\vee + \frac{1}{4}|\theta_-|^\vee\right)(\gamma_I u^\vee)^3 - \beta^{-3}\left(\frac{1}{2}\tau^\wedge + \frac{1}{8}|\theta_-|^\wedge\right)(B^\wedge)^6(\gamma_I u^\vee)^{-3} \geqslant 0, \tag{6.20}$$

$$\beta\Lambda_3^\vee - g^\wedge \geqslant 0. \tag{6.21}$$

We choose β sufficiently large so that Eqs. (6.20)-(6.21) are nonnegative. Then choose k_1 so that

$$\beta\Lambda_1^\vee - \beta^5 K_1 k_3^{12}(u^\wedge)^5 > 0. \tag{6.22}$$

Finally, choose a_ρ, a_σ and k_2 sufficiently small so that

$$\beta\Lambda_1^\vee - \beta^5 K_1 k_3^{12}(u^\wedge)^5 - \beta^{-3}a_\rho^\wedge(u^\vee)^{-3} - \beta^{-7}K_2(u^\vee)^{-7} \geqslant 0. \tag{6.23}$$

For this choice of data, $\phi_+ = \beta u$ is a global super solution. □

Remark 6.6. *As we mentioned in Remark 6.3, we require that $B > \|\phi_+\|_\infty$ in order for the condition $S(\nu,\nu) = (2\tau + |\theta_-|)B^6$ to imply that the marginally trapped surface condition* (2.33). *We may again choose choose B to be a large constant and require that $\|\tau\|_{e-1-\frac{1}{q},q;\Sigma_I}$ and $\|\theta_-\|_{e-1-\frac{1}{q},q;\Sigma_I}$ be sufficiently small so that the coefficient K_2 in* (6.23) *remains unchanged and $(2\tau^\wedge + |\theta_-|^\wedge/2)B^6$ decreases in size. This ensures that we can choose $B > \|\phi_+\|_\infty$ in the above construction.*

Remark 6.7. *The requirement that k_1 be sufficiently small places a restriction on the size of $b_\tau = \frac{2}{3}D\tau$. While this is not ideal, the above result still allows for τ to have zeroes and not satisfy the inequality*

$$\frac{\|D\tau\|_z}{\tau^\vee} \leqslant C < \infty,$$

where $z \geqslant 1$. This is the near-CMC condition. So the above barrier construction holds in the far-CMC setting, even though it places some restrictions on $D\tau$.

Recalling that $g = \delta(c + \mathcal{O}(R^{-3}))$ for $\delta > 0$, we show in the next Lemma that we may construct global super-solutions if we replace the assumption that $D\tau$ be small with the assumption that δ can be chosen arbitrarily small

Lemma 6.8. **(Non-Constant Global Super-Solution with small δ)** *Let the assumptions of Lemma 6.5 hold, with the exception that no smallness assumptions are placed on k_1. Then if $\delta > 0$ can be chosen sufficiently small, there exists a $\beta > 0$ such that if $B = \beta u$, then $\phi_+ = \beta u$ is a positive global super-solution of the Hamiltonian constraint equation (5.5).*

Proof. If we set $B = \beta u$, and follow the same process as in the proof of Lemma 6.5, we find that the following three inequalities must be satisfied in order for βu to be a global super-solution:

$$\beta\Lambda_1^\vee - \beta^5 K_1 k_3^{12}(u^\wedge)^5 - \beta^{-3}a_\rho^\wedge(u^\vee)^{-3} - \beta^{-7}K_2(u^\vee)^{-7} \geqslant 0, \tag{6.24}$$

$$\beta\Lambda_2^\vee + \beta^3\frac{1}{8}|\theta_-|^\vee(\gamma_I u^\vee)^3 \geqslant 0, \tag{6.25}$$

$$\beta\Lambda_3^\vee - g^\wedge \geqslant 0. \tag{6.26}$$

Eq. (6.25) is always true for positive u, and we may now choose $\beta > 0$, a_σ, k_2, a_ρ sufficiently small so that (6.24) is true. Finally, choosing δ sufficiently small will ensure that (6.26) is satisfied. □

Remark 6.9. *We note that the choice of B in Lemma 6.8 ensures that enforcing the condition $S(\nu,\nu) = (2\tau + |\theta_-|)B^6$ will imply the marginally trapped surface condition* (2.33).

Lemma 6.10. (Non-Constant Global Sub-solution) *Let $(\mathcal{M}, h)$ be a 3-dimensional, smooth, compact Riemannian manifold with metric $h \in W^{s,p}$, $s > \frac{3}{p}$ and non-empty boundary satisfying the conditions* (3.1)*. Assume that $\tau \in L^\infty$, $(4\tau^\vee + |\theta_-|^\vee) > 0$ on Σ_I, $g^\vee > 0$ and that ϕ_+ is the global super-solution obtained from Lemma 6.5 by solving* (6.18)*. We have the following two cases:*

(a) If $a_\rho^\vee > 0$, then there exists $\alpha > 0$ sufficiently small so that $\phi_- = \alpha\phi_+ < \phi_+$ is a global sub-solution.

(b) Suppose that $a_\sigma^\wedge > \mathtt{k}(\phi_+)$, where $\mathtt{k}$ is as in (6.2)*. Then there exists $\alpha > 0$ sufficiently small so that $\phi_- = \alpha\phi_+ < \phi_+$ is a global sub-solution.*

Proof. Evaluating at ϕ_-, where α is to be determined, we have that

$$A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) = \left(\begin{array}{c} \alpha\beta\Lambda_1 + a_\tau\phi_-^5 - a_\rho\phi_-^{-3} - a_{\boldsymbol{w}}\phi_-^{-7} \\ \alpha\beta\Lambda_2 + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)(\gamma_I\phi_-)^3 - \frac{1}{4}S(\nu,\nu)(\gamma_I\phi_-)^{-3} \\ \alpha\beta\Lambda_3 - g \end{array} \right).$$

This implies that

$$A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) \tag{6.27}$$

$$\leqslant \left(\begin{array}{c} \alpha\beta\Lambda_1 + \alpha^5 a_\tau^\wedge\phi_+^5 - \alpha^{-3}a_\rho^\vee\phi_+^{-3} - \alpha^{-7}a_{\boldsymbol{w}}^\vee\phi_+^{-7} \\ \alpha\beta\Lambda_2 + \alpha^3\left(\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge\right)(\gamma_I\phi_+)^3 - \alpha^{-3}\left(\frac{1}{2}\tau^\vee + \frac{1}{8}|\theta_-|^\vee\right)(B^\vee)^6(\gamma_I\phi_+)^{-3} \\ \alpha\beta\Lambda_3 - g^\vee \end{array} \right).$$

In case(a), because $\tau^\wedge < \infty, a_\rho^\vee > 0,\ (4\tau^\vee + |\theta_-|^\vee) > 0$ and $g^\vee > 0$, we may choose α sufficiently small so that each of the equations in (6.27) is non-positive. This implies that $A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) \leqslant 0$ and that $\phi_- = \alpha\phi_+$ is a global sub-solution.

In case(b) we have that $a_\rho^\vee = 0$, so we have that

$$A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) \tag{6.28}$$

$$\leqslant \left(\begin{array}{c} \alpha\beta\Lambda_1 + \alpha^5 a_\tau^\wedge\phi_+^5 - \alpha^{-7}a_{\boldsymbol{w}}^\vee\phi_+^{-7} \\ \alpha\beta\Lambda_2 + \alpha^3\left(\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge\right)(\gamma_I\phi_+)^3 - \alpha^{-3}\left(\frac{1}{2}\tau^\vee + \frac{1}{8}|\theta_-|^\vee\right)(B^\vee)^6(\gamma_I\phi_+)^{-3} \\ \alpha\beta\Lambda_3 - g^\vee \end{array} \right).$$

The equations corresponding to the lower bounds for the Robin operators in (6.28) remain unchanged. So in order to guarantee that we can choose $\alpha > 0$ sufficiently small so that ϕ_- is a global sub-solution, we must show that $a_{\boldsymbol{w}}^\vee > 0$ given the assumption that $a_\sigma^\vee > k(\phi_+)$. For $\epsilon > 0$, the inequality $2|\sigma_{ab}(\mathcal{L}\boldsymbol{w})^{ab}| \leqslant \epsilon\sigma^2 + \frac{1}{\epsilon}(\mathcal{L}\boldsymbol{w})^2$ implies that

$$a_{\boldsymbol{w}}^\vee \geqslant (1-\epsilon)a_\sigma^\vee - \left(\frac{1}{\epsilon} - 1\right)a_{\mathcal{L}\boldsymbol{w}}^\wedge \geqslant (1-\epsilon)a_\sigma^\vee - \left(\frac{1}{\epsilon} - 1\right)\mathtt{k}(\phi_+) = \mathtt{K}_\epsilon, \tag{6.29}$$

where a_σ and $a_{\mathcal{L}\boldsymbol{w}}$ are defined in the paragraph preceding (6.1) and $\mathtt{k}$ is the bound (6.2) on the momentum constraint. Requiring that $\epsilon \in (0,1)$ and that $a_{\mathcal{L}\boldsymbol{w}}^\vee > 0$, we find that $\epsilon \in (\mathtt{k}(\phi_+)/a_\sigma^\wedge, 1)$, which is nonempty provided that $a_\sigma^\vee > k(\phi_+)$. Therefore, choosing

$\epsilon \in (\mathtt{k}(\phi_+)/a_\sigma^\wedge, 1)$ we have that $a_{\mathcal{L}\boldsymbol{w}}^\vee > 0$, which allows us to choose α sufficiently small so that $A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) \leqslant 0$, which implies that $\phi_- = \alpha\phi_+$ is a global sub-solution. □

Remark 6.11. *In Lemma 6.12, we assume that δ can be taken arbitrarily small and we set $B = \beta u$, where u solves* (6.18)*. The global sub-solution constructed in Lemma 6.10 does not work for this choice of B. We instead use the following Lemma to obtain a global sub-solution when δ is small.*

Lemma 6.12. (Global Sub-Solution for small δ) *Let $(\mathcal{M}, h)$ be a 3-dimensional, smooth, compact Riemannian manifold with metric $h \in W^{s,p}$, $s > \frac{3}{p}$ and non-empty boundary satisfying the conditions* (3.1)*. Assume that $\tau \in L^\infty$, $(4\tau^\vee + |\theta_-|^\vee) > 0$ on Σ_I, $g^\vee > 0$ and that $B = \phi_+ = \beta u$ is the global super-solution obtained from Lemma 6.5 by solving* (6.18)*. Suppose $v \in W^{s,p}$ is a positive solution to*

$$\begin{aligned} -\Delta v + a_R v &= \lambda_1 > 0, \\ \partial_\nu v + \frac{1}{2}Hv &= \lambda_2 > 0, \\ \partial_\nu v + cv &= \lambda_3 > 0, \end{aligned} \tag{6.30}$$

where $\lambda_1, \lambda_2, \lambda_3$ are positive functions chosen so that v is distinct from u. We have the following two cases:

(a) If $a_\rho^\vee > 0$, then there exists $\alpha > 0$ sufficiently small so that $\phi_- = \alpha v < \phi_+$ is a global sub-solution.

(b) Suppose that $a_\sigma^\wedge > \mathtt{k}(\phi_+)$, where $\mathtt{k}$ is as in (6.2)*. Then there exists $\alpha > 0$ sufficiently small so that $\phi_- = \alpha v < \phi_+$ is a global sub-solution.*

Proof. The existence of a positive v solving (6.30) follows from Theorem 2.1 in [13]. Evaluating at ϕ_-, where α is to be determined, we have that

$$A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) = \begin{pmatrix} \alpha\beta\lambda_1 + a_\tau \phi_-^5 - a_\rho \phi_-^{-3} - a_{\boldsymbol{w}}\phi_-^{-7} \\ \alpha\beta\lambda_2 + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)(\gamma_I\phi_-)^3 - \frac{1}{4}S(\nu,\nu)(\gamma_I\phi_-)^{-3} \\ \alpha\beta\lambda_3 - g \end{pmatrix}.$$

This implies that

$$\begin{aligned} &A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) \\ &\leqslant \begin{pmatrix} \alpha\beta\lambda_1 + \alpha^5 a_\tau^\wedge v^5 - \alpha^{-3} a_\rho^\vee v^{-3} - \alpha^{-7} a_{\boldsymbol{w}}^\vee v^{-7} \\ \alpha\beta\lambda_2 + \alpha^3\left(\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge\right)(\gamma_I v)^3 - \alpha^{-3}\left(\frac{1}{2}\tau^\vee + \frac{1}{8}|\theta_-|^\vee\right)B^6(\gamma_I v)^{-3} \\ \alpha\beta\lambda_3 - g^\vee \end{pmatrix}. \end{aligned} \tag{6.31}$$

In case(a), because $\tau^\wedge < \infty, a_\rho^\vee > 0,\ (4\tau^\vee + |\theta_-|^\vee) > 0$, and $g^\vee > 0$, we may choose α sufficiently small so that each of the equations in (6.27) is non-positive. This implies that $A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) \leqslant 0$ and that $\phi_- = \alpha\phi_+$ is a global sub-solution. Moreover, because $u^\vee > 0$, we can choose $\alpha > 0$ so that $\phi_- = \alpha v < \phi_+ = \beta u$.

In case (b), the proof follows by making an argument similar to case (b) in the proof of Lemma 6.10, with $\phi_- = \alpha v$. □

Remark 6.13. *In practice, it will be impossible to construct the global sub-solution in Lemma 6.10(b) from using the global super-solution obtained in Lemma 6.5 given the smallness assumptions on a_σ and the dependence of $\mathtt{k}_2$ on σ. We include the construction here for completeness as an alternative to the condition that $\rho \neq 0$. We note that in the closed case, it was shown by Maxwell in* [20] *that under suitable smoothness assumptions* *on the metric (that it is in atleast $W^{2,p}$), the known decay and other properties of the Green's function for the Laplace-Beltrami operator (cf.* [2]*) implied that it was sufficient to construct only a global super-solution for completion of the Schauder argument in Theorem 3.5. This allowed Maxwell in* [20] *to partially extend the far-from-CMC results in* [12] *for closed manifolds to the vacuum case ($\rho = 0$) when the metric is in $W^{2,p}$ or better; the vacuum case for rough metrics on closed manifolds remains open. In* [7]*, Dilts followed closely Maxwell's argument in* [20] *and showed that under the same smoothness assumptions on the background metric, and by also making smoothness assumptions on the boundary and exploiting some additional results from* [13]*, the standard estimates for the Green's function from* [2] *can again be used to exploit Maxwell's technique for avoiding the sub-solution on compact manifolds with boundary that have smooth metrics. Additional assumptions can be placed on the other data to avoid assuming that $\rho \neq 0$ to obtain a global sub-solution. The following Lemma, based on barrier constructions pioneered in* [17] *(see also* [12] *for a detailed discussion of related constructions based on auxiliary problems), provides a method to obtain a global sub-solution in vacuum with additional, mild assumptions on a_R, σ and a_τ.*

Lemma 6.14. (Global Sub-solution with $\rho \equiv 0$) *Let $(\mathcal{M}, h)$ be a 3-dimensional, smooth, compact Riemannian manifold with metric $h \in W^{s,p}$, $s > \frac{3}{p}$ and non-empty boundary satisfying the conditions* (3.1)*. Assume that $\tau \in L^\infty$, $(4\tau^\vee + |\theta_-|^\vee) > 0$ on Σ_I, $g^\vee > 0$ and $|\sigma| > 0$. Additionally assume that there exists $\gamma_1 > 0$ such that $a_R + \gamma_1 a_\tau \geqslant 0$ and γ_2 so that $H + \gamma_2(2\tau + |\theta_-|) \geqslant 0$. Let $v \in W^{s,p}$ be a positive solution to*

$$
\begin{aligned}
-\Delta v + (a_R + \gamma_1 a_\tau) v &= a_{\mathbf{w}} > 0 \qquad (6.32)\\
\partial_\nu v + \frac{1}{2}(H + \gamma_2(2\tau + |\theta_-|))v &= \eta_2 > 0,\\
\partial_\nu v + cv &= \eta_3 > 0,
\end{aligned}
$$

where η_2, η_3 are positive functions. Then there exists $\alpha > 0$ such that $\phi_- = \alpha v \leqslant \phi_+$ is a global sub-solution, where ϕ_+ is any positive global super-solution.

Proof. The function v exists and is positive by Lemmas B.7 and B.8 in [13]. Evaluating at ϕ_-, where α will be determined, we have that

$$
A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) = \begin{pmatrix} \alpha a_{\boldsymbol{w}} - \alpha\gamma_1 a_\tau v + a_\tau \phi_-^5 - a_{\boldsymbol{w}} \phi_-^{-7} \\ \alpha\eta_2 - \alpha\gamma_2(2\tau + |\theta_-|)v + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)\phi_-^3 - \frac{1}{4}S(\nu,\nu)\phi_-^{-3} \\ \alpha\eta_3 - g \end{pmatrix}.
$$

This implies that

$$
\begin{aligned}
&A_L(\phi_-) + F(\phi_-, \boldsymbol{w}) \qquad (6.33)\\
&\quad \leqslant \begin{pmatrix} (\alpha^5 (v^\wedge)^5 - \alpha\gamma_1 v^\vee) a_\tau^\wedge + (\alpha - \alpha^{-7}(v^\vee)^{-7}) a_{\boldsymbol{w}}^\vee \\ \alpha(\eta_2^\wedge - \gamma_2(2\tau^\vee + |\theta_-|^\vee)v^\vee) + \alpha^3\left(\frac{1}{2}\tau^\wedge + \frac{1}{4}|\theta_-|^\wedge\right)(v^\wedge)^3 - \Theta \\ \alpha\eta_3^\wedge - g^\vee \end{pmatrix},
\end{aligned}
$$

where $\Theta = \alpha^{-3}\left(\frac{1}{2}\tau^\vee + \frac{1}{8}|\theta_-|^\vee\right)(B^\vee)^6 (v^\wedge)^{-3}$. By choosing $\alpha > 0$ sufficiently small, each of the components in (6.33) can be made non-positive. Moreover, for any positive super-solution $\phi_+ > 0$, α can be chosen sufficiently small so that $\phi_- = \alpha v < \phi_+$. □

6.3. **Obstacles to Global Barriers for Arbitrary $h \in \mathcal{Y}^+$ and τ.** In Section 6.2, we showed that we can obtain global super solutions provided that certain data is sufficiently small and that either h has sufficiently large scalar curvature and τ is arbitrary, or that

$h \in \mathcal{Y}^+$ and $D\tau$ is sufficiently small. However, it has proven to be extremely difficult to construct global super solutions where both h and τ are freely specifiable. We now give an analysis which helps to explain why this is the case.

As we saw in Theorem 6.5, in order for $\phi_+ = \beta u$ to be a global super solution, β must satisfy

$$\beta\Lambda_1^\vee - \beta^5 K_1 k_3^{12}(u^\wedge)^5 > 0, \quad \text{and} \quad \beta\Lambda_3^\vee - g^\wedge \geqslant 0,$$

where $k_3 = u^\wedge/u^\vee$. This implies that

$$\frac{g^\wedge}{\Lambda_3^\vee} \leqslant \beta < \frac{(\Lambda_1^\vee)^{\frac{1}{4}}(u^\vee)^3}{K_1^{\frac{1}{4}}(u^\wedge)^{\frac{17}{4}}}. \tag{6.34}$$

As we mentioned earlier, it is impossible to choose $g^\wedge$ small without affecting the size of $\Lambda_3^\vee$ (which ultimately depends on u). Therefore, if we cannot choose $D\tau$ to be small (which makes K_1 small), there is no guarantee that β can be chosen to satisfy the above conditions without knowing more about $u^\wedge, u^\vee$.

In an attempt to at least partially resolve this issue, we consider the following auxiliary problem: find u that solves

$$\begin{aligned} -\Delta u + a_R u &= f_1(u, \Lambda), \\ \gamma_I \partial_\nu u + \frac{1}{2}H\gamma_I u &= f_2(\gamma_I u, \Lambda), \\ \gamma_E \partial_\nu u + c\gamma_E u &= f_3(\gamma_E u, \Lambda), \end{aligned} \tag{6.35}$$

where Λ is a real valued parameter and f_1, f_2 and f_3 are positive functions. The idea is to choose the functions f_1, f_2 and f_3 so that we can solve (6.35) using the method of sub-and super-solutions. This will allow us to obtain a solution u that is point-wise bounded by the sub-and super-solutions, which will give us some control of $u^\vee$ and $u^\wedge$. The easiest approach to constructing barriers for (6.35) is to look for constant barriers. Therefore we require that f_1, f_2 and f_3 satisfy one of the two following conditions:

1) The functions

$$\begin{aligned} g_1(x) &= a_R^\wedge x - f_1(x, \Lambda), \\ g_2(x) &= \frac{1}{2}H^\wedge x - f_2(x, \Lambda), \\ g_3(x) &= c^\wedge x - f_3(x, \Lambda), \\ h_1(x) &= a_R^\vee x - f_1(x, \Lambda), \\ h_2(x) &= \frac{1}{2}H^\vee x - f_2(x, \Lambda), \\ h_3(x) &= c^\vee x - f_3(x, \Lambda), \end{aligned}$$

all have at exactly one positive root.

2) All of the functions g_i and h_i have at least two roots, where α_{i1}, α_{i2} are the two smallest positive roots for each g_i and γ_{i1} and γ_{i2} are the two smallest positive roots for each h_i. Additionally assume that $\gamma_{j1} < \gamma_{i2}$ and $\alpha_{i1} < \alpha_{j2}$ for each $1 \leqslant i, j \leqslant 3$.

We observe that an unfortunate consequence of requiring that one of the above two conditions be satisfied is that R, H and c must be positive, given that f_1, f_2 and f_3 are strictly positive. Therefore we are not entirely free to specify h in this construction. While this limitation is not ideal, the auxiliary problem (6.35) is a natural starting point in our attempts at constructing a global super-solution with a freely specifiable τ and as few conditions on h as possible.

In case **1**, let α by the smallest root of the g_i and let γ be the largest root of the h_i, and in case **2** let α be the smallest root of the g_i and γ be the smallest root of the h_i. It is easily checked that α and γ are sub- and super-solutions for (6.35). Therefore, using the techniques outlined in this paper, this problem can be solved to obtain u which satisfies $\alpha \leqslant u^\vee$ and $u^\wedge \leqslant \gamma$. Moreover, based on the definition of α and γ, $g_i(u) \geqslant 0$ and $h_i(u) \geqslant 0$ for each $1 \leqslant i \leqslant 3$.

Remark 6.15. *We note that in general, in order to solve* (6.35) *we only require that the functions f_1, f_2, f_3 be chosen so that there exists an interval I_1 such that the functions h_i are nonnegative on this interval. Similarly, we also require that there exist and interval I_2 with $\sup I_2 < \inf I_1$ such that the g_i are non-positive on I_2. Then way may choose a super-solution $\beta \in I_1$ and a sub-solution in $\alpha \in I_2$. For this more general collection of f_i, it is unclear whether $h_i(u) > 0$ or $g_i(u) > 0$ for $\alpha \leqslant u \leqslant \gamma$. This is not a necessary condition, and the following discussion suggests that this is not ideal. However, this assumption allows for the following heuristic that illustrates the difficulties with constructing barriers with minimal assumptions on τ and h.*

So in addition to the above conditions, if we can choose f_1, f_2, f_3, Λ and β so that

$$\frac{g^\wedge}{f_3(u,\Lambda)^\vee} \leqslant \beta < \frac{(f_1(u,\Lambda)^\vee)^{\frac{1}{4}}(u^\vee)^3}{K_1^{\frac{1}{4}}(u^\wedge)^{\frac{17}{4}}},$$

we will have a global super solution with freely specifiable τ and h with positive scalar curvature. Setting $\beta = \frac{g^\wedge}{f_3(u,\Lambda)^\vee}$, we find that we need to choose our functions and parameters to satisfy

$$g^\wedge < \frac{f_3(u,\Lambda)^\vee (f_1(u,\Lambda)^\vee)^{\frac{1}{4}}(u^\vee)^3}{K_1^{\frac{1}{4}}(u^\wedge)^{\frac{17}{4}}}. \tag{6.36}$$

Implicitly α and γ are functions of Λ. So the hope is that one can choose f_1, f_2 and f_3 and utilize the point-wise estimates $\alpha(\Lambda)$ and $\gamma(\Lambda)$ to determine if the above expression can be made sufficiently large by varying Λ . In particular, the uneven exponents on $u^\vee$ and $u^\wedge$ suggest that if one can choose f_1, f_2, f_3 so that $\alpha(\Lambda) \to 0$, $\gamma(\Lambda) \to 0$ and $\alpha(\Lambda) \sim \gamma(\Lambda)$ as $\Lambda \to 0$, and

$$\lim_{\Lambda\to 0} \frac{f_3(u,\Lambda)^\vee (f_1(u,\Lambda))^\vee}{\gamma(\Lambda)^{\frac{5}{4}}} = \infty,$$

then we can obtain our global super solution. However, we observe that

$$f_1(u,\Lambda)^\vee \leqslant a_R^\vee u^\vee, \quad \text{and} \quad f_3(u,\Lambda)^\vee \leqslant c^\vee u^\vee,$$

given that $g_i(u) \geqslant 0$ and $h_i(u) \geqslant 0$ for $\alpha \leqslant u \leqslant \gamma$. Therefore,

$$\frac{f_3(u,\Lambda)^\vee (f_1(u,\Lambda)^\vee)^{\frac{1}{4}}(u^\vee)^3}{K_1^{\frac{1}{4}}(u^\wedge)^{\frac{17}{4}}} \leqslant \frac{c^\vee (a_R^\vee)^{\frac{1}{4}}(u^\vee)^{\frac{17}{4}}}{K_1^{\frac{1}{4}}(u^\wedge)^{\frac{17}{4}}}. \tag{6.37}$$

Clearly $u^\vee/u^\wedge \leqslant 1$, and given that $c - g = \mathcal{O}(R^{-3})$ it is highly likely that $c^\vee \leqslant g^\wedge$. So without a largeness assumption on R or a smallness assumption on $D\tau$, it will not always be the case that g satisfies

$$g^\wedge < \frac{c^\vee (a_R^\vee)^{\frac{1}{4}}(u^\vee)^{\frac{17}{4}}}{K_1^{\frac{1}{4}}(u^\wedge)^{\frac{17}{4}}},$$

much less (6.36).

The attempted construction above shows that an auxiliary problem of the form (6.35), with positive f_1, f_2 and f_3 satisfying **1** or **2**, will not work in general if one hopes to obtain global super solutions with freely specifiable τ and h with positive scalar curvature. Semilinear problems such a this are a natural place to start when attempting to construct barriers with minimal assumptions on τ and h given that *a priori* estimates and sub- and super-solutions are readily attained. The above discussion suggests that one might require a variational approach such as in Theorem 2.1 in [13]. However, the drawback of such an approach is that there are no standard techniques for determining point-wise estimates of the solution u, which makes it difficult to verify the inequality (6.34) without additional assumptions on τ or h.

7. PROOF OF THE MAIN RESULTS

We now use the global barriers that we were able to construct above, together with the results from Section 4 and 5, to apply the coupled fixed point Theorem 3.5 to prove Theorems 3.2 and 3.3. We first prove Theorem 3.2. The proof of Theorem 3.3 involves only minor modifications of the proof of Theorem 3.2.

7.1. **Proof of Theorem 3.2.** Our strategy will be to prove the theorem first for the case $s \leqslant 2$, and then to bootstrap to include the higher regularity cases.

Step 1: The choice of function spaces. We have the (reflexive) Banach spaces $X = W^{s,p}$ and $Y = \boldsymbol{W}^{e,q}$, where $p, q \in (3, \frac{\alpha+1}{3})$, $(\alpha > 8)$, $s = s(p) \in (1 + \frac{3}{p}, 2]$, and $e = e(p,s,q) \in (1,s] \cap (1 + \frac{3}{q}, s - \frac{3}{p} + \frac{3}{q}]$. We have the ordered Banach space $Z = W^{\tilde{s},p}$ with the compact embedding $X = W^{s,p} \hookrightarrow W^{\tilde{s},p} = Z$, for $\tilde{s} \in (1 + \frac{3}{p} - \frac{4}{\alpha}, 1 + \frac{3}{p})$. The interval $[\phi_-, \phi_+]_{\tilde{s},p}$ is nonempty (by compatibility of the barriers we will choose below), and by Lemma 3.7 on page 18 it is also convex with respect to the vector space structure of $W^{\tilde{s},p}$ and closed with respect to the norm topology of $W^{\tilde{s},p}$. We then take $U = [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M$ for sufficiently large M (to be determined below), where $\overline{B}_M$ is the closed ball in $Z = W^{\tilde{s},p}$ of radius M about the origin, ensuring that U is non-empty, convex, closed, and bounded as a subset of $Z = W^{\tilde{s},p}$.

Step 2: Construction of the mapping S. We have $\boldsymbol{b}_j \in \boldsymbol{W}^{e-2,q}$, and $\boldsymbol{b}_\tau \in \boldsymbol{L}^z$ with $z = \frac{3p}{3+(2-s)p}$. The assumptions on e imply that $\boldsymbol{L}^z \hookrightarrow \boldsymbol{W}^{e-2,q}$. Similarly, $\gamma_E \tau \in W^{1-\frac{1}{z},z}(\Sigma_I) \hookrightarrow W^{e-1-\frac{1}{q},q}(\Sigma_I)$ and $\theta_- \in W^{s-1-\frac{1}{p},p}(\Sigma_I) \hookrightarrow W^{e-1-\frac{1}{q},q}(\Sigma_I)$. Because $\gamma_I(\phi_+) \in W^{s-\frac{1}{p},p}(\Sigma_I)$ and

$$W^{s-\frac{1}{p},p}(\Sigma_I) \otimes W^{e-1-\frac{1}{q},q}(\Sigma_I) \to W^{e-1-\frac{1}{q},q}(\Sigma_I)$$

is point-wise bounded by Corollary A.5(a) in [13] or Corollary 3(a) in [12], we have that $\mathbf{V}^a \nu_a \in W^{e-1-\frac{1}{q},q}(\Sigma_I)$. Moreover, by Theorems 4.4 and 4.5 the momentum constraint equation with boundary conditions (3.5) and (3.7) is has a unique solution $\boldsymbol{w} \in W^{e,q}$ for any "source" $\phi \in [\phi_-, \phi_+]_{\tilde{s},p}$. The ranges for the exponents ensure that Lemma 4.7 holds, so that the momentum constraint solution map

$$S : [\phi_-, \phi_+]_{\tilde{s},p} \to \boldsymbol{W}^{e,q} = Y,$$

is continuous.

Step 3: Construction of the mapping T. Define $r = \frac{3p}{3+(2-s)p}$, so that the continuous embedding $L^r \hookrightarrow W^{s-2,p}$ holds. Since the pointwise multiplication is bounded on $L^{2r} \otimes L^{2r} \to L^r$, and $\boldsymbol{w} \in \boldsymbol{W}^{e,q} \hookrightarrow \boldsymbol{W}^{1,2r}$, we have $a_{\boldsymbol{w}} \in W^{s-2,p}$ by $\sigma \in L^{2r}$. The embeddings $W^{1,z} \hookrightarrow W^{e-1,q} \hookrightarrow L^{2r}$ also guarantee that $a_\tau = \frac{1}{12}\tau^2 \in W^{s-2,p}$. We have the scalar curvature $R \in W^{s-2,p}$, and these considerations show that the Hamiltonian constraint equation is well defined with $[\phi_-, \phi_+]_{s,p}$ as the space of solutions. Similarly, $\gamma_I \tau \in W^{1-\frac{1}{z},z}(\Sigma_I) \hookrightarrow W^{s-1-\frac{1}{p},p}(\Sigma_I)$ and the fact that

$$W^{s-\frac{1}{p},p}(\Sigma_I) \otimes W^{s-1-\frac{1}{p},p}(\Sigma_I) \to W^{s-1-\frac{1}{p},p}(\Sigma_I)$$

is a point-wise bounded map imply that the Robin boundary conditions are well-defined provided $\phi \in [\phi_-, \phi_+]_{s,p}$.

Suppose for the moment that the scalar curvature R of the background metric h is continuous, and by using the map T^s introduced in Lemma 5.1, define the map T by $T(\phi, \boldsymbol{w}) = T^s(\phi, a_{\boldsymbol{w}})$, where $a_{\boldsymbol{w}}$ is now considered as an expression depending on $\boldsymbol{w}$. Then Lemma 5.1 implies that the map $T : [\phi_-, \phi_+]_{\tilde{s},p} \times \boldsymbol{W}^{e,q} \to W^{s,p}$ is continuous for any reasonable shift a_s, which, by Lemma 5.2, can be chosen so that T is monotone in the first variable. Combining the monotonicity with Lemma 5.3, we infer that the interval $[\phi_-, \phi_+]_{\tilde{s},p}$ is invariant under $T(\cdot, a_{\boldsymbol{w}})$ if $\boldsymbol{w} \in S([\phi_-, \phi_+]_{\tilde{s},p})$. Since $\boldsymbol{L}^z \hookrightarrow \boldsymbol{W}^{e-2,q}$, from Theorem 4.5 we have

$$\begin{aligned} \|\mathcal{L}\boldsymbol{w}\|_\infty \leqslant C \Big(& \|\phi\|_\infty^6 \|\mathbf{b}_\tau\|_z + \|\mathbf{b}_j\|_{e-2,q} + \|(2\tau + |\theta_-|/2)\|_{e-1-\frac{1}{q},q;\Sigma_I} \|\gamma_I(\phi_+)\|_\infty^6 \\ & + \|\sigma(\nu,\nu)\|_{e-1-\frac{1}{q},p;\Sigma_I} + \|\mathbf{W}\|_{e-1-\frac{1}{q},p;\Sigma_I} \Big), \end{aligned}$$

for any $\boldsymbol{w} \in S([\phi_-, \phi_+]_{\tilde{s},p})$. In view of Lemma 5.4, this shows that there exists a closed ball $\overline{B}_M \subset W^{\tilde{s},p}$ such that

$$\phi \in [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M, \quad \boldsymbol{w} \in S([\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M) \quad \Rightarrow \quad T(\phi, \boldsymbol{w}) \in \overline{B}_M.$$

Under the conditions in the above displayed formula, from the invariance of the interval $[\phi_-, \phi_+]_{\tilde{s},p}$, we indeed have $T(\phi, \boldsymbol{w}) \in U = [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M$.

However, the scalar curvature of h may be not continuous, and in general it is not clear how to introduce a shift so that the resulting operator is monotone. Nevertheless, we can conformally transform the metric into a metric with continuous, positive scalar curvature and positive boundary mean curvature by Theorem 2.2(c) in [13]. By using the conformal covariance of the Hamiltonian constraint (cf. Lemma A.2), we will be able to construct an appropriate mapping T. Let $\tilde{h} = \psi^4 h$ be a metric with continuous positive scalar curvature $\tilde{R}$ and boundary mean curvature $\tilde{H}$, where $\psi \in W^{s,p}$ is the (positive) conformal factor of the scaling satisfying $\gamma_E \partial_\nu \psi = 0$. Such a conformal factor exists by adapting the proof of Theorem 2.1 in [13] to allow for the specified boundary condition $\gamma_E \partial_\nu \psi = 0$. Let $\tilde{T}^s$ be the mapping introduced in Lemma 5.1, corresponding to the Hamiltonian constraint equation with the background metric $\tilde{h}$, coefficients $\tilde{a}_\tau = a_\tau$, $\tilde{a}_\rho = \psi^{-8} a_\rho$, and Robin boundary conditions given by

$$\gamma_I \partial_\nu \phi + \frac{1}{2} \tilde{H} \gamma_I \phi + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_- \right) \phi^3 - \psi^{-6} S(\nu,\nu)(\gamma_I \phi)^{-3} = 0, \tag{7.1}$$

$$\gamma_E \partial_\nu \phi + \psi^{-2} c \gamma_E \phi - \psi^{-3} g = 0. \tag{7.2}$$

With $\tilde{a}_{\boldsymbol{w}} = \psi^{-12} a_{\boldsymbol{w}}$, this *scaled* Hamiltonian constraint equation has sub- and super-solutions $\psi^{-1}\phi_-$ and $\psi^{-1}\phi_+$, respectively, as long as ϕ_- and ϕ_+ are sub- and super-solutions respectively of the original Hamiltonian constraint equation (see [13]). We choose the shift in $\tilde{T}^s$ so that it is monotone in $[\psi^{-1}\phi_-, \psi^{-1}\phi_+]_{\tilde{s},p}$. Then by the monotonicity and the above mentioned sub- and super-solution property under conformal scaling, for $\boldsymbol{w} \in S([\phi_-, \phi_+]_{\tilde{s},p})$, $\tilde{T}^s(\cdot, \psi^{-12} a_{\boldsymbol{w}})$ is invariant on $[\psi^{-1}\phi_-, \psi^{-1}\phi_+]_{\tilde{s},p}$. Finally, we define

$$T(\phi, \boldsymbol{w}) = \psi \tilde{T}^s(\psi^{-1}\phi, \psi^{-12} a_{\boldsymbol{w}}),$$

where, as before, $a_{\boldsymbol{w}}$ is considered as an expression depending on $\boldsymbol{w}$. From the pointwise multiplication properties of ψ and ψ^{-1}, the map $T : [\phi_-, \phi_+]_{\tilde{s},p} \times \boldsymbol{W}^{e,q} \to W^{s,p}$ is continuous, and from the monotonicity and Lemma 5.4 , $T(\cdot, \boldsymbol{w})$ is invariant on $U = [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M$ for $\boldsymbol{w} \in S(U)$, where M is taken to be sufficiently large. Moreover, if the fixed point equation

$$\phi = \psi \tilde{T}^s(\psi^{-1}\phi, \psi^{-12}a_{\boldsymbol{w}}),$$

is satisfied, then $\psi^{-1}\phi$ is a solution to the scaled Hamiltonian constraint equation with $\tilde{a}_{\boldsymbol{w}} = \psi^{-12}a_{\boldsymbol{w}}$, and so by conformal covariance, ϕ is a solution to the original Hamiltonian constraint equation (see [13]).

Step 4: Barrier choices and application of the fixed point theorem. At this point, Theorem 3.5 implies the Main Theorem 3.2, provided that we have an admissible pair of barriers for the Hamiltonian constraint and we can choose B so that $B > \|\phi_+\|_\infty$ so that the marginally trapped surface conditions (2.33) are satisfied. The ranges for the exponents ensure through Theorems 4.4 and 4.5 that we can use the estimate (6.2); see the discussion following the estimate on page 29. In this case we use the global super-solution constructed in Lemma 6.2(a) and the global sub-solution constructed in Lemma 6.4(a) or (b) or Lemma 6.14, depending on whether $a_\rho \neq 0$ or $a_\sigma^\vee > k(\phi_+)$. Remark 6.3 implies that we can choose B so that $B > \|\phi_+\|_\infty$. This concludes the proof for the case $s \leqslant 2$.

Step 5: Bootstrap. Now suppose that $s > 2$. First of all we need to show that the equations are well defined in the sense that the involved operators are bounded in appropriate spaces. All other conditions being obviously satisfied, we will show that the Hamiltonian constraint is well-defined by showing that $a_{\boldsymbol{w}} \in W^{s-2,p}$ for any $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$. Since σ and $\mathcal{L}\boldsymbol{w}$ belong to $W^{e-1,q}$, it suffices to show that the pointwise multiplication is bounded on $W^{e-1,q} \otimes W^{e-1,q} \to W^{s-2,p}$, and by employing Corollary A.5(b) in [13], we are done as long as $s-2 \leqslant e-1 \geqslant 0$, $s-2-\frac{3}{p} < 2(e-1-\frac{3}{q})$, and $s-2-\frac{3}{p} \leqslant e-1-\frac{3}{q}$. After a rearrangement these conditions read as $e \geqslant 1$, $e \geqslant s-1$, $e > \frac{3}{q}+\frac{d}{2}$, and $e \geqslant \frac{3}{q}+d-1$, with the shorthand $d = s-\frac{3}{p} > 1$, the latter inequality by the hypothesis of the theorem. We have $d-1 > \frac{d}{2}$ for $d > 2$, and $1 \geqslant \frac{d}{2}$ for $d \leqslant 2$, meaning that the condition $e > \frac{3}{q}+\frac{d}{2}$ is implied by the hypotheses $e \geqslant \frac{3}{q}+d-1$ and $e > 1+\frac{3}{q}$. Similarly, given that $S(\nu,\nu) = (|\theta_-|/2)\gamma_I(\phi_+)^6 \in W^{s-1-\frac{1}{p},p}(\Sigma_I)$ and pointwise multiplication is bounded on $W^{s-1-\frac{1}{p},p}(\Sigma_I) \otimes W^{s-1,p}(\Sigma_I) \to W^{s-1-\frac{1}{p},p}(\Sigma_I)$, the Robin boundary operators are well-defined. So we conclude that the constraint equations with the specified Robin boundary conditions are well defined.

Next, we will treat the equations as equations defined with $s = e = 2$ and with p and q appropriately chosen. This is possible, since if the quadruple (p,s,q,e) satisfies the hypotheses of the theorem, then $(\tilde{p}, \tilde{s} = 2, \tilde{q}, \tilde{e} = 2)$ satisfies the hypotheses too, provided that $2-\frac{3}{\tilde{p}} \leqslant s-\frac{3}{p}$, and $1 < 2-\frac{3}{\tilde{q}} \leqslant e-\frac{3}{q}$. Since the latter conditions reflect the Sobolev embeddings $W^{s,p} \hookrightarrow W^{2,\tilde{p}}$ and $W^{e,q} \hookrightarrow W^{2,\tilde{q}} \hookrightarrow W^{1,\infty}$, the coefficients of the equations can also be shown to satisfy sufficient conditions for posing the problem for $(\tilde{p}, 2, \tilde{q}, 2)$. Finally, we have $\tau \in W^{s-1,p} \hookrightarrow W^{1,\tilde{p}} = W^{1,z}$ since $z = \tilde{p}$ by $\tilde{s} = 2$ for this new formulation. Now, by the special case $s \leqslant 2$ of this theorem that is proven in the above steps, under the remaining hypotheses including the conditions on the metric and the near-CMC condition, we have $\phi \in W^{2,\tilde{p}}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{2,\tilde{q}}$ solution to the coupled system.

To complete the proof we only need to show that these solutions indeed satisfy $\phi \in W^{s,p}$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$. Suppose that $\phi \in W^{s_1,p_1}$ and $\boldsymbol{w} \in \boldsymbol{W}^{e_1,q_1}$, with $1 < s_1 - \frac{3}{p_1} \leqslant s - \frac{3}{p}$,

$1 < e_1 - \frac{3}{q_1} \leqslant e - \frac{3}{q}$, $\max\{2, s-2\} \leqslant s_1 \leqslant s$, and $\max\{2, e-2\} \leqslant e_1 \leqslant \min\{e, s\}$. Then we have $\boldsymbol{b}_\tau \phi^6 + \boldsymbol{b}_j \in \boldsymbol{W}^{e-2,q}$, and so Corollary B.4 in [13] implies that $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$. This implies that $a_{\boldsymbol{w}} \in W^{s-2,p}$, and by employing Corollary B.4 in [13] once again, we get $\phi \in W^{s,p}$. The proof is completed by induction. □

7.2. **Proof of Theorem 3.3.** The proof is identical to the proof of Theorem 3.2, except for the particular barriers used. In the proof of Theorem 3.2, the near-CMC condition is used to construct global barriers satisfying

$$0 < \phi_- \leqslant \phi_+ < \infty,$$

for all three Yamabe classes, and then the supporting results for the operators S and T established in §4 and §5 are used to reduce the proof to invoking the Coupled Fixed-Point Theorem 3.5. The construction of ϕ_+ is in fact the only place in the proof of Theorem 3.2 that requires the near-CMC condition.

Cases (b) and (c). Here, the proof is identical to that of Theorem 3.2, except that the additional conditions made on the background metric h_{ab} (that it be in $\mathcal{Y}^+(\mathcal{M})$), and on the data (the smallness conditions on $|\theta_-|, D\tau, \sigma$, ρ, and j) allow us to make use of the alternative construction of a global super-solution given in Lemma 6.5, together with compatible global sub-solutions given in Lemma 6.10(a) or Lemma 6.14, depending on whether $\rho \neq 0$. Therefore we can apply Theorem 3.5 to solve the coupled conformal equations (3.2)-(3.3) with boundary conditions (3.4)-(3.8), where $S(\nu, \nu) = (2\tau + |\theta_-|)B^6$ and $B \in (W^{s,p}_+ \backslash \{0\}) \cap L^\infty$ is freely specified. Furthermore, Remark 6.6 implies that we may choose B to be constant such that $B > \|\phi_+\|_\infty$ so that the marginally trapped surface conditions (2.33) are satisfied.

Case (a). Again, the proof is identical to that of Theorem 3.2, except that the additional conditions made on the background metric h_{ab} (that it be in $\mathcal{Y}^+(\mathcal{M})$), and on the data (the smallness conditions on $|\theta_-|, \delta, \sigma$, ρ, and j) allow us to make use of the alternative construction of a global super-solution given in Lemma 6.8, together with compatible global sub-solutions given in Lemma 6.12 or Lemma 6.14, depending on whether $\rho \neq 0$. Therefore we can apply Theorem 3.5 to solve the coupled conformal equations (3.2)-(3.3) with boundary conditions (3.4)-(3.8), where $S(\nu, \nu) = (2\tau + |\theta_-|)B^6$ and $B = \beta u \in (W^{s,p}_+ \backslash \{0\}) \cap L^\infty$ is obtained by solving (6.18). Furthermore, Remark 6.9 implies that this choice of B ensures that the marginally trapped surface conditions (2.33) are satisfied. Theorem 3.3 now follows. □

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APPENDIX A. SOME KEY TECHNICAL TOOLS AND SOME SUPPORTING RESULTS

The results in this article leverage and then build on the analysis framework and the supporting technical tools developed in our two previous articles [12, 13], including the material contained in the appendices of both works. We have made an effort to use completely consistent notation with these two prior works, and have also endeavored to avoid as much a possible any replication of the technical tools. In particular, we have made use of a number of results from [12, 13] on: topological fixed-point theorems, ordered Banach spaces, monotone increasing maps, Sobolev spaces on closed manifolds, elliptic operators and maximum principles, Yamabe classification of non-smooth metrics, and conformal covariance of the Hamiltonian constraint. Although these technical tools represent the bulk of the results we need in order to extablish the main results of the paper, we will need the two additional sets of results below.

A priori estimates for the auxillary problem. The first result we need are a priori L^∞-estimates for solutions to a class of auxilliary problems.

Lemma A.1. *Let the assumptions for Lemma B.7 in* [13] *hold, and and let* f *and* g *playing the roles of* α *and* β*, respectively in Lemma B.7(a) in* [13]*. Then the solution* u *to the boundary value problem*

$$\begin{aligned} -\Delta u + fu &= \Lambda_1 > 0, \\ \gamma_N \partial_\nu u + g\gamma_N u &= \Lambda_2 > 0, \\ \gamma_D u &= \lambda > 0, \end{aligned} \tag{A.1}$$

satisfies the following inequalities,

$$\begin{aligned} &u^\wedge \leq \beta < \infty \quad \textit{if} \ \ f^\vee > 0 \ \ \textit{and} \ \ g^\vee > 0\ , \\ &u^\vee \geq \alpha > 0 \quad \textit{if} \ \ f^\wedge < \infty \ \ \textit{and} \ \ g^\wedge < \infty\ , \end{aligned} \tag{A.2}$$

where

$$\beta = \max\left\{\frac{\Lambda_1^\wedge}{f^\vee}, \frac{\Lambda_2^\wedge}{g^\vee}, \lambda^\wedge\right\} \quad \textit{and} \quad \alpha = \min\left\{\frac{\Lambda_1^\vee}{f^\wedge}, \frac{\Lambda_2^\vee}{g^\wedge}, \lambda^\vee\right\}. \tag{A.3}$$

Proof. The fact that u exists and is positive follows from Lemmas B.8 and B.7 in [13]. Define

$$H^1_{0,D} = \{w \in W^{1,2} \ : \ w = 0 \text{ on } \Sigma_D\}.$$

Then the functions $(u-\beta)^+$ and $(u-\alpha)^-$ are in $H^1_{0,D}$ given the definition of α and β. Define the sets $\mathcal{Y}^+ = \{x \in \overline{\mathcal{M}} : u \geq \beta\}$, $\mathcal{Y}^- = \{x \in \overline{\mathcal{M}} : u \leq \alpha\}$. Let dx be the measure induced by the metric and ds the corresponding boundary meaure, we have

$$\begin{aligned} &\|\nabla(u-\beta)^+\|_2 \\ &\quad = \int_{\mathcal{M}} \nabla(u-\beta)^+ \nabla(u-\beta)^+ \, dx \\ &\quad = \int_{\mathcal{M}\cap\mathcal{Y}^+} \nabla u \nabla(u-\beta)^+ \, dx \\ &\quad = \int_{\mathcal{M}\cap\mathcal{Y}^+} (\Lambda_1 - fu)(u-\beta)\, dx + \int_{\Sigma_N\cap\mathcal{Y}^+} (\Lambda_2 - g\gamma_N(u))\gamma_N(u-\beta)\, ds \\ &\quad \leqslant \int_{\mathcal{M}\cap\mathcal{Y}^+} (\Lambda_1^\wedge - f^\vee u)(u-\beta)\, dx + \int_{\Sigma_N\cap\mathcal{Y}^+} (\Lambda_2^\wedge - g^\vee\gamma_N(u))\gamma_N(u-\beta)\, ds \\ &\quad \leqslant 0, \end{aligned} \tag{A.4}$$

where the above quantity is non-positive by the definition of β. Therefore we may conclude that $(u-\beta)^+$ is constant and that either $u \leq \beta\ a.e$ or u is a constant larger than β. But this is impossible given that $\gamma_D u = \lambda \leq \beta$. So $u \leq \beta\ a.e.$ We may use a similar argument involving $(u-\alpha)^-$ and the set $\mathcal{Y}^-$ to conclude that $u \geq \alpha\ a.e.$ □

Conformal invariance of the Hamiltonian constraint. The second result we need is a modification of Lemma 4.1 in [13], which concerns conformal invariance of the Hamiltonian constraint equation on compact manifolds with certain types of boundary conditions.

Let $\mathcal{M}$ be a smooth, compact, connected n-dimensional manifold with boundary $\Sigma = \Sigma_I \cup \Sigma_E$, $\Sigma_I \cap \Sigma_E = \varnothing$, equipped with a Riemannian metric $h_{ab} \in W^{s,p}_{\mathrm{loc}}$, where we assume throughout this section that $p \in (1, \infty)$, $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and that $n \geqslant 3$. Let γ_I and γ_E be the trace operators on Σ_I and Σ_E respectively. We consider the following model for the Hamiltonian constraint with Robin boundary conditions on Σ_I an Σ_E:

$$F(\phi) := \begin{pmatrix} -\Delta\phi + \frac{n-2}{4(n-1)} R\phi + a\phi^t \\ \gamma_I \partial_\nu \phi + \frac{n-2}{2} H \gamma_I \phi + b(\gamma_I \phi)^e \\ \gamma_E \partial_\nu \phi - c\gamma_E \phi - f \end{pmatrix} = 0,$$

where $t, e \in \mathbb{R}$ are constants, $R \in W^{s-2,p}(\mathcal{M})$ and $H \in W^{s-1-\frac{1}{p},p}(\Sigma_I)$ are respectively the scalar and mean curvatures of the metric g, and the other coefficients satisfy $a \in W^{s-2,p}(\mathcal{M})$, $b \in W^{s-1-\frac{1}{p},p}(\Sigma_I)$, and $c, f \in W^{s-\frac{1}{p},p}(\Sigma_E)$. Setting $r = \frac{4}{n-2}$, we will be interested in the transformation properties of F under the conformal change $\tilde{h}_{ab} = \theta^r h_{ab}$ of the metric with the conformal factor $\theta \in W^{s,p}(\mathcal{M})$ satisfying $\theta > 0$. To this end, we consider

$$\tilde{F}(\psi) := \begin{pmatrix} -\tilde{\Delta}\psi + \frac{n-2}{4(n-1)} \tilde{R}\psi + \tilde{a}\psi^t \\ \gamma_I \partial_{\tilde{\nu}} \psi + \frac{n-2}{2} \tilde{H} \gamma_I \psi + \tilde{b}(\gamma_I \psi)^e \\ \gamma_E \partial_\nu \phi - \tilde{c}\gamma_E \phi - \tilde{f} \end{pmatrix} = 0,$$

where $\tilde{\Delta}$ is the Laplace-Beltrami operator associated to the metric $\tilde{g}$, $\tilde{\nu}$ is the outer normal to Σ with respect to $\tilde{h}$, $\tilde{R} \in W^{s-2,p}(\mathcal{M})$ and $\tilde{H} \in W^{s-1-\frac{1}{p},p}(\Sigma)$ are respectively the scalar and mean curvatures of $\tilde{h}$, and $\tilde{a} \in W^{s-2,p}(\mathcal{M})$, $\tilde{b} \in W^{s-1-\frac{1}{p},p}(\Sigma_I)$, and $\tilde{c}, \tilde{f} \in W^{s-\frac{1}{p},p}(\Sigma_E)$. The following is a variation of Lemma 4.1 in [13] which we need to incoporate the exterior boundary condition.

Lemma A.2. *Let* $\tilde{a} = \theta^{t-r-1} a$, $\tilde{b} = \theta^{e-\frac{r}{2}-1} b$, *and* $\tilde{c} = \theta^{-\frac{r}{2}} c, \tilde{f} = \theta^{-\frac{r}{2}-1} f$. *Then if* $\gamma_E \partial_\nu \theta = 0$, *we have*

$$\begin{aligned} \tilde{F}(\psi) = 0 \quad &\Leftrightarrow \quad F(\theta\psi) = 0, \\ \tilde{F}(\psi) \geqslant 0 \quad &\Leftrightarrow \quad F(\theta\psi) \geqslant 0, \\ \tilde{F}(\psi) \leqslant 0 \quad &\Leftrightarrow \quad F(\theta\psi) \leqslant 0. \end{aligned}$$

Proof. One can derive the following relations

$$\begin{aligned} \tilde{R} &= \theta^{-r} R - \tfrac{4(n-1)}{n-2} \theta^{-r-1} \Delta\theta, \\ \tilde{\Delta}\psi &= \theta^{-r} \Delta\psi + 2\theta^{-r-1} \nabla^a \theta \nabla_a \psi. \end{aligned}$$

Combining these relations with

$$\Delta(\theta\psi) = \theta\Delta\psi + \psi\Delta\theta + 2\nabla^a\theta\nabla_a\psi,$$

we obtain

$$-\tilde{\Delta}\psi + \tfrac{n-2}{4(n-1)} \tilde{R}\psi = \theta^{-r-1} \left(-\Delta(\theta\psi) + \tfrac{n-2}{4(n-1)} R\theta\psi \right).$$

On the other hand, we have

$$\begin{aligned} \tilde{H} &= \theta^{-\frac{r}{2}} H + \tfrac{2}{n-2} \theta^{-\frac{r}{2}-1} \partial_\nu \theta, \\ \partial_{\tilde{\nu}} \psi &= \theta^{-\frac{r}{2}} \partial_\nu \psi, \end{aligned}$$

where traces are understood in the necessary places. The above imply that

$$\partial_{\tilde{\nu}}\psi + \tfrac{n-2}{2}\tilde{H}\psi = \theta^{-\frac{r}{2}-1}\left(\partial_\nu(\theta\psi) + \tfrac{n-2}{2}H\theta\psi\right),$$

and the proof follows. □

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