CONVERGENCE AND OPTIMALITY OF ADAPTIVE METHODS IN THE FINITE ELEMENT EXTERIOR CALCULUS FRAMEWORK

MICHAEL HOLST, ADAM MIHALIK, AND RYAN SZYPOWSKI

ABSTRACT. Finite Element Exterior Calculus (FEEC) was developed by Arnold, Falk, Winther and others over the last decade to exploit the observation that mixed variational problems can be posed on a Hilbert Complex, and Galerkin-type mixed methods can then be obtained by solving finite-dimensional subcomplex problems. Stability and consistency of the resulting methods then follow directly from the framework by establishing the existence of operators connecting the Hilbert complex with its subcomplex, giving a essentially a "recipe" for well-behaved methods. In 2012, Demlow and Hirani developed a posteriori error indicators for driving adaptive methods in the FEEC framework. While adaptive techniques have been used successfully with mixed methods for years, convergence theory for such techniques has not been fully developed. The main difficulty is lack of error orthogonality. In 2009, Chen, Holst, and Xu established convergence and optimality of an adaptive mixed finite element method for the Poisson equation (the Hodge-Laplace problem for $k = n = 2$) on simply connected polygonal domains in two dimensions. Their argument used a type of quasi-orthogonality result, exploiting the fact that the error was orthogonal to the divergence free subspace, while the part of the error not divergence free was bounded by data oscillation through a discrete stability result. In this paper, we use the FEEC framework to extend these convergence and complexity results for mixed methods on simply connected domains in two dimensions to more general domains. While our main results are for the Hodge-Laplace problem ($k = n$) on domains of arbitrarily topology and spatial dimension, a number of our supporting results also hold for the more general $\mathfrak{B}$-Hodge-Laplace problem ($k \neq n$).

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1. Introduction

An idea that has had a major influence on the development of numerical methods for PDE applications is that of *mixed finite elements*, whose early success in areas such as computational electromagnetics was later found to have surprising connections with the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [8, 26, 27, 16]. A core idea underlying these developments is the *Helmholtz-Hodge* orthogonal decomposition of an arbitrary vector field $f \in L^2(\Omega)$ into curl-free, divergence-free, and harmonic functions:

$$f = \nabla p + \nabla \times q + h,$$

where h is harmonic (divergence- and curl-free). The mixed formulation is explicitly computing the decomposition for $h = 0$, and finite element methods based on mixed formulations exploit this. There is a connection between this decomposition and *de Rham cohomology*; the space of harmonic forms is isomorphic to the first *de Rham cohomology* of the domain Ω, with the number of holes in Ω giving the first Betti number, and creating obstacles to well-posed formulations of elliptic problems. A natural question is then: What is an appropriate mathematical framework for understanding this abstractly, that will allow for a methodical construction of "good" finite element methods for these types of problems? The answer turns out to be theory of *Hilbert Complexes*. Hilbert complexes were originally studied in [9] as a way to generalize certain properties of elliptic complexes, particularly the Hodge decomposition and other aspects of Hodge theory. The *Finite Element Exterior Calculus* (or *FEEC*) [3, 4] was developed to exploit this abstraction. A key insight was that from a functional-analytic point of view, a mixed variational problem can be posed on a Hilbert complex: a differential complex of Hilbert spaces, in the sense of [9]. Galerkin-type mixed methods are then obtained by solving the variational problem on a finite-dimensional subcomplex. Stability and consistency of the resulting method, often shown using complex and case-specific arguments, are reduced by the framework to simply establishing existence of operators with certain properties that connect the Hilbert complex with its subcomplex, essentially giving a "recipe" for the development of provably well-behaved methods.

Due to the pioneering work of Babuska and Rheinboldt [5], adaptive finite element methods (AFEM) based on *a posteriori* error estimators have become standard tools in solving PDE problems arising in science and engineering (cf. [1, 36, 29]). A standard adaptive algorithm has the general iterative structure:

$$\textsf{Solve} \longrightarrow \textsf{Estimate} \longrightarrow \textsf{Mark} \longrightarrow \textsf{Refine} \tag{1.1}$$

where Solve computes the discrete solution u_k in a subspace $X_k \subset X$; Estimate computes certain error estimators based on u_k, which are reliable and efficient in the sense that they are good approximation of the true error $u - u_k$ in the energy norm; Mark applies certain marking strategies based on the estimators; and finally, Refine divides each marked element and completes the mesh to to obtain a new partition, and subsequently an enriched subspace X_{k+1}. The fundamental problem with the adaptive procedure (1.1) is guaranteeing convergence of the solution sequence. The first convergence result for (1.1) was obtained by Babuska and Vogelius [6] for linear elliptic problems in one space dimension. The multi-dimensional case was open until Dörfler [14] proved convergence of (1.1) for Poisson equation, under the assumption that the initial mesh was fine enough to resolve the influence of data oscillation. This result was improved by Morin, Nochetto, and Siebert [22], in which the convergence was proved without conditions on the initial mesh, but requiring the so-called *interior node property*, together with an additional marking step driven by data oscillation. These results were then improved and generalized in several respects [25, 24, 32]. In another direction, it was shown by Binev, Dahmen and DeVore [7] for the first time that AFEM for Poisson equation in the plane has optimal computational complexity by using a special coarsening step. This result was improved by Stevenson [33] by showing the optimal complexity in general spatial dimension without a coarsening step. These error reduction and optimal complexity results were improved recently in several aspects in [10]. In their analysis, the artificial assumptions of interior node and extra marking due to data oscillation were removed, and the convergence result is applicable to general linear elliptic equations. The main ingredients of this new convergence analysis are the global upper bound on the error given by the *a posteriori* estimator, orthogonality (or possibly only quasi-orthogonality) of the underlying bilinear form arising from the linear problem, and a type of error indicator reduction produced by each step of AFEM. We refer to [28] for a recent survey of convergence analysis of AFEM for linear elliptic PDE problems which gives an overview of all of these results through late 2009. See also [19] or an overview of various extensions to nonlinear problems.

Of particular relevance here is the 2009 article of Chen, Holst, and Xu [11], where convergence and optimality of an adaptive mixed finite element method for the Poisson equation on simply connected polygonal domains in two dimensions was established. The main difficulty for mixed finite element methods is the lack of minimization principle, and thus the failure of orthogonality. A quasi-orthogonality property is proved on the $\|\sigma - \sigma_h\|_{L^2}$ error in [11] using the fact that the error is orthogonal to the divergence free subspace, while the part of the error that is not divergence free was bounded by the data oscillation using a discrete stability result. This discrete stability result was then also used to get a localized discrete upper bound, which was the key to giving a proof of optimality of the resulting adaptive method. A key technical tool was the use of the error indicator developed by Alonso in [2]. In this paper, we will generalize the approach taken in [11] by analyzing the $\|\sigma - \sigma_h\|_{L^2\Lambda^{k-1}(\Omega)}$ error in the FEEC framework, which will allow us extend the convergence and complexity results for simply connected domains in two dimensions in [11] to domains of arbitrary topology and spatial dimension. In FEEC terminology, the problem considered in [11] is equivalent to the Hodge-Laplace problem when $k = n = 2$. As described in more detail in Section 2 below, Hodge-Laplace problems on the complex H^k, $k = n$ are a subset of the more general $\mathfrak{B}$-Hodge-Laplace problem. Our main result will apply to the $k = n$ case for arbitrary n and domains which are not necessarily simply connected. However, a number of our supporting results also hold for the more general $\mathfrak{B}$-Hodge-Laplace problem ($k \neq n$). For each result, we will indicate whether it holds for all $\mathfrak{B}$-Hodge-Laplace problems, or just the case $k = n$.

In mixed finite element methods σ is often the variable of interest, and the error measured in the natural norm can be broken into two components,

$$\|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} = \|\sigma - \sigma_h\|_{L^2\Lambda^{k-1}(\Omega)} + \|d(\sigma - \sigma_h)\|_{L^2\Lambda^{k-1}(\Omega)}.$$

In the general $\mathfrak{B}$ problems we have $d(\sigma - \sigma_h) = f - f_h$, and standard interpolation techniques can be used to efficiently reduce this error. Our results will focus on the first term involving $\sigma - \sigma_h$, the quantity that is often of interest, yet typically cannot be calculated explicitly.

The remainder of the paper is organized as follows. In Section 2 we introduce the notational and technical tools needed for the paper. The first part of Section 2 follows the ideas of [4] in introducing general Hilbert complexes, the de Rham complex, and properties of specific mappings between the complexes. We then give a brief overview of a standard adaptive finite element algorithm. In Section 3 we follow the ideas in [11] and develop a quasi-orthogonality result. In Section 4, we prove discrete stability (which was needed for proving quasi-orthogonality in Section 3), and also establish a continuous stability result, which will be needed for deriving an upper bound on the error. In Section 5 we begin by introducing an error indicator and then derive bounds and a type of continuity result for this indicator. An adaptive algorithm is then presented in Section 6, and we then combine the results from the previous sections to prove both convergence and optimality. Finally, we draw some conclusions in 7.

2. PRELIMINARIES

In this section we first review abstract Hilbert complexes. We then examine the particular case of the de-Rham complex. We follow closely the notation and the general development of Arnold, Faulk and Winther in [3, 4]. We also discuss results from Demlow and Hirani in [13]. (See also [17, 18] for a concise summary of Hilbert Complexes in a yet more general setting.) We then give an overview of the basics of Adaptive Finite Element Methods (AFEM), and the ingredients we will need to prove convergence and optimality within the FEEC framework.

2.1. Hilbert complexes.

We begin with a quick summary of some basic concepts and definitions. A *Hilbert complex* $(W, \mathrm{d}\,)$ is a sequence of Hilbert spaces W^k equipped with closed, densely defined linear operators, d^k, which map their domain, $V^k \subset W^k$ to the kernel of d^{k+1} in W^{k+1}. A Hilbert complex is bounded if each d^k is a *bounded* linear map from W^k to W^{k+1} A Hilbert complex is *closed* if the range of each d^k is closed in W^{k+1}. Given a Hilbert complex (W, d), the subspaces $V^k \subset W^k$ endowed with the graph inner product

$$\langle u, v\rangle_{V^k} = \langle u, v\rangle_{W^k} + \langle d^k u, d^k v\rangle_{W^{k+1}},$$

form a Hilbert complex (V, d) known as the *domain complex*. By definition $\mathrm{d}^{k+1}\circ\, \mathrm{d}^k = 0$, thus (V, d) is a bounded Hilbert complex. Additionally, (V, d) is closed if (W, d) is closed.

The range of d^{k-1} in V^k will be represented by $\mathfrak{B}^k$, and the null space will be represented by $\mathfrak{Z}^k$. Clearly, $\mathfrak{B}^k \subset \mathfrak{Z}^k$. The elements of $\mathfrak{Z}^k$ orthogonal to $\mathfrak{B}^k$ are the space of harmonic forms, represented by $\mathfrak{H}^k$. For a closed Hilbert complex we can write the *Hodge decomposition* of W^k and V^k,

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W}, \tag{2.1}$$

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}. \tag{2.2}$$

Following notation common in the literature, we will write simply $\mathfrak{Z}^{k\perp}$ for $\mathfrak{Z}^{k\perp_W}$ or $\mathfrak{Z}^{k\perp_V}$, when clear from the context. For closed Hilbert complexes, an important result will be the *Poincaré inequality*,

$$\|v\|_V \leq c_P \|d^k v\|_W, \ \ v \in \mathfrak{Z}^{k\perp}. \tag{2.3}$$

The de Rham complex is the practical complex where abstract results proved on a general Hilbert complex will be made useful. The de-Rham complex satisfies an important compactness discussed in [4], and therefore the compactness property is assumed in the abstract analysis.

The abstract Hodge Laplacian. Given a Hilbert complex (W, d), the operator $L = dd^* + d^*d$, $W^k \to W^{k+1}$ will be referred to as the abstract Hodge Laplacian. For a given $f \in W^k$, the Hodge Laplacian problem can be written, find u such that

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, v \in V^k \cap V_k^*.$$

The above formulation has some undesirable properties from a computation perspective. The space finite element spaces $V^k \cap V_k^*$, where V_k^* is the domain of d^*, can be difficult to implement, and the problem will not be well-posed in the presence of a nontrivial harmonic space, $\mathfrak{H}^k$. In order to circumvent these issues, a well posed (cf. [3, 4]) *mixed formulation of the abstract Hodge Laplacian* is introduced as the problem of finding $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$, such that:

$$\begin{aligned} \langle \sigma, \tau\rangle - \langle d\tau, u\rangle &= 0, && \forall \tau \in V^{k-1}, \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, && \forall v \in V^k, \\ \langle u, q\rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{2.4}$$

Sub-complexes and approximate solutions to the Hodge Laplacian. In [3, 4] a theory of approximate solutions to the Hodge-Laplace problem is developed by using finite dimensional approximating Hilbert complexes. Let (W, d) be a Hilbert complex with domain complex (V, d). An approximating subcomplex is a set of finite dimensional Hilbert spaces, $V_h^k \subset V^k$ with the property that $dV_h^k \subset V_h^{k+1}$. Since V_h is a Hilbert complex, V_h has a corresponding Hodge decomposition,

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp_V}.$$

By this construction, (V_h, d) is an abstract Hilbert complex with a well posed Hodge Laplace problem. Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, such that

$$\begin{aligned} \langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle &= 0, && \forall \tau \in V_h^{k-1}, \\ \langle d\sigma_h, v\rangle + \langle du_h, dv\rangle + \langle p_h, v\rangle &= \langle f, v\rangle, && \forall v \in V_h^k, \\ \langle u_h, q\rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{2.5}$$

An assumption made in [4] in developing their theory is the existence of a bounded cochain projection, $\pi_h : V \to V_h$, which commutes with the differential operator.

In [4], a convergence result is developed between the approximating and original solutions. The result relies on the approximating complex getting sufficiently close to the original complex in the sense that $\inf_{v\in V_h^k} \|u - v\|_V$ can be assumed to be sufficiently small for relevant $u \in V^k$. Adaptive methods, on the other hand, gain computational efficiency by limiting the degrees of freedom used in areas of the domain where it does not significantly impact the quality of the solution.

2.2. **The de Rham complex and approximation properties.** The de Rham complex is a cochain complex with many desirable attributes, which, under certain assumptions, allows the results from the previous section to become useful computational tools. This section provides a short review of concepts and definitions related to the de Rham complex, and how they relate to finite elements. This introduction will be brief and and mostly follows the notation from the more in-depth discussion in [4].

For the remainder of the paper we assume a bounded Lipschitz polyhedral domain, $\Omega \in \mathbb{R}^n, n \geq 2$. Let $\Lambda^k(\Omega)$ be the space of smooth k-forms on Ω, and let $L^2\Lambda^k(\Omega)$ be the completion of $\Lambda^k(\Omega)$ with respect to the L^2 inner-product. There are no non-zero harmonic forms in $L^2\Lambda^n(\Omega)$ (see [3], Theorem 2.4) which will be key in simplifying the analysis in our primary case of interest, $k = n$. In general, such a property cannot be assumed for any other $0 < k < n-1$ (where σ will lives in the Hodge-Laplace problem/mixed finite element method), therefore interaction with the harmonics isn't completely avoided. Note that the results in [11] hold only for polygonal and simply connected domains, so in the case $k = n = 2$, $\mathfrak{H}^{k-1}$ contains no non-zero elements, and thus completely avoids harmonics.

The de Rham complex. Let d be the exterior derivative acting as an operator from $L^2\Lambda(\Omega)$ to $L^2\Lambda^{k+1}(\Omega)$. The L^2 inner-product will define the W-norm, and the V-norm will be defined as the graph inner-product

$$\langle u, \omega \rangle_{V^k} = \langle u, \omega \rangle_{L^2} + \langle du, d\omega \rangle_{L^2}.$$

This forms a Hilbert complex ($L^2\Lambda^k(\Omega)$, d), with domain complex $H\Lambda^k(\Omega)$, defined as the set of elements in $L^2\Lambda^k(\Omega)$ with exterior derivatives in $L^2\Lambda^{k+1}(\Omega)$. It can be shown that the compactness property is satisfied, and the therefore results proved on abstract Hilbert complexes can be applied.

Clearly, given the first equation of the Hodge Laplace problem, understanding properties of the adjoint operator are important. Defining the coderivative operator, $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$, and two particular spaces, will be helpful in this respect.

$$\star\delta\omega = (-1)^k d \star \omega, \tag{2.6}$$

$$\mathring{H}\Lambda^k(\Omega) = \{\omega \in H\Lambda^k(\Omega) \big| \operatorname{tr}_{\partial\Omega}\omega = 0\}, \tag{2.7}$$

$$\mathring{H}^*\Lambda^k(\Omega) := \star\mathring{H}\Lambda^{n-k}. \tag{2.8}$$

And combining δ with Stoke's theorem gives a useful version of integration by parts

$$\langle d\omega, \mu \rangle = \langle \omega, \delta\mu \rangle + \int_{\partial\Omega} \operatorname{tr} \omega \wedge \operatorname{tr} \star \mu, \ \ \omega \in \Lambda^{k-1}, \ \mu \in \Lambda^k. \tag{2.9}$$

The following result uses the above concepts and is helpful in understanding the mixed Hodge Laplace problem on the de Rham complex.

Theorem 2.1. *(Theorem 4.1 from* [4]*) Let d be the exterior derivative viewed as an unbounded operator $L^2\Lambda^{k-1}(\Omega) \to L^2\Lambda^k(\Omega)$ with domain $H\Lambda^k(\Omega)$. The the adjoint d*, as an unbounded operator $L^2\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega)$, has $\mathring{H}^*\Lambda^k(\Omega)$ as its domain and coincides with the operator δ defined in* (2.6).

Applying the results from the previous section and Theorem 2.1, we get the mixed Hodge Laplace problem on the de Rham complex: find the unique $(\sigma, u, p) \in H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega) \times \mathfrak{H}^k$ such that

$$\begin{aligned} \sigma = \delta u,\ d\sigma + \delta du = f - p \quad & \text{in } \Omega, \\ \text{tr} \star u = 0,\ \text{tr} \star du = 0 \quad & \text{on } \partial\Omega, \\ u \perp \mathfrak{H}^k. & \end{aligned} \tag{2.10}$$

The general complex can be described by the following diagram

$$0 \to H^1(\Omega) \xrightarrow{d} \cdots \to H(div;\Omega) \xrightarrow{div} L^2(\Omega) \xrightarrow{d} 0, \tag{2.11}$$

and thus in the case $k = n$, (2.10) is equivalent (up to a sign) to the solution of Poisson equation with natural Dirichlet boundary conditions. In this case du = 0 and p = 0, thus the mixed Hodge Laplace problem on the de Rham complex simplifies to: find the unique $(\sigma, u, p) \in H\Lambda^{n-1}(\Omega) \times H\Lambda^n(\Omega) \times \mathfrak{H}^n$ such that

$$\begin{aligned} \sigma = \delta u,\ d\sigma = f \quad & \text{in } \Omega, \\ \text{tr} \star u = 0, \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.12}$$

Let (Λ_n, d) be a finite dimensional subcomplex of the de Rham complex, then a discrete version of (2.12) can be written: find the unique $(\sigma_h, u_h, p_h) \in \Lambda_h^{n-1}(\Omega) \times \Lambda_h^n(\Omega) \times \mathfrak{H}_h^n$ such that

$$\begin{aligned} \sigma_h = \delta_h u_h,\ d\sigma_h = f \quad & \text{in } \Omega, \\ \text{tr} \star u_h = 0, \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.13}$$

Here it is worth noting that δ_h is distinct from δ, and follows from the definition of the abstract discrete problem.

Finite element differential forms. For the remainder of the paper it is assumed that the approximating sub-complexes of the de Rham complex are constructed as combinations the polynomial spaces of k-forms, $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$. For a detailed discussion on these spaces and construction of Hilbert complexes using these spaces, see [4]. We also have a useful property in the case $k = n$,

$$\begin{aligned} \mathcal{P}_r^-\Lambda^n &= \mathcal{P}_{r-1}\Lambda^n, \\ \mathcal{P}_r^-\Lambda^0 &= \mathcal{P}_r\Lambda^0. \end{aligned}$$

For a shape-regular, conforming triangulation $\mathcal{T}_h$ of $\Omega, \Lambda_h^k(\Omega) \subset \Lambda^k(\Omega)$ will denote space of k-forms constructed using specific combinations of the these spaces on $\mathcal{T}_h$. For an element $T \in \mathcal{T}_h$, we set $h_T := \text{diam}(T)$. We do not discuss the details of these spaces further, but when needed we will mention and explain specific properties.

Bounded Cochain Projections. Bounded cochain projections and their approximation properties are necessary in the analysis of both uniform and adaptive FEMs in the FEEC framework. Properties of three different interpolation operators will be important in our analysis. The three operators and respective notation that we will use are as follows: the canonical projections I_h defined in [3, 4], the smoothed projection operator π_h from [4], and the commuting quasi-interpolant Π_h, as defined in [13] with ideas similar to [30, 31, 12]. Some cases will require a simple projection, and $P_h f$ also written f_h, will denote the L^2-projection of f on to the discrete space parameterized by h. $f_{\mathfrak{B}_h}$ will denote the L^2 projection of f onto the $\mathfrak{B}$ component of the discrete space parameterized by h. Note $f_{\mathfrak{B}_h} = f_h$ when $k = n$.

For the remainder of the paper, $\|\cdot\|$ will denote the $L^2\Lambda^k(\Omega)$ norm, and when taken on specific elements of the domain, T, we write $\|\cdot\|_T$. For all other norms, such as $H\Lambda^k(\Omega)$ and $H^1\Lambda^k(\Omega)$ we write $\|\cdot\|_{H\Lambda^k(\Omega)}$ and $\|\cdot\|_{H^1\Lambda^k(\Omega)}$ respectively.

Lemma 2.2. *Suppose $\tau \in H^1\Lambda^{n-1}(\Omega)$, and I_h is the canonical projection operator defined in* [3, 4]. *Let $\Lambda_h^{n-1}(\Omega)$ and $\Lambda_h^n(\Omega)$ be taken as above. Then I_h is a projection onto $\Lambda_h^n(\Omega), \Lambda_h^{n-1}(\Omega)$ and satisfies*

$$\|\tau - I_h\tau\|_T \leq Ch_T\|\tau\|_{H^1\Lambda^k(T)}, \quad \forall T \in \mathcal{T}_h, \tag{2.14}$$

$$I_h d = dI_h \tag{2.15}$$

Proof. The first part is comes from Equation (5.4) in [3]. The second part follows the construction of I_h. □

Given $f_h, u_h \in \Lambda_h^n(\Omega)$, let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$. Let $T \in \mathcal{T}_H$, then the two projection lemmas hold:

Lemma 2.3.

$$\int_T (f_h - I_H f_h) = 0. \tag{2.16}$$

Proof. Since we are dealing with the case $k = n$, the canonical projections are L^2 bounded.

Let $\omega \in P_r\Lambda^n(T)$. This is sufficient to cover pertinent cases as $P_r^-\Lambda^n(T) = P_{r-1}\Lambda^n(T)$. By definition [3](pg 57):

$$\int_T (\omega - I_H\omega) \wedge \eta = 0, \;\; \eta \in P_r^-\Lambda^0(T) = P_r\Lambda^0(T).$$

Set $\eta = 1$ and this completes proof. □

Lemma 2.4.

$$\langle (I_h - I_H)u_h, f_h\rangle_T = \langle u_h, (I_h - I_H)f_h\rangle_T. \tag{2.17}$$

Proof.

$$\int_T (u_h - I_H u_h) \wedge \eta = 0, \quad \eta \in P_r\Lambda^0(T) = \star P_r\Lambda^n(T).$$

Thus,

$$\langle (I_h - I_H)u_h, f_h\rangle_T = \langle (I_h - I_H)u_h, (I_h - I_H)f_h\rangle_T.$$

Using the same logic we can get rid of $I_H u_h$. □

The next lemma is a portion of a lemma taken directly from [13], it will be a key tool in bounding the error.

Lemma 2.5. *Assume $1 \leq k \leq n$, and $\phi \in H\Lambda^{k-1}(\Omega)$ with $\|\phi\| \leq 1$. Then there exists $\varphi \in H^1\Lambda^{k-1}(\Omega)$ such that $d\varphi = d\phi, \Pi_H d\phi = d\Pi_H\phi = d\Pi_H\varphi$, and*

$$\sum_{T\in T_h} h_T^{-2}\|\varphi - \Pi_H\varphi\|_T^2 + h_T^{-1}\|\mathrm{tr}(\varphi - \Pi_H\varphi)\|_{\partial T}^2 \leq C.$$

Proof. See Lemma 6 in [13]. □

The following theorem is a special case of Theorem 3.5 from [4]. Rather than showing the result on a general Hilbert Complex with a general cochain projection, we use the de Rham complex and the smoothed projection operator π_h in order to use uniform boundedness of the cochain projection.

Theorem 2.6. *Assume $\Lambda_h^k(\Omega)$ is a subcomplex of $H\Lambda^k(\Omega)$ as described above, and let π_h be the smoothed projection operator. Then*

$$\|(I - P_{\mathfrak{H}})q\|_V \leq \|(I - \pi_h^k)P_{\mathfrak{H}}q\|_V, \quad q \in \mathfrak{H}_h^k, \tag{2.18}$$

then combining the above with the triangle inequality,

$$\|q\|_V \leq c\|P_{\mathfrak{H}}q\|_V, \quad q \in \mathfrak{H}_h^k. \tag{2.19}$$

Proof. Since the de Rham complex is a bounded closed Hilbert complex, (2.18) is directly from [4]. (2.18) with the triangle inequality implies

$$\|q\|_V - \|P_{\mathfrak{H}}q\|_V \leq \|(I - \pi_h^k)\|\|P_{\mathfrak{H}}q\|_V, \quad q \in \mathfrak{H}_h^k,$$

thus,

$$\|q\|_V \leq (\|(I - \pi_h^k)\| + 1)\|P_{\mathfrak{H}}q\|_V, \quad q \in \mathfrak{H}_h^k.$$

By construction the smoothed projection operator π_h is a uniformly bounded operator with respect to h and therefore $(\|(I - \pi_h^k)\| + 1)$ can be replaced with a generic constant not dependent on the triangulation. In Corollary 2.8 a similar result will be required to deal with the harmonics on two discrete complexes. In this case the canonical projection I_h can be used as a map between the two complexes, and $(I - I_h)$ is clearly uniformly bounded with respect to h. □

Theorem 2.7 will be essential in dealing with the harmonic forms in the proof of a continuous upper-bound. The corollary will be used identically when proving a discrete upper-bound. For use in our next two results we introduce an operator δ and one of its important properties. Let A, B be $n < \infty$ dimensional, closed subspaces of a Hilbert space W, and let

$$\delta(A, B) = \sup_{x \in A, \|x\|=1} \|x - P_B x\|,$$

then [13], Lemma 2 which takes the original ideas from [20], shows

$$\delta(A, B) = \delta(B, A). \tag{2.20}$$

Theorem 2.7. *Assume $\mathfrak{H}_H$ and $\mathfrak{H}$ have the same finite dimensionality. The there exist a constant $C_{\mathfrak{H}}$ dependent only on $\mathcal{T}_0$, such that*

$$\delta(\mathfrak{H}, \mathfrak{H}_H) = \delta(\mathfrak{H}_H, \mathfrak{H}) \leq C_{\mathfrak{H}} < 1. \tag{2.21}$$

Proof. Given that $\mathfrak{H}_H$ and $\mathfrak{H}$ have the same finite dimensionality we can apply (2.20) to prove the first equality.

For the second part,

$$\delta(\mathfrak{H}_H, \mathfrak{H}) = \sup_{x \in \mathfrak{H}_H, \|x\|=1} \|x - P_{\mathfrak{H}}x\|,$$

for any $x \in \mathfrak{H}_H$ with $\|x\| = 1$, (2.19) implies,

$$C \leq \|P_{\mathfrak{H}}x\|, \quad 0 < C < 1.$$

Now, by orthogonality of the projection, we have

$$\delta(\mathfrak{H}_H, \mathfrak{H}) \leq \sqrt{1 - C^2} = C_{\mathfrak{H}} < 1.$$

□

Corollary 2.8.

$$\delta(\mathfrak{H}_h, \mathfrak{H}_H) = \delta(\mathfrak{H}_H, \mathfrak{H}_h) \leq \tilde{C}_{\mathfrak{H}} < 1. \tag{2.22}$$

Proof. In the proof uses the same logic as that of Theorem 2.7. The difference is that the harmonics are compared on two discrete complexes $\mathfrak{H}_h$ and $\mathfrak{H}_H$, and therefore I_h is used rather than π_h. □

2.3. **Adaptive Finite Elements Methods.** Given an initial triangulation, $\mathcal{T}_0$, the adaptive procedure will generate a nested sequence of triangulations $\mathcal{T}_k$ and discrete solutions σ_h, by looping through the following steps:

$$\textsf{Solve} \longrightarrow \textsf{Estimate} \longrightarrow \textsf{Mark} \longrightarrow \textsf{Refine} \tag{2.23}$$

The following subsection will describe details of these steps.

Approximation Procedure. We assume access to a routine SOLVE, which can produce solution to (2.5) given a triangulation, problem data, and a desired level of accuracy. For the ESTIMATE step we will introduce error indicators η_T on each element T in trigulation $\mathcal{T}_h$. In the MARK step we will use Dörfler Marking strategy [15]. An essential feature of the marking process is that the summation of the error indicators on the marked elements exceeds a user defined marking parameter θ.

We assume access to an algorithm REFINE in which marked elements are subdivided into two elements of the same size, resulting in a conforming, shape-regular mesh. Triangles outside of the original marked set may be refined in order to maintain conformity. Bounding the number of such refinements is important in showing optimality of the method. Along these lines, Stevenson [34] showed certain bisection algorithms developed in two-dimensions can be extended to n-simplices of arbitrary dimension satisfying

$(1)\{\mathcal{T}_k\}$ is shape regular and the shape regularity depends only on $\mathcal{T}_0$.

$(2)\#\mathcal{T}_k \leq \#\mathcal{T}_0 + C\#M.$

Approximation of the Data. A measure of data approximation will be necessary in establishing a quasi-orthogonality result. Following ideas of [22], data oscillation will defined as follows

Definition 2.9. *(*Data oscillation*) Let f* $\in L^2\Lambda^k(\Omega)$*, and* $\mathcal{T}_h$ *be a conforming triangulation of* Ω*. Let* h_T *be the mesh-size size for a given* $T \in \mathcal{T}_h$*. We define*

$$\operatorname{osc}(f, \mathcal{T}_h) := \|h(f - f_{\mathfrak{B}_h})\|_{\mathcal{T}_{\mathcal{H}}} := \Big(\sum_{T \in T_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2\Big)^{1/2}.$$

Stevenson [34] generalized the ideas of [7] generalizes the APPROX algorithm of [7] to arbitrary dimensions,

Theorem 2.10. (Generalized Binev, Dahmen and DeVore) *Given a tolerance* ϵ*, an f* $\in L^2\Lambda^n(\Omega)$ *and a shape regular triangulation* $\mathcal{T}_0$*, there exists an algorithm*

$$\mathcal{T}_H = \textsf{APPROX}(f, \mathcal{T}_0, \epsilon),$$

such that

$$\operatorname{osc}(f, \mathcal{T}_{\mathcal{H}}) \leq \epsilon, \quad and \quad \#\mathcal{T}_H - \#\mathcal{T}_0 \leq C\|f\|_{\mathcal{A}_o^{1/s}}^{1/s} \epsilon^{-1/s}$$

As in the case of [11], the analysis of convergence and procedure will follow [10], and the optimality will follow [33]

3. QUASI-ORTHOGONALITY

The main difficulty for mixed finite element methods is the lack of minimization principle, and thus the failure of orthogonality. In [11], a quasi-orthogonality property is proven using the fact that the error is orthogonal to the divergence free subspace. In this section we follow much of the same logic in proving a quasi-orthogonality result in the solutions to (2.12) and (2.13). Analogous to [11], our result uses the fact that $\sigma - \sigma_h$ is orthogonal to the $\mathfrak{Z}_h^{n-1} \subset H\Lambda_h^{n-1}(\Omega)$.

Solutions of Hodge Laplace problems on nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ will frequently be compared. Nested in the sense that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. For a given $f \in H\Lambda^n(\Omega)$, let $\mathcal{L}^{-1}f$ denote the solutions of (2.12). Let $\mathcal{L}_h^{-1}f_{\mathfrak{B}_h}$ and $\mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$ denote the solution of (2.13) on $\mathcal{T}_h$ and $\mathcal{T}_H$ respectively. Set the following triples, $(u, \sigma, p) = \mathcal{L}^{-1}f$, $(u_h, \sigma_h, p_h) = \mathcal{L}_h^{-1}f_{\mathfrak{B}_h}$, $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) = \mathcal{L}_h^{-1}f_{\mathfrak{B}_H}$ and $(u_H, \sigma_H, p_H) = \mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$. As with general $\mathfrak{B}$ problems, the harmonic component will be zero in each of these solutions. When we are only interested in σ we will abuse this notation by witting $\sigma = \mathcal{L}^{-1}f$.

Lemma 3.1. *Given $f \in L^2\Lambda^k(\Omega)$, such that $f \in \mathfrak{B}^k$, and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, then*

$$\langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle = 0. \tag{3.1}$$

Proof. Since $\tilde{\sigma}_h - \sigma_H \in V_h^{k-1} \subset V^{k-1}$, (2.4) implies

$$\langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle = \langle u - u_h, d(\tilde{\sigma}_h - \sigma_H) \rangle,$$

and the harmonic terms are zero since these are $\mathfrak{B}$ problems,

$$\begin{aligned} &= \langle u - u_h, f_{\mathfrak{B}_H} - f_{\mathfrak{B}_H} \rangle \\ &= 0. \end{aligned}$$

□

Theorem 3.2. *Given $f \in L^2\Lambda^n(\Omega)$ and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, then*

$$\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle \leq \sqrt{C_0}\|\sigma - \sigma_h\|\operatorname{osc}(f_{\mathfrak{B}_h}, \mathcal{T}_H), \tag{3.2}$$

and for any $\delta > 0$,

$$(1 - \delta)\|\sigma - \sigma_h\|^2 \leq \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \frac{C_0}{\delta}\operatorname{osc}^2(f_{\mathfrak{B}_h}, \mathcal{T}_H). \tag{3.3}$$

Proof. By (3.1) we have

$$\begin{aligned} \langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle &= \langle \sigma - \sigma_h, \sigma_h - \tilde{\sigma}_h \rangle + \langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle \\ &= \langle \sigma - \sigma_h, \sigma_h - \tilde{\sigma}_h \rangle \\ &\leq \|\sigma - \sigma_h\|\|\sigma_h - \tilde{\sigma}_h\|. \end{aligned}$$

And then by the discrete stability result, Theorem 4.4, we have

$$\leq \sqrt{C_0}\|\sigma - \sigma_h\|\mathrm{osc}(f_{\mathfrak{B}_h}, \mathcal{T}_H).$$

(3.3) follows standard arguments and is identical to [11] (3.4) □

4. CONTINUOUS/DISCRETE STABILITY

In this section we will prove stability results for approximate solutions to the σ portion of the Hodge Laplace problem. Theorem 4.1 gives a stability result for particular solutions of the Hodge de Rham problem that will be useful in bounding the approximation error in Section 5. Theorem 4.4 will prove the discrete stability result used in Theorem 3.2. The basic structure of these arguments will follow [11], but key modifications are introduced in order to generalize the dimensionality and topology of the main results.

Theorem 4.1. (Continuous Stability Result) *Given $f \in L^2\Lambda^n(\Omega)$, let $\mathcal{T}_h$ be a triangulation of Ω. Set $(\sigma, u, p) = \mathcal{L}^{-1}f$ and $(\tilde{\sigma}, \tilde{u}, \tilde{p}) = \mathcal{L}^{-1}f_{\mathfrak{B}_h}$, then*

$$\|\sigma - \tilde{\sigma}\| \leq C\mathrm{osc}(f, \mathcal{T}_h). \tag{4.1}$$

Proof. The harmonic terms are zero since $f, f_{\mathfrak{B}_h} \in \mathfrak{B}^k$, thus

$$\|\sigma - \tilde{\sigma}\|^2 = \langle d(\sigma - \tilde{\sigma}), u - \tilde{u}\rangle = \langle f - f_{\mathfrak{B}_h}, u - \tilde{u}\rangle.$$

Let $v = u - \tilde{u}$. Since $v \in \mathfrak{B}^k$ and $\|\delta v\| = \|\mathrm{grad}\ v\| = \|\sigma - \tilde{\sigma}\|$, we have $v \in H^1\Lambda^n(\Omega)$. Restricting v to an element $T \in \mathcal{T}_h$, we have $v \in H^1\Lambda^n(T)$, thus

$$\|\sigma - \tilde{\sigma}\|^2 = \langle f - f_{\mathfrak{B}_h}, v\rangle = \sum_{T \in T_h} \langle f - f_{\mathfrak{B}_h}, v - I_h v\rangle_T.$$

Applying (2.14),

$$\begin{aligned}
&\leq C \sum_{T \in T_h} h_T \|f - f_{\mathfrak{B}_h}\|_T \|v\|_{H^1\Lambda^n(T)} \\
&= C \sum_{T \in T_h} h_T \|f - f_{\mathfrak{B}_h}\|_T (\|u - \tilde{u}\|_T + \|\delta(u - \tilde{u})\|_T) \\
&\leq C(\sum_{T \in T_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2} (\sum_{T \in T_h} (\|u - \tilde{u}\|_T + \|\delta(u - \tilde{u})\|_T)^2)^{1/2},
\end{aligned}$$

and $v \in H^1\Lambda^n(\Omega)$ allows us to to combine terms of the summation,

$$\leq C(\sum_{T \in T_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2} (\|u - \tilde{u}\| + \|\delta(u - \tilde{u})\|).$$

Since $u - \tilde{u} \in \mathfrak{B}^k$, $\|u - \tilde{u}\| = \langle u - \tilde{u}, d\tau\rangle$ for some $\tau \in \mathfrak{Z}^\perp$ with $\|d\tau\| = 1$, thus

$$= C(\sum_{T \in T_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2} \langle(\sigma - \tilde{\sigma}), \tau\rangle_\Omega + \|\sigma - \tilde{\sigma}\|).$$

Then applying Poincaré on τ:

$$= C\|\sigma - \tilde{\sigma}\|(\sum_{T \in T_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2}.$$

Dividing through by $\|\sigma - \tilde{\sigma}\|$ to complete proof. □

The following is Lemma 4 in [13], and is a special case of Theorem 1.5 of [21]. It is related to the bounded invertibility of d, and will be an important tool in proving discrete stability.

Lemma 4.2. *Assume that B is a bounded Lipschitz domain in R^n that is homeomorphic to a ball. Then the boundary value problem d$\varphi = g \in L_2\Lambda^k(B)$ in B, $\mathrm{tr}\,\varphi = 0$ on ∂B has a solution $\varphi \in H^1_0\Lambda^{k-1}(B)$ with $\|\varphi\|_{H^1\Lambda^{k-1}(B)} \leq C\|g\|_B$ if and only if dg = 0 in B, and in addition,* tr *g = 0 on ∂B if 0 $\leq k \leq n-1$ and $\int_B g = 0$ if $k = n$.*

The next lemma is an intermediate step in proving the discrete stability result. The general structure follows [11] and applies Lemma 4.2 in order to find a sufficiently smooth function that essentially is a bounded inverse of d for the approxmation error of u_h on $\mathcal{T}_H$.

Lemma 4.3. *Let $\mathcal{T}_h$, $\mathcal{T}_H$ be nested conforming triangulations and let σ_h, σ_H be the respective solutions to* (2.5) *with data $f \in L^2\Lambda^n(\Omega)$. Then for any $T \in \mathcal{T}_H$*

$$\|u_h - I_H u_h\|_T \leq \sqrt{C_0} h_T \|\sigma_h\|_T. \tag{4.2}$$

Proof. Let $g_\Omega = u_h - I_H u_h = (I_h - I_H)u_h \in L^2\Lambda^n(\Omega)$. Then, for any $T \in \mathcal{T}_H$ let $g = \mathrm{tr}_T g_\Omega \in L^2\Lambda^n(T)$, and by Lemma 2.3, $\int_T g = 0$. Thus Lemma 4.2 can be applied to find $\tau \in H^1_0\Lambda^{n-1}(T)$, such that:

$$d\tau = (I_h - I_H)u_h, \text{ on } T$$
$$\|\tau\|_{H^1\Lambda^{n-1}(T)} \leq C\|(I_h - I_H)u_h\|_T.$$

Extend τ to $H^1\Lambda^{n-1}(\Omega)$ by zero and then, by Lemma 2.4,

$$\|(I_h - I_H)u_h\|^2_T = \langle (I_h - I_H)u_h, d\tau\rangle_T = \langle u_h, (I_h - I_H)d\tau\rangle_T.$$

Then by Lemma 2.2, and locality of τ,

$$= \langle u_h, d(I_h - I_H)\tau\rangle_\Omega = \langle \sigma_h, (I_h - I_H)\tau\rangle_\Omega.$$

Then again by locality of τ,

$$= \langle \sigma_h, (I_h - I_H)\tau\rangle_T \leq \|\sigma_h\|_T(\|\tau - I_h\tau\|_T + \|\tau - I_H\tau\|_T).$$

Since I_h is uniformly bounded with respect to h on $\Lambda^n_h(\Omega)$,

$$\leq Ch_T\|\sigma_h\|_{0,T}\|\tau\|_{H^1\Lambda^{n-1}(T)} \leq Ch_T\|\sigma_h\|_T\|(I_h - I_H)u_h\|_T.$$

Cancel one power of $\|(I_h - I_H)u_h\|_T$ to complete the proof.

□

Theorem 4.4. (Discrete Stability Result) *Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be nested conforming triangulations. Let $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) = \mathcal{L}_h^{-1} f_{\mathfrak{B}_H}$ and $(u_h, \sigma_h, p_h) = \mathcal{L}_h^{-1} f_{\mathfrak{B}_h}$, with $f \in L^2\Lambda^n(\Omega)$. Then there exists a constant such that*

$$\|\sigma_h - \tilde{\sigma}_h\| \leq C\mathrm{osc}(f_{\mathfrak{B}_h}, \mathcal{T}_H) \tag{4.3}$$

Proof. From 2.5, and since $p_h, \tilde{p}_h = 0$, we have

$$\langle \sigma_h - \tilde{\sigma}_h, \tau_h \rangle = \langle u_h - \tilde{u}_h, d\tau_h \rangle, \quad \forall \tau_h \in \Lambda_h^{k-1}, \tag{4.4}$$

$$\langle d(\sigma_h - \tilde{\sigma}_h), v_h \rangle = \langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h \rangle, \quad \forall v_h \in \Lambda_h^k. \tag{4.5}$$

Next set $\tau_h = \sigma_h - \tilde{\sigma}_h$ in (4.4), and $v_h = u_h - \tilde{u}_h$ in (4.5) to obtain:

$$\|\sigma_h - \tilde{\sigma}_h\|^2 = \langle u_h - \tilde{u}_h, d(\sigma_h - \tilde{\sigma}_h) \rangle = \langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h \rangle,$$

and since $(f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}) \perp L^2\Lambda_H^k(\Omega)$ when $k = n$, we have

$$\langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h \rangle = \langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h - I_H v_h \rangle.$$

Then by Lemma 4.3, we have:

$$\begin{aligned}
\|\sigma_h - \tilde{\sigma}_h\|^2 &= \sum_{T \in T_H} \langle v_h - I_H v_h, f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H} \rangle_T \\
&\leq \sum_{T \in T_H} \|f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}\|_T \|v_h - I_H v_h\|_T \\
&\leq C \sum_{T \in T_H} h_T \|f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}\|_T \|\sigma_h - \tilde{\sigma}_h\|_T \\
&\leq C (\sum_{T \in T_H} h_T^2 \|f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}\|_T^2)^{1/2} \|\sigma_h - \tilde{\sigma}_h\|
\end{aligned}$$

Then cancel one $\|\sigma_h - \tilde{\sigma}_h\|$ to complete the proof.

□

5. Error Estimator, Upper and Lower bounds

In this section we introduce the a posteriori error estimators used in our adaptive algorithm. The first two terms of the estimator follow [2, 11], and a third term is introduced in order to construct a more practical and efficient algorithm. Next, we prove bounds on these estimators and a continuity result, both of which are key ingredients in showing the convergence and optimality of our adaptive method.

5.1. **Error Estimator: Definition, Lower bound and Continuity.**

Definition 5.1. (Element Error Estimator) *Let $T \in \mathcal{T}_H$. Let the jump in τ over an element face be denoted by $[[\tau]]$. For element faces on $\delta\Omega$ we set $[[\tau]] = \tau$. The element error indicator is defined as*

$$\eta_T^2(\sigma_H) = h_T \|[[\mathrm{tr} \star \sigma_H]]\|_{\partial T}^2 + h_T^2 \|\delta \sigma_H\|_T^2 + h_T^2 \|d(\sigma - \sigma_H)\|_T^2$$

For a subset $\tilde{\mathcal{T}}_H \subset \mathcal{T}_H$, define

$$\eta^2(\sigma_H, \tilde{\mathcal{T}}_H) := \sum_{T \in \tilde{\mathcal{T}}_H} \eta_T^2(\sigma_H)$$

Theorem 5.2. (Lower Bound) *Given $f \in L^2\Lambda^k(\Omega)$ and a shape regular triangulation $\mathcal{T}_H$, let $\sigma = \mathcal{L}^{-1} f$ and $\sigma_H = \mathcal{L}_H^{-1} f$. Then there exists a constant dependent only on the shape regularity of $\mathcal{T}_H$ such that*

$$C_2\eta^2(\sigma_H, \mathcal{T}_H) \le \|\sigma - \sigma_H\|^2 + C_2\text{osc}^2(f, \mathcal{T}_H). \tag{5.1}$$

Proof. In proving a lower bound, in [13] it is shown that

$$h_T\|\delta\sigma_H\|_T \le C\|\sigma - \sigma_H\|_T,$$
$$h_T^{1/2}\|[[\text{tr} \star \sigma_H]]\|_{\partial T} \le C\|\sigma - \sigma_H\|_{\mathcal{T}_t},$$

where $\mathcal{T}_t$ is the set of all triangles sharing a boundary with T. The first is equation (5.7) and the second is a result of equation (5.12).

Sum terms and add oscillation to both sides to complete the error-indicator term. Notice, by conformity of the triangulation, the summation of the $\|\sigma - \sigma_H\|_{\mathcal{T}_t}$ terms can at most be some multiple of $\|\sigma - \sigma_H\|_{\mathcal{T}_H}$ depending on the dimensionality of the problem. □

The following lemma will be important in proving a continuity result used in showing convergence of our adaptive algorithm. It is nearly identical to an estimator efficiency proof in [13], but the subtle difference is that we make use of σ_H, the solution on the less refined mesh, and σ is not used in our arguments.

Lemma 5.3. *Given $f \in L^2\Lambda^k(\Omega)$ and nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let $\sigma_h = \mathcal{L}_h^{-1}f_{\mathfrak{B}_h}$ and $\sigma_H = \mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$. Then for $T \in \mathcal{T}_h$*

$$C_2\sum_{T\in\mathcal{T}_h}(h_T\|[[\text{tr } \star(\sigma_h - \sigma_H)]]\|_{\partial T}^2 + h_T^2\|\delta(\sigma_h - \sigma_H)\|_T^2) \le \|\sigma_h - \sigma_H\|^2. \tag{5.2}$$

Proof. Here we will closely follow [13] in applying the “bubble function” technique of Verfürth[35] in order to bound residual terms in the FEEC framework. For $T \in \mathcal{T}_h$ one can construct a corresponding bubble function $b_T \in W_\infty^1(\Omega)$ with supp(b_T) = T, and the property that for any polynomial form v of arbitrary but uniformly bounded degree defined on T, we have

$$\|v\|_T \simeq \|\sqrt{b_T}v\|_T. \tag{5.3}$$

For $n-1$ dimensional faces $e = T_1 \cap T_2$, with $T_1, T_2 \in \mathcal{T}_h$, and T_2 void (see [13]) on $\partial\Omega$, one can construct a corresponding edge bubble function $b_e \in W_\infty^1(\Omega)$ with supp(b_e) = $T_1 \cup T_2$ and the property that for any polynomial form v of arbitrary but uniformly bounded degree defined on e, we have

$$\|v\|_e \simeq \|\sqrt{b_e}v\|_e. \tag{5.4}$$

Given a k-form v defined on an $n-1$ dimensional face $e = T_1 \cap T_2$, one can construct χ_v to be a polynomial extension of v to $T_1 \cup T_2$ such that

$$\|\chi_v\|_{0,(T_1\cup T_2)} \le Ch_T^{1/2}\|v\|_e, \tag{5.5}$$

where h_T can be either h_{T_1} or h_{T_2} since they are neighbors which have sizes related by a shape regularity constant

Let $\psi = b_T(\delta(\sigma_h - \sigma_H))$, which by construction of b_T will be zero on ∂T. Applying integration by parts, we have

$$\|\delta(\sigma_h - \sigma_H)\|_T^2 \simeq \langle\delta(\sigma_h - \sigma_H), \psi\rangle = \langle\sigma_h - \sigma_H, d\psi\rangle, \tag{5.6}$$

and then applying an inverse inequality $\|d\psi\|_T \leq Ch_T^{-1}\|\psi\|_T$,

$$h_T\|\delta(\sigma_h - \sigma_H)\| \leq C\|\sigma_h - \sigma_H\|_T. \tag{5.7}$$

For an element face e, shared by elements T_1, T_2, we have

$$\begin{aligned}\|[[\text{tr } \star(\sigma_h - \sigma_H)]]\|_e^2 &\simeq \langle b_e \star \psi, [[\text{tr } \star(\sigma_h - \sigma_H)]]\rangle \\ &= \int_e \text{tr } (b_e\chi_\psi) \wedge [[\text{tr } \star(\sigma_h - \sigma_H)]] \\ &= \langle d(b_e\chi_\psi), \sigma_h - \sigma_H\rangle_{T_1\cup T_2} - \langle b_e\chi_\psi, \delta_h(\sigma_h - \sigma_H)\rangle_{T_1\cup T_2},\end{aligned} \tag{5.8}$$

where δ_h is defined to be δ evaluated elementwise on elements of $\mathcal{T}_h$. The necessity of this additional definition is that neither σ_h or σ_H are in $H^*\Lambda^{k-1}(\Omega)$ globally, but σ_h and σ_H are in $H^*\Lambda^{k-1}$ when restricted to individual elements of $\mathcal{T}_h$. Next, using the inverse inequality $\|d(b_e\chi_\psi)\|_T \leq Ch_T^{-1}\|b_e\chi_\psi\|_T$, we have

$$\leq C\|[[\text{tr}\star(\sigma_h - \sigma_H)]]\|_e(h_T^{-1/2}\|\sigma_h - \sigma_H\|_{T_1\cup T_2} + h_T^{1/2}\|\delta_h(\sigma_h - \sigma_H)\|_{T_1\cup T_2}), \tag{5.9}$$

where h_T can be either h_{T_1} or h_{T_2} for the same reasons as mentioned above. Applying (5.7) we have

$$h_T^{1/2}\|[[\text{tr}\star(\sigma_h - \sigma_H)]]\|_e \leq C\|\sigma_h - \sigma_H\|_{T_1\cup T_2}. \tag{5.10}$$

Squaring and summing (5.7) and (5.10) for every element will complete the proof. The edges not on the boundary of Ω will be included twice in the summation, and the overlap of the $C\|\sigma_h - \sigma_H\|_{T_1\cup T_2}$ terms can be bounded by a multiple depending on n. □

Theorem 5.4. (Continuity of the Error Estimator) *Given $f \in L^2\Lambda^n(\Omega)$ and nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let $\sigma_h = \mathcal{L}_h^{-1}f_{\mathfrak{B}_h}$ and $\sigma_H = \mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$. Then we have:*

$$\beta(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h)) \leq \|\sigma_h - \sigma_H\|^2 + \text{osc}^2(f_h, \mathcal{T}_H) \tag{5.11}$$

Proof. Applying the triangle inequality to (5.2) gives

$$\begin{aligned}\|\sigma_h - \sigma_H\|^2 \geq C\Big(&\sum_{T\in\mathcal{T}_h}(h_T\|[[\text{tr } \star(\sigma_h)]]\|_{\partial T}^2 + h_T^2\|\delta(\sigma_h)\|_T^2) \\ &- \sum_{T\in\mathcal{T}_h}(h_T\|[[\text{tr } \star(\sigma_H)]]\|_{\partial T}^2 + h_T^2\|\delta(\sigma_H)\|_T^2)\Big).\end{aligned}$$

In terms of the error indicator this can be written

$$\begin{aligned}\|\sigma_h - \sigma_H\|^2 \geq C\big(&\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h) \\ &- \sum_{T\in\mathcal{T}_h} h_T^2\|d(\sigma - \sigma_h)\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|d(\sigma - \sigma_H)\|_T^2\big).\end{aligned}$$

An additional application of the triangle inequality yields

$$\|\sigma_h - \sigma_H\|^2 \geq C(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h) - \sum_{T\in\mathcal{T}_h} h_T^2\|d(\sigma_h - \sigma_H)\|_T^2)).$$

Since $f_h, f_H \in L^2\Lambda^k(\Omega)$ globally, using the summation on the coarser mesh completes the proof

$$\|\sigma_h - \sigma_H\|^2 + \sum_{T\in\mathcal{T}_H} h_T^2\|d(\sigma_h - \sigma_H)\|_T^2 \geq C(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h)).$$

□

5.2. **Continuous and Discrete Upper Bounds.** The following proofs have a similar structure to the continuous and discrete upper bounds proved in [2, 11]. A key element of the proof will be comparisons between the discrete solution $\sigma_H = \mathcal{L}_H^{-1} f$ and the solution to the intermediate problem, $\tilde{\sigma} = \mathcal{L}^{-1} f_H$. We begin by looking the orthogonal decomposition of $\tilde{\sigma} - \sigma_H$,

$$\tilde{\sigma} - \sigma_H = (\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H) - P_{\mathfrak{B}^{k-1}}\sigma_H - P_{\mathfrak{H}^{k-1}}\sigma_H$$

which allows the norm to be rewritten

$$\|\tilde{\sigma} - \sigma_H\|^2 = \|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 + \|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2 + \|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2.$$

Lemmas 5.5, 5.6 and 5.7 will each bound a portion of this orthogonal decomposition. Then Theorem 5.8 will combine these results in proving the desired error bound.

Lemma 5.5. *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\tilde{\sigma} = \mathcal{L}^{-1} f_{\mathfrak{B}_H}$ and $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then*

$$\|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 = 0. \tag{5.12}$$

Proof. Since we are only dealing with $\mathfrak{Z}^\perp$, we have

$$\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H = \delta v,\ v \in H\Lambda^k(\Omega).$$

Thus,

$$\|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 = \langle \tilde{\sigma} - \sigma_H, \delta v\rangle = \langle d(\tilde{\sigma} - \sigma_H), v\rangle.$$

In the case of $\mathfrak{B}$ problems the harmonics are void and

$$\langle d(\tilde{\sigma} - \sigma_H), v\rangle = \langle f_{\mathfrak{B}_H} - f_{\mathfrak{B}_H}, v\rangle = 0.$$

□

The next lemma uses the quasi-interpolant Π_H described in [13], and also applies integration by parts in the same standard fashion that [13] use in bounding error measured in the natural norm, $\|u - u_h\|_{H\Lambda^k(\Omega)} + \|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} + \|p - p_h\|$. In [13], coercivity of the bilinear-form is used to separate components of the error, whereas here we simply analyze the orthogonal decomposition of $\sigma - \sigma_H$. In [13], they employ Galerkin orthogonality implied by taking the difference between the continuous and discrete problems, allowing them to make use of Π_h. Here we are able to introduce the quasi-interpolant by simply using the fact that $\sigma_H \perp \mathfrak{B}_H^{k-1}$.

Lemma 5.6. *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then*

$$\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2 \leq C\eta^2(\sigma_H, \mathcal{T}_H). \tag{5.13}$$

Proof.

$$\|P_{\mathfrak{B}^{k-1}}\sigma_H\| = \langle \sigma_H, \frac{P_{\mathfrak{B}^{k-1}}\sigma_H}{\|P_{\mathfrak{B}^{k-1}}\sigma_H\|}\rangle = \langle -\sigma_H, d\phi\rangle,\ \ \phi \in \mathfrak{Z}^{\perp k-2}$$

By the the Poincaré inequality $\|\phi\|$ can be bounded from above by a constant. ϕ can then be replaced with φ satisfying the properties of Lemma 2.5, and noting $\sigma_H \perp \mathfrak{B}_H^{k-1}$,

$$= \langle -\sigma_H, d(\varphi - \Pi_H\varphi)\rangle.$$

The problem is now reduced to a case handled in [13], when they bound a portion of their η_{-1} estimator. We follow their ideas to complete to proof. Applying the integration by parts formula we have

$$= \sum_{T\in T_H} [\int_{\partial T} (\text{tr} \star \sigma_H \wedge \text{tr}\, (\varphi - \Pi_H\varphi)) + \langle \delta\sigma_H, \varphi - \Pi_H\varphi\rangle_T]$$

Noting tr$(\varphi - \Pi_H\varphi)$ is single-valued on the element boundaries, this can be reduced to

$$\le C \sum_{T\in T_H} \|\text{tr}(\varphi - \Pi_H\varphi)\|_{\partial T}\|[[\text{tr}\ \star\sigma_H]]\|_{\partial T} + \|\varphi - \Pi_H\varphi\|_T\|\delta\sigma_H\|_T$$

$$\le C \sum_{T\in T_H} (h_T^{1/2}\|[[\text{tr}\ \star\sigma_H]]\|_{\partial T} + h_T\|\delta\sigma_H\|_T)(h_T^{-1/2}\|\text{tr}(\varphi - \Pi_H\varphi)\|_{\partial T} + h_T\|\varphi - \Pi_H\varphi\|_T)$$

Which written using the definition of the error indicator simplifies to

$$\le C\eta(\sigma_H, \mathcal{T}_H) \sum_{T\in T_H} (h_T\|\text{tr}(\varphi - \Pi_H\varphi)\|^2_{\partial T} + h_T^2\|\varphi - \Pi_H\varphi\|^2_T)$$

The proof is then complete by applying the bounds from Lemma 2.5, and squaring both sides.

□

Lemma 5.7. *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\tilde\sigma = \mathcal{L}^{-1}f_{\mathfrak{B}_H}$ and $\sigma_H = \mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$. Then*

$$\|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2 \le C\|\tilde\sigma - \sigma_H\|^2, \qquad C < 1. \tag{5.14}$$

Proof. Since $\tilde\sigma \perp \mathfrak{Z}^{k-1}$ and $\sigma_H \perp \mathfrak{Z}_H^{k-1}$, we follow [13] and write

$$\begin{aligned}\|P_{\mathfrak{H}^{k-1}}\sigma_H\| &= \sup\nolimits_{v\in\mathfrak{H},\|v\|=1}(\sigma_H - \tilde\sigma, v - P_{\mathfrak{H}_H}v)\\ &\le \sup\nolimits_{v\in\mathfrak{H},\|v\|=1}(\|v - P_{\mathfrak{H}_H}v\|)\|\sigma_H - \tilde\sigma\|\\ &= \delta(\mathfrak{H}, \mathfrak{H}_H)\|\sigma_H - \tilde\sigma\|.\end{aligned}$$

Applying Theorem 2.7, and then squaring both sides we get

$$\|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2 \le C\|\tilde\sigma - \sigma_H\|^2, \quad C < 1.$$

□

Now we have the tools to prove the continuous upper bound for the $\mathfrak{B}$ problems.

Theorem 5.8. (Continuous Upper-Bound) *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\tilde\sigma = \mathcal{L}^{-1}f_{\mathfrak{B}_H}$ and $\sigma_H = \mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$. Then*

$$\|\sigma - \sigma_H\|^2 \le C_1\eta^2(\sigma_H, \mathcal{T}_H). \tag{5.15}$$

Proof. Since these are $\mathfrak{B}$ problems, $p, \tilde p = 0$, and

$$\begin{aligned}\langle \sigma - \tilde\sigma, \tilde\sigma - \sigma_H\rangle &= \langle u - \tilde u, d(\tilde\sigma - \sigma_H)\rangle\\ &= \langle u - \tilde u, f_{\mathfrak{B}_H} - f_{\mathfrak{B}_H}\rangle\\ &= 0.\end{aligned}$$

Thus, by applying (5.14), (5.12), (5.13) and Theorem 4.1,

$$\begin{aligned}\|\sigma-\sigma_H\|^2 &= \|\tilde\sigma-\sigma_H\|^2+\|\sigma-\tilde\sigma\|^2\\ &\le C(\|(\tilde\sigma-P_{\mathfrak{Z}^\perp}\sigma_H)\|^2+\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2)+\|\sigma-\tilde\sigma\|^2\\ &\le C(\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2)+\|\sigma-\tilde\sigma\|^2\\ &\le C_1(\eta^2(\sigma_H,\mathcal{T}_H))+C_0\mathrm{osc}^2(f,\mathcal{T}_H)\\ &\le C\eta^2(\sigma_H,\mathcal{T}_H).\end{aligned}$$

□

Theorem 5.9. (Discrete Upper-Bound) *Given $f\in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}$ and nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let $\sigma_h=\mathcal{L}_h^{-1}f_{\mathfrak{B}_h}$ and $\sigma_H=\mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$. Then*

$$\|\sigma_h-\sigma_H\|^2\le C_1\eta^2(\sigma_H,T_H). \tag{5.16}$$

Proof. The proof requires the same ingredients needed to prove the Continuous Upper bound.

The same intermediate steps are taken by performing analysis on the W_h^{k-1} orthogonal decomposition of $\tilde\sigma_h-\sigma_H$.

$$\tilde\sigma_h-\sigma_H=(\tilde\sigma_h-P_{\mathfrak{Z}_h^\perp}\sigma_H)-P_{\mathfrak{B}_h^{k-1}}\sigma_H-P_{\mathfrak{H}_h^{k-1}}\sigma_H.$$

The discrete version of Lemma 5.5 uses δ_h rather than δ, but is otherwise identical. The discrete version of Lemma 5.6 is identical. The discrete version of Lemma 5.7 follows the same structure but makes use of Corollary 2.8. The final step in the proof uses the discrete stability result, Theorem 4.4.

□

6. Convergence of AMFEM

After presenting the adaptive algorithm, the remainder of this section proves convergence and then optimality. The results in this section follow ideas already in the literature [33, 22, 23, 15, 11], with Theorem 6.3 building on these ideas by proving reduction in a quasi-error using relationships between data oscillation and reduction of a second type of quasi-error. The algorithm and analysis to follow only handle the case $k=n$. In presenting our algorithm we replace h with iteration counter k.

Algorithm:[$\mathcal{T}_N,\sigma_N$] = AMFEM($\mathcal{T}_0,f,\epsilon,\theta$): Given a initial shape-regular triangulation $\mathcal{T}_0$ and marking parameter θ, set $k=0$ and iterate the following steps until a desired decrease in the error-estimator is achieved:

$$\begin{aligned}&(1)(u_k,\sigma_k,p_k)=SOLVE(\mathcal{T}_k)\\ &(2)\{\eta_T\}=ESTIMATE(\sigma_k,\mathcal{T}_k)\\ &(3)\mathcal{M}_k=MARK(\{\eta_T\},\mathcal{T}_k,\theta)\\ &(4)\mathcal{T}_{k+1}=REFINE(\mathcal{T}_k,\mathcal{M}_k)\end{aligned}$$

6.1. **Convergence of AMFEM.** The following notation will be used in the proofs and discussion of this section:

$$e_k = \|\sigma - \sigma_k\|^2, \qquad E_k = \|\sigma_{k+1} - \sigma_k\|^2, \qquad \eta_k = \eta^2(\sigma_k, \mathcal{T}_k),$$
$$o_k = \mathrm{osc}^2(f, \mathcal{T}_k), \qquad \hat{o}_k = \mathrm{osc}^2(f_{k+1}, \mathcal{T}_k),$$

where $f_k = P_k f = P_{\mathfrak{B}_k} f$ since $k = n$.

Lemma 6.1.

$$\beta\eta_{k+1} \le \beta(1 - \lambda\theta)\eta_k + E_k + \hat{o}_k. \tag{6.1}$$

Proof. This follows from continuity of the error estimator (5.11), and properties of the marking strategy, i.e. reduction of the summation on a finer mesh due to smaller element sizes on refined elements. The proof can be found in [11]. $\lambda < 1$ is a constant dependent on the dimensionality of the problem. □

For convenience, we recall the quasi-orthogonality (3.3) the continuous upper-bound (5.15) equations,

$$(1 - \delta)e_{k+1} \le e_k - E_k + C_0\hat{o}_k, \text{ for any } \delta > 0,$$
$$e_k \le C_1\eta_k.$$

With these three ingredients, basic algebra leads to the following result,

Theorem 6.2. *When*

$$0 < \delta < \min\{\frac{\beta}{2C_1}\theta, 1\}, \tag{6.2}$$

there exists $\alpha \in$ (0,1) *and* C_δ *such that*

$$(1 - \delta)e_{k+1} + \beta\eta_{k+1} \le \alpha[(1 - \delta)e_k + \beta\eta_k] + C_\delta\hat{o}_k. \tag{6.3}$$

Proof. Follows the same steps as [11]. □

With the above result we next prove convergence.

Theorem 6.3. (Termination in Finite Steps) *Let σ_k be the solution obtained in the kth loop in the algorithm AMFEM, then for any $0 < \delta < \min\{\frac{\beta}{2C_1}\theta, 1\}$, there exists positive constants C_δ and 0< $\gamma_\delta < 1$ depending only on given data and the initial grid such that,*

$$(1 - \delta)\|\sigma - \sigma_k\|^2 + \beta\eta^2(\sigma_k, \mathcal{T}_k) + \zeta\mathrm{osc}^2(f, \mathcal{T}_k) \le C_q\gamma_\delta^k,$$

and the algorithm will terminate in finite steps.

Proof. The following proof will be broken into two cases, depending on the relative size of $\hat{o}_k$. For ease of reading, let $q_k = (1 - \delta)\|\sigma - \sigma_k\|^2 + \beta\eta^2(\sigma_k, \mathcal{T}_k)$.

Case 1. Suppose the case $C_\delta\hat{o}_k \le (\frac{1-\alpha}{2})q_k$. Thus for an arbitrary positive constant C, (6.3) yields

$$q_{k+1} + Co_{k+1} \le (\alpha + \frac{1 - \alpha}{2})q_k + Co_k.$$

Since $\beta o_k \le q_k$,

$$q_{k+1} + Co_{k+1} \le (\hat{\alpha} + \frac{1 - \hat{\alpha}}{2})q_k + \frac{C - \frac{\beta(1-\hat{\alpha})}{2}}{C}Co_k, \tag{6.4}$$

where

$$\hat{\alpha} = (\alpha + \frac{1-\alpha}{2}) < 1.$$

Case 2. Suppose the case $C_\delta \hat{o}_k \geq (\frac{1-\alpha}{2}) q_k$. We then have,

$$\begin{aligned} o_{k+1} &\leq \kappa o_k, \kappa < 1, \\ \hat{o}_k &\leq o_k. \end{aligned}$$

This implies

$$o_{k+1} \leq (\kappa + \frac{1-\kappa}{2}) o_k - \frac{1-\kappa}{2} \hat{o}_k.$$

Combined with (6.3) we have

$$q_{k+1} + \frac{2C_\delta}{1-\kappa} o_{k+1} \leq \alpha q_k + \hat{\kappa} \frac{2C_\delta}{1-\kappa} o_k, \tag{6.5}$$

where $\hat{\kappa} = (\kappa + \frac{1-\kappa}{2}) < 1$. The proof is completed by taking $\frac{2C_\delta}{1-\kappa}$ for the constant in (6.4), and then combining with (6.5). The rate of decay will be determined by

$$\gamma_\delta = \max\left\{\hat{\kappa}, \frac{C - \frac{\beta(1-\hat{\alpha})}{2}}{C}, \hat{\alpha} + \frac{1-\hat{\alpha}}{2}\right\} < 1. \tag{6.6}$$

□

The methods used above to prove convergence have many similarities to prior work. Our treatment of oscillation, however, uses properties of $\hat{o}_k$ that create distinct implementation and efficiency improvements. To clarify this point, next we compare our convergence proof with two from the literature, [10, 11]. In order to make the differences clear, we focus on the basic properties of the three equations that are at the core of the convergence analysis.

Convergence is essentially proved by manipulating the equations for quasi-orthogonality, continuity of the error estimator and the upper-bound. For that reason they will be the focus of our discussion, and for ease of comparison, we present our results together in a simplified form.

$$\begin{aligned} (1-\delta)e_{k+1} &\leq e_k - E_k + \tfrac{C_0}{\delta}\hat{o}_k, &&\text{for any } \delta > 0, \\ \beta\eta_{k+1} &\leq \beta\lambda\eta_k + E_k + \hat{o}_k, && \lambda < 1, \\ e_k &\leq C_1\eta_k. \end{aligned} \tag{6.7}$$

In [10], an orthogonality result, $e_{k+1} = e_k - E_k$, is possible since they are not working with a mixed merthod. In addition, a similar estimator continuity result is proved without the need for the $\hat{o}_k$ term. Since oscillation is not present, convergence is proved without the additional analysis used in the proof of Theorem 6.3.

For the purpose of comparison, we now present a simplified version of the equivalent equations from [11],

$$\begin{aligned} (1-\delta)e_{k+1} &\leq e_k - E_k + \tfrac{C_0}{\delta} o_k, &&\text{for any } \delta > 0, \\ \beta\eta_{k+1} &\leq \beta\lambda\eta_k + E_k, && \lambda < 1, \\ e_k &\leq C_1\eta_k + C_0 o_k. \end{aligned} \tag{6.8}$$

In [11], oscillation is not included in the error indicator and therefore is needed in the upper bound. Once o_k is used for the upper-bound, it is used out of simplicity in the quasi-orthogonality result as $o_k \geq \hat{o}_k$. The issue with including o_k versus $\hat{o}_k$ is that o_k can be significant in steps where oscillation is not reduced. Whereas the value of $\hat{o}_k$ indicates oscillation reduction and thus reduces the impact of data oscillation on remaining iterations. In order to manage this situation, the algorithm in [11] marks separately for η and oscillation. This is a disadvantage from an implementation point of view, and is also inefficient in cases when η and oscillation are different orders of magnitude.

6.2. **Optimality of AMFEM.** Once Theorem 4.1, Theorem 2.10, and the Lemma 6.1 are established, optimality can be proved independent of dimension following the proof of Theorem 5.3 in [33].

Theorem 6.4. (Optimality) *For any $f \in L^2\Lambda^n(\Omega)$, shape regular $\mathcal{T}_0$ and $\epsilon > 0$, let $\sigma = \mathcal{L}^{-1}f$ and [$\sigma_N, \mathcal{T}_N$] =* AMFEM*($\mathcal{T}_H, f_H, \epsilon/2$,0, 1). Where [$\mathcal{T}_H, f_H$] =* APPROX*(f, $\mathcal{T}_0, \epsilon/2$). If $\sigma \in \mathcal{A}^s$ and $f \in \mathcal{A}^s_o$, then*

$$\|\sigma - \sigma_N\| \leq C(\|\sigma\|_{\mathcal{A}^s} + \|f\|_{\mathcal{A}^s_o})(\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s}. \tag{6.9}$$

Proof. Follows directly from [11]. □

7. Conclusion and Future Work

In this paper, we have focused on the error $\|\sigma - \sigma_h\|$ for the Hodge Laplacian in the specific case $k = n$. Next, we outline how this work relates to further generalizations.

The $\mathfrak{B}$ problems are of particular interest, as a convergent algorithm for $\|\sigma - \sigma_h\|$ in $\mathfrak{B}$ problems would inherently retain the same convergence properties when applied to generic Hodge Laplace problems since $P_{\mathfrak{B}^k_h} f = P_{\mathfrak{B}^k_h}(P^k_{\mathfrak{B}} f)$ and $d\sigma_h = P_{\mathfrak{B}^k_h} f$. With the exception of the stability results, the methods used to prove convergence only relied on properties inherent to the more general $\mathfrak{B}$-problems. The issue with stability is that we cannot assume $H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \subset H^1\Lambda^k(\Omega)$ [4]. However, $\mathring{H}^*\Lambda^n(\Omega) \subset H^1\Lambda^n(\Omega)$, thus we have the desired interpolation properties of $u - \tilde{u}$ in the case $k = n$.

Adaptivity focusing on the natural norm, $\|u - u_h\|_{H\Lambda^k(\Omega)} + \|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} + \|p - p_h\|$, is another direction of interest. Error indicators related to this norm are analyzed in [13], yet difficulties still arise in attempt to gain full generality (see [13] for a detailed discussion). Additionally, a quasi-orthogonality result would be useful. The quasi-orthogonality proved here used analysis tailored specifically to the norm of interest. A generalized quasi-orthogonality and convergence result would likely require a different line of of reasoning and a specific analysis regarding a refinement strategy takes into account the approximation of the harmonic forms.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, LA JOLLA CA 92093
E-mail address: mholst@math.ucsd.edu, amihalik@math.ucsd.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, CAL POLY POMONA, POMONA CA 91768
E-mail address: rsszypowski@csupomona.edu