FINITE ELEMENT EXTERIOR CALCULUS FOR EVOLUTION PROBLEMS

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ABSTRACT. Arnold, Falk, and Winther [*Bull. Amer. Math. Soc.* **47** (2010), 281–354] recently showed that mixed variational problems, and their numerical approximation by mixed methods, could be most completely understood using the ideas and tools of *Hilbert complexes*. This led to the development of the Finite Element Exterior Calculus (FEEC) for a large class of linear elliptic problems. More recently, Holst and Stern [arXiv:1005.4455,arXiv:1010.6127] extended the FEEC framework to semi-linear problems, and to problems containing *variational crimes*, allowing for the analysis and numerical approximation of linear and nonlinear geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension, generalizing surface finite element approximation theory. In this article, we develop another distinct extension to the FEEC, namely to parabolic and hyperbolic evolution systems, allowing for the treatment of geometric and other evolution problems. Our approach is to combine the recent work on the FEEC for elliptic problems with a classical approach to solving evolution problems via semi-discrete finite element methods, by viewing solutions to the evolution problem as lying in time-parameterized Hilbert spaces (or *Bochner* spaces). Building on classical approaches by Thomée for parabolic problems and Geveci for hyperbolic problems, we establish *a priori* error estimates for Galerkin FEM approximation in the natural parametrized Hilbert space norms. In particular, we recover the results of Thomée and Geveci for two-dimensional domains and lowest-order mixed methods as special cases, effectively extending their results to arbitrary spatial dimension and to an entire family of mixed methods. We also show how the Holst and Stern framework allows for extensions of these results to certain semi-linear evolution problems.

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1. INTRODUCTION

More than two decades of research on linear mixed variational problems, and their numerical approximation by mixed methods, recently culminated in the seminal work of Arnold, Falk, and Winther Arnold, Falk, and Winther [3]. They showed that these problems could be most completely understood using the ideas and tools of *Hilbert complexes*, leading to the development of the Finite Element Exterior Calculus (FEEC) for elliptic problems. In two related articles [17, 18], Holst and Stern extended the Arnold–Falk–Winther framework to include *variational crimes*, allowing for the analysis and numerical approximation of linear and nonlinear geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension, generalizing the existing surface finite element approximation theory in several directions. In the current article, we extend the FEEC in another direction, namely to parabolic and hyperbolic evolution systems. Our approach is to combine the recent work on the FEEC for elliptic problems with a classical approach to solving evolution problems using semi-discrete finite element methods, by viewing solutions to the evolution problem as lying in time-parameterized Banach (or *Bochner*) spaces. Building on classical approaches by Thomée for parabolic problems and Geveci for hyperbolic problems, we establish *a priori* error estimates for Galerkin FEM approximation in the natural natural parametrized Hilbert space norms. In particular, we recover the results of Thomée and Geveci for two-dimensional domains and the lowest-order mixed method as a special case, effectively extending their results to arbitrary spatial dimension and to an entire family of mixed methods. We also show how the Holst and Stern framework allows for extensions of these results to certain semi-linear evolution problems.

To understand why the finite element exterior calculus (FEEC) has emerged in a natural way to become a major mathematical tool in the development of numerical methods for PDE, we recall one of the many examples presented at length in [3]. Consider the vector Laplacian:

$$-\Delta u = -\text{ grad } \text{div}\, u + \text{ curl } \text{ curl}\, u,$$

and a natural variational formulation: Find $u \in H(\text{curl};\Omega) \cap H_0(\text{div};\Omega)$ s.t.

$$\int_\Omega [(\nabla\cdot u)(\nabla\cdot v)+(\nabla\times u)\cdot(\nabla\times v)]\, dx = \int_\Omega f\cdot v\, dx,\ \forall v \in H(\text{curl};\Omega)\cap H_0(\text{div};\Omega). \quad (1.1)$$

A *mixed* formulation is a natural alternative: Find $(\sigma, u) \in H^1(\Omega) \times H(\text{curl};\Omega)$ s.t.

$$\int_\Omega (\sigma\tau - u\cdot\nabla\tau)\, dx = 0, \quad \forall \tau \in H^1(\Omega), \quad (1.2)$$

$$\int_\Omega [\nabla\sigma\cdot v + (\nabla\times u)\cdot(\nabla\times v)]\, dx = \int_\Omega f\cdot v\, dx, \quad \forall v \in H(\text{curl};\Omega). \quad (1.3)$$

Using the standard finite element approach based on the non-mixed formulation (1.1) can yield incorrect results if the domain has certain geometric features (e.g. domains with corners) or topological features (e.g. non-simply connected domains). A standard finite element approach based on the mixed formulation (1.2)-(1.3), on the other hand, suffers neither of these difficulties and typically works extremely well.

The explanation for why one approach fails and the other succeeds lies in the fundamental mathematical structures underlying the finite element method. The error due to geometric features can be traced to a problem of *inconsistency*, i.e. that the discrete approximation of the operators and data do not approximate the continuous problem correctly as the mesh size is taken to zero. The error due to topological features can be traced to the presence of non-zero *harmonic vector fields* on the domain, i.e. vector fields which are both curl-free and divergence-free. The mixed formulation turns out to be both consistent and respectful of non-zero harmonic vector fields while the standard formulation does not. A natural question is then: What is an appropriate mathematical framework for understanding these problems abstractly so that a methodical construction of "good" finite element methods can be carried out for these and similar PDE problems?

The answer turns out to be *Hilbert Complexes*. Hilbert complexes were originally studied in [8] as a way to generalize certain properties of elliptic complexes, particularly the Hodge decomposition and other aspects of Hodge theory. A *Hilbert complex* (W, d) consists of a sequence of Hilbert spaces W^k, along with closed, densely-defined linear maps $\mathrm{d}^k \colon V^k \subset W^k \to V^{k+1} \subset W^{k+1}$, possibly unbounded, such that $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$ for each k.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{\mathrm{d}^{k-1}} V^k \xrightarrow{\mathrm{d}^k} V^{k+1} \longrightarrow \cdots$$

This Hilbert complex is said to be *bounded* if d^k is a bounded linear map from W^k to W^{k+1} for each k, i.e., (W, d) is a cochain complex in the category of Hilbert spaces. It is said to be *closed* if the image $\mathrm{d}^k V^k$ is closed in W^{k+1} for each k. It was shown in [2, 3] that Hilbert complexes are also a convenient abstract setting for mixed variational problems and their numerical approximation by mixed finite element methods, providing the foundation of a framework called *finite element exterior calculus*. This line of research is the culmination of several decades of work on mixed finite element methods and computational electromagnetics [6, 15, 22, 23]. The most important example of a Hilbert complex for our purposes of the FEEC arises from the de Rham complex of smooth differential forms on a domain or manifold.

The main developments in FEEC to date have been for linear (and now semi-linear) elliptic problems such as Poisson's equation

$$-\Delta u = f.$$

Our goal here is expand the scope of this analysis to include parabolic linear (and semi-linear) equations such as the heat equation,

$$(\partial_t - \Delta) u = f,$$

and hyperbolic equations such as the wave equation,

$$(\partial_{tt} - \Delta) u = f.$$

The exterior calculus framework treats Δ as $(d + \delta)^2$, where d is the exterior derivative operator and δ its adjoint. The incorporation of the time derivative operation ∂_t into this framework, however, has not been previously considered. To remedy this, we develop the most natural extension of FEEC theory to evolution problems: a generalization of the semi-discrete method often called the 'method of lines.' This approach involves the discretization of the spatial part of the differential operator, leaving the time variable continuous. It can be viewed as introducing a time parameter into the discrete (Hilbert) spaces that have been developed for elliptic problems. These parametrized Hilbert spaces are particular kinds of *Bochner spaces* and we will review work by Renardy and Rogers [26] that makes obvious the well-posedness of the problems we consider. Moreover, the accompanying Bochner space norms, when coupled with FEEC notation for Hilbert complexes, provide a clear and consistent notation for bounding errors in mixed methods accumulated over a finite time interval.

We note that there is another approach to solving evolution problems with finite elements, namely using a complete discretization of space-time. This tactic allows for the dynamical change of the underlying discrete approximation spaces in both space and time. Such an approach gives rise to space-time adaptivity, and is potentially the most flexible and powerful approach to the numerical treatment of parabolic and hyperbolic evolution problems. This approach, which we will consider in a second article, is most naturally formulated using geometric calculus, a well-studied mathematical structure for time-dependent problems. In the current article, we focus on extending FEEC to semi-discrete methods using Bochner norm estimates for the method-of-lines approach.

Finally, we note that the work presented here was developed simultaneously and independently from a related project by Arnold and Chen [1] for generalized Hodge-Laplacian style linear parabolic problems. Our focus in this work is to extend the *scalar* Hodge-Laplacian to both linear and semi-linear parabolic problems as well as linear hyperbolic problems, as this touches the existing literature on semi-discrete methods in the broadest fashion. The pairing of these two results will lead to further insight in a variety of research directions.

Summary of the paper. The remainder of the paper is structured as follows. In Section 2, we review the classical semi-discrete mixed finite element method error estimates for parabolic problems (due to Thomée [30] and others) and for hyperbolic problems (due to Geveci [13] and others). In Section 3, we give a very brief overview the Finite Element Exterior Calculus and recall some relevant results. In Section 4, we formulate abstract parabolic and hyperbolic problems in Bochner spaces and state some standard results on the existence and uniqueness of strong and weak solutions. In Section 5, we combine the classical approach to semi-discrete methods with modern FEEC theory to establish some basic *a priori* error estimates for Galerkin mixed finite element methods for parabolic problems. The main result is Theorem 5.2, which exploits the FEEC framework to obtain a classification of spatial finite element spaces that give optimal order convergence rates in Bochner norms. In Section 6, we carry out a similar analysis for hyperbolic problems, resulting in the error estimate given in Theorem 6.2, a simultaneous sharpening of the result by Geveci for problems in two dimensional domains and a generalization to problems on n dimensional domains. Our results recover the estimates of Thomée and Geveci for two-dimensional domains and the lowest-order mixed method as a special case, effectively extending their results to arbitrary spatial dimension and to an entire family of mixed methods. In Section 7, we employ the results of Holst and Stern [18] to extend our parabolic estimates to a class of semi-linear evolution PDE. Finally, in Section 8, we draw conclusions and make remarks on future directions.

2. Semi-Discrete FEM Error Estimates for Evolution Problems

We begin by reviewing semi-discrete finite element methods and their *a priori* error estimates for parabolic and hyperbolic PDE systems. We focus in each case on a relatively simple, well-studied system of interest to modeling communities, namely, the heat equation (parabolic) and the wave equation (hyperbolic). The **heat equation** is: find $u(x,t)$ such that

$$\begin{aligned} u_t - \Delta u = f \quad & \text{in } \Omega, \quad \text{for } t>0 \\ u = 0 \quad & \text{on } \partial\Omega, \quad \text{for } t>0 \quad \text{with } u(\cdot,0)=g \text{ in } \Omega. \end{aligned} \tag{2.1}$$

We review the approach to Galerkin methods for this problem as presented in Thomée [30] for domains $\Omega \subset \mathbb{R}^2$. His approach is based on work with Johnson [19] and builds upon prior analysis of elliptic projection [7]. A similar approach, restricted to $\Omega \subset \mathbb{R}^2$, was carried out by Garcia in [12]. Similar work Let $\sigma = \nabla u$ and define the mixed, weak form problem: Find $(u,\sigma) \in L^2 \times H(\text{div})$ such that

$$\begin{aligned} (u_t,\phi) - (\text{div}\sigma,\phi) &= (f,\phi), \quad \forall\, \phi \in L^2, \qquad t>0 \\ (\sigma,\omega) + (u,\text{div}\omega) &= 0, \qquad \forall\, \omega \in H(\text{div}), \quad t>0, \quad u(0)=g. \end{aligned} \tag{2.2}$$

The semi-discrete problem is then to find $(u_h,\sigma_h) \in S_h \times H_h \subset L^2 \times H(\text{div})$ such that

$$\begin{aligned} (u_{h,t},\phi_h) - (\text{div}\sigma_h,\phi_h) &= (f,\phi_h), \quad \forall\, \phi_h \in S_h, \quad t>0 \\ (\sigma_h,\omega_h) + (u_h,\text{div}\omega_h) &= 0, \qquad \forall\, \omega_h \in H_h, \quad t>0, \quad u_h(0)=g_h. \end{aligned} \tag{2.3}$$

where g_h is an approximation of g in S_h. With bases for S_h and H_h, the matrix form of the discrete problem is

$$\begin{aligned} AU_t - B\Sigma &= F, \\ B^TU + D\Sigma &= 0, \quad \text{for } t>0, \quad U(0) \text{ given}, \end{aligned}$$

where U and Σ are vectors corresponding to u_h and σ_h. It is easily seen that the matrices A and D are positive definite. Eliminating Σ, we have the system of ODEs

$$AU_t + BD^{-1}B^TU = F, \quad \text{for } t>0, \quad U(0) \text{ given},$$

which by standard results in ODE theory has a unique solution.

Thomée uses discontinuous linear elements for S_h and piecewise quadratic elements for H_h. He defines the solution operator $T_h : L^2 \to S_h$ given by $T_hf = u_h$ for the corresponding elliptic problem and sets

$$g_h := R_hg := -T_h\Delta g.$$

For $g_h = R_hg$ and $t \geq 0$, Thomée derives the estimates

$$||u_h(t) - u(t)||_{L^2} \leq ch^2 \left(||u(t)||_{H^2} + \int_0^t ||u_t||_{H^2}\, ds \right), \tag{2.4}$$

$$||\sigma_h(t) - \sigma(t)||_{L^2} \leq ch^2 \left(||u(t)||_{H^3} + \left(\int_0^t ||u_t||_{H^2}^2\, ds \right)^{1/2} \right). \tag{2.5}$$

Note that these estimates are for a fixed time value t and restricted to a particular choice of finite elements in 2D.

We now turn to the **wave equation**: find $u(x,t)$ such that

$$\begin{aligned} u_{tt} - \Delta u = f \quad &\text{in } \Omega, \quad \text{for } t>0, \\ u = 0 \quad &\text{on } \partial\Omega, \quad \text{for } t>0 \quad \text{with } u(\cdot,0) = u_0 \text{ in } \Omega, \\ & \qquad\qquad\qquad\qquad \text{and } u_t(\cdot,0) = u_1 \text{ in } \Omega \end{aligned} \tag{2.6}$$

There are two approaches to defining a mixed weak form of this problem. The first is very similar to the parabolic case: given f, u_0, and u_1, find (u,σ) such that

$$\begin{aligned} (u_{tt},\phi) - (\text{div}\,\sigma,\phi) &= (f,\phi), \quad \forall\, \phi \in L^2, \qquad t>0, \\ (\sigma,\omega) + (u,\text{div}\,\omega) &= 0, \qquad \forall\, \omega \in H(\text{div}), \quad t>0, \\ u(0) &= u_0, \\ u_t(0) &= u_1. \end{aligned} \tag{2.7}$$

It is difficult to derive estimates for the numerical approximation of (2.7) akin to those found in the parabolic case due to the second derivatives appearing in the formulation. Some attempts at estimates along these lines for $\Omega \subset \mathbb{R}^2$ have been given by Baker [4] and Cowsar, Dupont and Wheeler [10, 11].

For the purpose of extending the FEEC framework, we find the **'velocity-stress' formulation** of the problem and the results of Geveci [13] to be more useful. This formulation solves for $\mu := u_t$ instead of u: Given f, u_0, and u_1, find $(\mu, \sigma) \in L^2 \times H(\text{div})$ such that

$$\begin{aligned}
(\mu_t, \phi) - (\text{div}\,\sigma, \phi) &= (f, \phi), \quad \forall\, \phi \in L^2, \qquad\quad t > 0, \\
(\sigma_t, \omega) + (\mu, \text{div}\,\omega) &= 0, \qquad\quad\ \ \forall\, \omega \in H(\text{div}), \quad t > 0, \\
\mu(0) &= u_1, \\
\sigma(0) &= \nabla u_0.
\end{aligned} \tag{2.8}$$

The semi-discrete problem is then to find $(\mu_h, \sigma_h) \in S_h \times H_h \subset L^2 \times H(\text{div})$ such that

$$\begin{aligned}
(\mu_{h,t}, \phi_h) - (\text{div}\,\sigma_h, \phi_h) &= (f, \phi_h), \quad \forall\, \phi_h \in S_h, \quad t > 0, \\
(\sigma_{h,t}, \omega_h) + (\mu_h, \text{div}\,\omega_h) &= 0, \qquad\quad\ \ \forall\, \omega_h \in H_h, \quad t > 0, \\
\mu_h(0) &= u_{1,h}, \\
\sigma_h(0) &= (\nabla u_0)_h,
\end{aligned} \tag{2.9}$$

where $u_{1,h}$ is an approximation of u_1 in S_h and $(\nabla u_0)_h$ is an approximation of ∇u_0. Again, bases for S_h and H_h reduce the discrete problem to a matrix formulation:

$$\begin{aligned}
AW_t - B\Sigma &= F, \\
B^T W + D\Sigma_t &= 0, \quad \text{for } t > 0, \quad W(0), \Sigma(0) \text{ given},
\end{aligned}$$

where W and Σ are vectors corresponding to μ_h and σ_h and A and D are symmetric, positive definite matrices. As Geveci [13, p. 248] explains, this can be reduced to a single iterative system of the form

$$(D + k^2 B^T A^{-1} B)\Sigma^{n+1} = G,$$

where k denotes the time step in an implicit Euler time-differencing scheme.

To derive an error estimate for the velocity-stress discretization, Geveci states the need for projection operators from $H(\text{div})$ to H_h and from L^2 to S_h satisfying certain approximation properties. He explains that such operators exists for a variety of finite element spaces in $\mathbb{R}^2$, e.g. the Raviart-Thomas spaces [25], allowing the following result. For $1 \leq s \leq r$ with $r \geq 2$,

$$\begin{aligned}
&||\mu_h(t) - \mu(t)||_{L^2} + ||\sigma_h(t) - \sigma(t)||_{L^2} \leq c\left(||u_1 - u_{1,h}||_{L^2} + ||\nabla u_0 - (\nabla u_0)_h||_{L^2}\right) + \\
&\qquad\qquad + ch^s\left(||u_1||_s + ||\nabla u_0||_s + \int_0^t (||\mu_t(\tau)||_s + ||\sigma_t(\tau)||_s\, d\tau\right).
\end{aligned} \tag{2.10}$$

Like estimates (2.4) and (2.5) for the parabolic problem, (2.10) says that the approximation error can be controlled in L^2 norm at any time t by the H^s norm of the initial conditions plus the accumulated norm of the variables up to time t. It is these types of estimates that the FEEC framework can refine, simplify, and generalize to arbitrary spatial dimension n.

3. The Finite Element Exterior Calculus

The finite element exterior calculus (FEEC) provides an elegant mathematical framework for deriving error estimates for a large class of elliptic PDE. We now give a brief overview of the notation and certain main results from FEEC which are relevant to this paper. We refer the reader to the seminal papers of Arnold, Falk, and Winther [2, 3] for additional explanation.

Let Ω be a bounded n-manifold embedded in $\mathbb{R}^n$ and assume Ω has a piecewise smooth, Lipschitz boundary. The space of L^2-bounded continuous differential k-forms on Ω is given by

$$L^2\Lambda^k(\Omega) := \left\{ \sum_I a_I dx_I \in \Lambda^k(\Omega) \ : \ a_I \in L^2(\Omega) \quad \forall\, I \right\},$$

where I ranges over all strictly increasing sequences of k indices chosen from $\{1,\ldots,n\}$. The exterior derivative operator $d_k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ acts on these spaces to form a Hilbert complex $(L^2\Lambda, d)$. The associated domain complex is the sequence of spaces $H\Lambda^k :=$ domain(d_k) $\subset L^2\Lambda^k(\Omega)$, commonly called the L^2 deRham complex:

$$0 \longrightarrow H\Lambda^0 \xrightarrow{d_0} H\Lambda^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} H\Lambda^n \longrightarrow 0.$$

The norm on each space is the graph norm associated to d, i.e.

$$(u,v)_{H^k(\Omega)} := (u,v)_{\Lambda^k(\Omega)} + (d_k u, d_k v)_{\Lambda^{k+1}(\Omega)}.$$

We note that in any dimension n, the beginning and end of the L^2 deRham complex can be understood in terms of traditional Sobolev spaces and differential operators:

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} \cdots \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0$$

A major conclusion of FEEC is that stable finite element methods for elliptic PDE must seek solutions in finite dimensional subspaces $\Lambda_h^k \subset H\Lambda^k$ that satisfy certain key approximation properties. First, the subspaces should form a subcomplex of the L^2 deRham complex, meaning $d\Lambda_h^k \subset \Lambda_h^{k+1}$. Second, Λ_h^k should have sufficient approximation that upper bounds on $\inf_{v\in\Lambda_h^k} ||u-v||_{H\Lambda^k}$ can be ensured for some or all $u \in H\Lambda^k$. Third, there must exist bounded cochain projections $\pi_h^k : H\Lambda^k \to \Lambda_h^k$ which are invariant on Λ_h^k, commute with the exterior derivative operators, and provide a bound $\left|\left|\pi_h^k v\right|\right|_{H\Lambda^k} \le c\,||v||_{H\Lambda^k}$ for all $v \in H\Lambda^k$.

In the context of the deRham complex, all these properties are shown to be provided for by two canonical classes of piecewise degree r polynomials associated to a simplicial mesh $\mathcal{T}$ of Ω. Let $\mathcal{P}_r$ denote polynomials in n variables of degree at most r and $\mathcal{H}_r \subset \mathcal{P}_r$ the subspace of homogeneous polynomials. The first class, denoted $\mathcal{P}_r\Lambda^k(\mathcal{T})$, consists of all k-forms with coefficients belonging to $\mathcal{P}_r$ on each n-simplex of $\mathcal{T}$. The second class, denoted $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$, interleaves with the first class, i.e.

$$\mathcal{P}_{r-1}\Lambda^k(\mathcal{T}) \subsetneq \mathcal{P}_r^-\Lambda^k(\mathcal{T}) \subsetneq \mathcal{P}_r\Lambda^k(\mathcal{T}).$$

To define $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$, first define X be the vector field on $\mathbb{R}^n$ such that $X(x)$ is the vector based at $x \in \mathbb{R}^n$ that points opposite to the origin with length $|x|$. Define $\mathcal{P}_r^-\Lambda^k(\mathcal{T}) := \mathcal{P}_r\Lambda^k \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$, a direct sum, where κ is defined by contraction with X. The map κ is called the *Koszul differential* and gives rise to the *Koszul complex*. This is elaborated upon in detail in the work of Arnold, Falk and Winther e.g. [3, p. 328].

For $n = 3$, we have the following correspondences between the FEEC notation of finite element spaces and traditional element spaces.

$$\begin{array}{rcl}\mathcal{P}_{r+1}\Lambda^2(\mathcal{T}) & = & \text{Nédélec 2nd-kind } H(\text{div}) \text{ elements of degree} \leq r+1 \text{ (see [23])} \\ \mathcal{P}^-_{r+1}\Lambda^2(\mathcal{T}) & = & \text{Nédélec 1st-kind } H(\text{div}) \text{ elements of order } r \text{ (see [22])} \\ \mathcal{P}^-_{r+1}\Lambda^3(\mathcal{T}) & = & \text{discontinuous elements of degree} \leq r \\ \mathcal{P}_{r}\Lambda^3(\mathcal{T}) & = & \text{discontinuous elements of degree} \leq r\end{array}$$

Hence, in the case of the deRham complex, FEEC recovers well-known finite element spaces while at the same time describing their generalization to arbitrary spatial dimensions.

The last piece of FEEC used in this work is the existence of smoothed projection operators

$$\pi_h^k : L^2\Lambda^k \to L^2\Lambda_h^k \quad \text{where } \Lambda_h^k \in \{\mathcal{P}_r\Lambda^k(\mathcal{T}),\ \mathcal{P}_r^-\Lambda^k(\mathcal{T})\}. \tag{3.1}$$

These operators are shown, by virtue of their construction, to be uniformly bounded (in $L^2\Lambda^k$) with respect to h. The following theorem asserts some key properties of these operators.

Theorem 3.1 ([3] Theorem 5.9)**.**

(i.) Let Λ_h^k be one of the spaces $\mathcal{P}^-_{r+1}\Lambda^k(\mathcal{T})$ or, if $r \geq 1$, $\mathcal{P}_r\Lambda^k(\mathcal{T})$. Then π_h^k is a projection onto Λ_h^k and satisfies

$$\left|\left|\omega - \pi_h^k\omega\right|\right|_{L^2\Lambda^k(\Omega)} \leq ch^s \left|\left|\omega\right|\right|_{H^s\Lambda^k(\Omega)}, \quad \omega \in H^s\Lambda^k(\Omega),$$

for $0 \leq s \leq r+1$. Moreover, for all $\omega \in L^2\Lambda^k(\Omega)$, $\pi_h^k\omega \to \omega$ in L^2 as $h \to 0$.

(ii.) Let Λ_h^k be one of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ with $r \geq 1$. Then

$$\left|\left|d(\omega - \pi_h^k\omega)\right|\right|_{L^2\Lambda^k(\Omega)} \leq ch^s \left|\left|d\omega\right|\right|_{H^s\Lambda^k(\Omega)}, \quad \omega \in H^s\Lambda^k(\Omega),$$

for $0 \leq s \leq r$.

(iii.) Let $\Lambda_h^{k-1} \in \left\{\mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}), \mathcal{P}^-_{r+1}\Lambda^{k-1}(\mathcal{T})\right\}$ and $\Lambda_h^k = \mathcal{P}^-_{r+1}\Lambda^k(\mathcal{T})$ or, if $r > 0$, $\mathcal{P}_r\Lambda^k(\mathcal{T})$. Then $d\pi_h^{k-1} = \pi_h^k d$.

An explicit construction of these operators can be found in the papers of Arnold, Falk, and Winther [2, 3].

4. Abstract Evolution Problems and Bochner Spaces

We now cast parabolic and hyperbolic problems into the abstract framework of parametrized Banach spaces. These types of spaces are also known as Bochner spaces, a term we will use to avoid the lengthy equivalent ‘parametrized Banach space.’ We follow prior approaches using this approach, especially [29, page 66] and [26].

Let X be a Banach space and $\mathring{I} := (0,T)$ an interval with closure $I := \overline{I} = [0,T]$. Define

$$C(\mathring{I}, X) := \{u : \mathring{I} \to X \ \mid\ u \text{ bounded and continuous}\}.$$

Equip this space with the norm

$$\left|\left|u\right|\right|_{C(\mathring{I},X)} := \sup_{t\in\mathring{I}} \left|\left|u(t)\right|\right|_X .$$

The **Bochner space** $L^P(\mathring{I}, X)$ is then defined to be the completion of $C(\mathring{I}, X)$ with respect to the norm

$$\left|\left|u\right|\right|_{L^p(\mathring{I},X)} := \left(\int_{\mathring{I}} \left|\left|u(t)\right|\right|_X^p dt\right)^{1/p}.$$

The space $H^1(\mathring{I}, X)$ has an analogous norm

$$||u||_{H^1(\mathring{I},X)} := \left(\int_{\mathring{I}} ||u(t)||_X^2 + \left|\left| \frac{d}{dt} u(t) \right|\right|_X^2 dt \right)^{1/2}.$$

We will commonly use $X = L^2\Lambda^k$ or $X = H^s\Lambda^k$ where it understood that the forms are defined over spatial domain is Ω.

We now consider abstractions of the two main types of evolution PDE.

Let H and V be real, separable Banach spaces such that V is continuously and densely embedded in H. This provides a Gelfand triple, also called a *rigged Hilbert space*:

$$V \subset H \subset V^*,$$

where H is also continuously and densely embedded in V^*. It should be noted that the isomorphism between V and V^* is in general *not* the same as the composition of the inclusion mappings.

Let $(\cdot, \cdot)$ denote the inner product on H as well as the natural pairing between V^* and V. Given $A(t) \in \mathcal{L}(V, V^*)$ depending continuously on $t \in I$, define a quadratic form

$$a(t, u, v) := -(A(t)u, v), \tag{4.1}$$

for $(t, u, v) \in \mathbb{R} \times V \times V$. Assume that a satisfies the coercivity condition

$$a(t, u, u) \geq c_1 ||u||_V^2 - c_2 ||u||_H^2, \tag{4.2}$$

with c_1, c_2 constants independent of $t \in I$. Consider the abstract parabolic problem

$$u_t = A(t)u + f(t), \quad t > 0, \tag{4.3}$$
$$u(0) = u_0, \tag{4.4}$$

and the abstract hyperbolic problem

$$\ddot{u} = A(t)u + f(t), \quad t > 0, \tag{4.5}$$
$$u(0) = u_0, \tag{4.6}$$
$$\dot{u}(0) = u_1. \tag{4.7}$$

These abstract formulations are well-posed in the following sense.

Theorem 4.1 (Existence of Unique Solution to the Abstract Parabolic Problem)**.** *Let $f \in L^2(I, V^*)$ and $u_0 \in H$, let $a(\cdot, \cdot, \cdot)$ be as in* (4.1)*, and let* (4.2) *hold. Then the abstract parabolic problem (4.3) has a unique solution*

$$u \in L^2(\mathring{I}, V) \cap H^1(\mathring{I}, V^*).$$

Moreover, the Sobolev embedding theorem implies $u \in C(I, V^*)$, allowing an interpretation of the initial condition $u(0) = u_0$.

Proof. See [26], page 382. □

The analogous result for the abstract hyperbolic case will require two additional conditions:

$$a(t, u, v) = a(t, v, u), \quad \forall\, u, v \in V, \tag{4.8}$$
$$A \in C^1(I, \mathcal{L}(V, V^*)). \tag{4.9}$$

Theorem 4.2 (Existence of Unique Weak Solution to the Abstract Hyperbolic Problem)**.** *Given* $f \in L^1(I, H)$*,* $u_0 \in V$*, and* $u_1 \in H$*. Let* $a(\cdot,\cdot,\cdot)$ *be as in* (4.1)*, and let* (4.2), (4.8)*, and* (4.9) *hold. Then the abstract hyperbolic problem (4.5) has a unique weak solution*

$$u \in C(I, V) \cap C^1(I, H).$$

Proof. See [26], page 389. □

These results give a concise and elegant way to prove that a wide class of PDE problems amenable to finite element methods are well-posed. In particular, we now explain how these abstract results apply in the cases of the heat equation (2.1) and wave equation (2.6) studied here. Fix the Gelfand triple $H_0^1 \subset L^2 \subset (H_0^1)^* =: H^{-1}$ and define $A : H_0^1 \to H^{-1}$ by $u \mapsto (\nabla u, \nabla \cdot)$. The bilinear form induced by A is the weak form of the Laplacian and is thus coercive, due to the Poincaré inequality being available on H_0^1. Interpreting u as a map from I into H_0^1, we have the **weak form of the heat equation** (2.1): Given $f \in L^2(I, H^{-1})$ and $u_0 \in L^2$, find $u : I \to H_0^1$ such that

$$\begin{aligned} (u_t, v) &= (\nabla u, \nabla v) + (f, v), \quad \forall v \in L^2(I, H_0^1), \quad t > 0, \\ (u(0), v) &= (u_0, v), \qquad\qquad\qquad \forall v \in L^2(I, H_0^1). \end{aligned} \tag{4.10}$$

Theorem 4.1 applies to the strong form of (4.10) and hence implies that there is a unique solution $u \in L^2(\mathring{I}, H_0^1) \cap H^1(\mathring{I}, H^{-1}) \cap C(I, H^{-1})$.

For the hyperbolic case, observe that the weak form of the Laplacian given by A satisfies the symmetry and smoothness conditions required for (4.8) and (4.9). The **weak form of the wave equation** (2.6) is: Given $f \in L^1(I, L^2)$, $u_0 \in H_0^1$, and $u_1 \in L^2$, find $u : I \to H_0^1$ such that

$$\begin{aligned} (u_{tt}, v) &= (\nabla u, \nabla v) + (f, v), \quad \forall v \in L^2(I, H_0^1), \quad t > 0, \\ (u(0), v) &= (u_0, v), \qquad\qquad\qquad \forall v \in L^2(I, H_0^1). \\ (u_t(0), v) &= (u_1, v), \qquad\qquad\qquad \forall v \in L^2(I, H_0^1). \end{aligned} \tag{4.11}$$

Theorem 4.2 thus says that there exists a unique solution $u \in C(I, H_0^1) \cap C^1(I, L^2)$.

5. *A Priori* Error Estimates for Parabolic Problems

We extend Thomée's error estimates from Section 2 to the broader class of elements and arbitrary spatial dimension allowed by FEEC using the abstract framework established in Section 4. Let $\Omega \subset \mathbb{R}^n$ and suppose that the kernel of div : $H\Lambda^{n-1} \to H\Lambda^n$ is trivial.[1] Define the **Bochner mixed weak parabolic** problem: Given $f \in L^2(I, H^{-1})$ and $g \in L^2$, find $(u, \sigma) : I \to H\Lambda^n \times H\Lambda^{n-1}$ such that

$$\begin{aligned} (u_t, \phi) - (\operatorname{div} \sigma, \phi) &= (f, \phi), \quad \forall\, \phi \in H\Lambda^n, \qquad t \in I, \\ (\sigma, \omega) + (u, \operatorname{div} \omega) &= 0, \qquad\quad \forall\, \omega \in H\Lambda^{n-1}, \quad t \in I, \\ u(0) &= g. \end{aligned} \tag{5.1}$$

Observe that (5.1) is the mixed form of (4.10) with the introduction of the variable σ defined by div $\sigma = u$ in a weak sense. As discussed at the end of Section 4, a unique solution for u exists, implying the existence of a solution for σ. Since we assumed that the kernel of div is trivial, σ is unique as well. Hence, (5.1) has a unique solution pair (u, σ) in the space $L^2(\mathring{I}, H\Lambda^n \times H\Lambda^{n-1}) \cap H^1(\mathring{I}, (H\Lambda^n \times H\Lambda^{n-1})^*) \cap C(I, (H\Lambda^n \times H\Lambda^{n-1})^*)$. Therefore, it makes sense to look for discrete approximations of (u, σ) as functionals

[1] The trivial divergence kernel assumption is satisfied by all contractible domains, for instance.

on finite dimensional subsets of $H\Lambda^n \times H\Lambda^{n-1}$, e.g. finite element spaces. The **semi-discrete Bochner parabolic problem** is thus: Find $(u_h, \sigma_h) : I \to \Lambda_h^n \times \Lambda_h^{n-1}$ such that

$$\begin{aligned} (u_{h,t}, \phi_h) - (\text{div}\, \sigma_h, \phi_h) &= (f, \phi_h), \quad \forall\, \phi_h \in \Lambda_h^n, \quad t \in I, \\ (\sigma_h, \omega_h) + (u_h, \text{div}\, \omega_h) &= 0, \qquad\quad \forall\, \omega_h \in \Lambda_h^{n-1}, \quad t \in I, \\ u_h(0) &= g_h. \end{aligned} \tag{5.2}$$

Define g_h to be the solution to the elliptic problem with load data $-\Delta g$, i.e.

$$\begin{aligned} (\text{div}\, \hat{\sigma}_h, \phi_h) + (\Delta g, \phi_h) &= 0 \quad \forall\, \phi_h \in \Lambda_h^n, \\ (\hat{\sigma}_h, \omega_h) + (g_h, \text{div}\, \omega_h) &= 0, \quad \forall\, \omega_h \in \Lambda_h^{n-1}. \end{aligned} \tag{5.3}$$

It is shown in [30] that a unique solution to (5.2) exists, based on the positive-definiteness of the solution operator $T_h : L^2 \to \Lambda_h^n$ for the elliptic problem. A more basic argument for this result can also be made by appealing to the existence of an adjoint to the discrete divergence operator.

Elliptic projection, an idea dating back to Wheeler [32], can be carried out for any fixed time value as we now discuss. For any $t_0 \in I$, define the **time-ignorant discrete elliptic problem**: Find $(\tilde{u}_h, \tilde{\sigma}_h) \in \Lambda_h^n \times \Lambda_h^{n-1}$ such that

$$\begin{aligned} (\text{div}\, \tilde{\sigma}_h, \phi_h) + (-\Delta u(t_0), \phi_h) &= 0, \quad \forall\, \phi_h \in \Lambda_h^n, \\ (\tilde{\sigma}_h, \omega_h) + (\tilde{u}_h, \text{div}\, \omega_h) &= 0, \quad \forall\, \omega_h \in \Lambda_h^{n-1}, \\ \tilde{u}_h(0) &= g_h. \end{aligned} \tag{5.4}$$

Note that the u appearing in the first equation of (5.4) is the solution to the continuous problem (5.1). Thus, we can view $\tilde{\sigma}_h$ and $\tilde{u}_h$ as functions of t with the understanding that they are defined for each t value by (5.4) alone; no continuity with respect to t is required, hence the moniker 'time-ignorant.'

For ease of notation, and in keeping with Thomée, define the error functions

$$\begin{aligned} \rho(t) &:= \tilde{u}_h(t) - u(t), \\ \theta(t) &:= u_h(t) - \tilde{u}_h(t), \\ \varepsilon(t) &:= \sigma_h(t) - \tilde{\sigma}_h(t). \end{aligned}$$

We now prove a lemma which will aid in our subsequent analysis. The result appears as part of the proof of Thomée [30, Theorem 17.2] but we expand it here for clarity.

Lemma 5.1 (Thomée [30])**.** *The error functions satisfy the semi-discrete formulation:*

$$\begin{aligned} (\theta_t, \phi_h) - (\text{div}\, \varepsilon, \phi_h) &= -(\rho_t, \phi_h), \quad \forall\, \phi_h \in \Lambda_h^n, \quad t \in I, \\ (\varepsilon, \omega_h) + (\theta, \text{div}\, \omega_h) &= 0, \qquad\qquad \forall\, \omega_h \in \Lambda_h^{n-1}, \quad t \in I. \end{aligned} \tag{5.5}$$

Proof. The second equation is immediate from the second equations in (5.2) and (5.4). The first equation can be written out as

$$(u_{h,t}, \phi_h) - (\text{div}\, \sigma_h, \phi_h) + (\text{div}\, \tilde{\sigma}_h, \phi_h) - (\tilde{u}_{h,t}, \phi_h) = (u_t, \phi_h) - (\tilde{u}_{h,t}, \phi_h)$$

which is reduced as follows:

$$\begin{aligned} (u_{h,t}, \phi_h) - (\text{div}\, \sigma_h, \phi_h) + (\text{div}\, \tilde{\sigma}_h, \phi_h) &= (u_t, \phi_h) && \text{cancel like terms} \\ (u_{h,t}, \phi_h) - (\text{div}\, \sigma_h, \phi_h) &= -(\Delta u, \phi_h) + (u_t, \phi_h) && \text{by (5.4)} \\ (f, \phi_h) &= -(\Delta u, \phi_h) + (u_t, \phi_h) && \text{by (5.2)} \end{aligned}$$

This says that the continuous problem $u_t - \Delta u = f$ should hold in a weak sense when tested against any of the functions in Λ_h^n. This is guaranteed to be true since we chose $\Lambda_h^n \subset \Lambda^n = L^2$. Thus, the error equations hold as stated. □

The following theorem says that if Λ_h^n and Λ_h^{n-1} are chosen according to the FEEC framework, then error estimates akin to (2.4) and (2.5) can be obtained. Note that in the semidiscrete setting, $(\Delta u)_t(t) = \partial_t \Delta u(t)$ since the time and spatial derivatives commute, allowing the simplified notation $\Delta u_t(t)$ used here.

Theorem 5.2. *Fix $\Omega \subset \mathbb{R}^n$ such that the kernel of* $\mathrm{div} : H\Lambda^{n-1} \to H\Lambda^n$ *is trivial (see footnote 1) and fix $I := [0, T]$. Suppose (u, σ) is the solution to (5.1) such that the regularity estimate*

$$||u(t)||_{H^{s+2}} + ||\sigma(t)||_{H^{s+1}} \leq c\, ||\Delta u(t)||_{H^s} \tag{5.6}$$

holds for $0 \leq s \leq s_{\max}$ and $t \in I$. Choose finite element spaces

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \textit{or} \\ \mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \mathcal{P}_{r+1}^{-}\Lambda^n(\mathcal{T}) \quad (= \mathcal{P}_r\Lambda^n(\mathcal{T}))$$

Then for $0 \leq s \leq s_{\max}$, g_h defined by (5.3), and (u_h, σ_h) the solution to (5.2), the following error estimates hold:

$$||u_h - u||_{L^2(I, L^2\Lambda^n)} \leq \begin{cases} ch\left(||\Delta u||_{L^2(I,L^2)} + \sqrt{T}\, ||\Delta u_t||_{L^1(I,L^2)}\right) & \textit{if } r = 0 \\ ch^{2+s}\left(||\Delta u||_{L^2(I,H^s)} + \sqrt{T}\, ||\Delta u_t||_{L^1(I,H^s)}\right) & \textit{for } r > 0, \\ & \textit{if } s \leq r-1 \end{cases} \tag{5.7}$$

$$||\sigma_h - \sigma||_{L^2(I, L^2\Lambda^{n-1})} \leq \begin{cases} ch\left(||\Delta u||_{L^2(I,H^s)} + \sqrt{T}\, ||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \textit{if } r = 0,\ s = 0,\ \Lambda_h^{n-1} = \mathcal{P}_1^{-}\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\, ||\Delta u||_{L^2(I,H^s)} + h\sqrt{T}\, ||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \textit{if } r = 0,\ s \leq 1,\ \Lambda_h^{n-1} = \mathcal{P}_1\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\, ||\Delta u||_{L^2(I,H^s)} + h^{(3/2)+s}\sqrt{T}\, ||\Delta u_t||_{L^2(I,H^s)}\right) \\ \qquad \textit{for } r > 0,\ \textit{if } s \leq r-1 \end{cases} \tag{5.8}$$

$$||\mathrm{div}(\sigma_h - \sigma)||_{L^2(I, L^2\Lambda^n)} \leq \begin{cases} c\left(h^s\, ||\Delta u||_{L^2(I,H^s)} + h\, ||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \textit{if } r = 0,\ s \leq 1 \\ c\left(h^s\, ||\Delta u||_{L^2(I,H^s)} + h^{2+s}\, ||\Delta u_t||_{L^2(I,H^s)}\right) \\ \qquad \textit{for } r > 0,\ \textit{if } s \leq r-1 \end{cases} \tag{5.9}$$

Remark 5.3. Previous literature on semi-discrete methods usually leaves regularity assumptions implied by the error estimates. For instance, if $||u(t)||_{H^3}$ appears on the right side, it is implicitly assumed that $u(t) \in H^3$ for all $t \in I$. We have stated the specific regularity assumption (5.6) to make clear what regularity must be assumed and to follow the presentation from Arnold, Falk, and Winther [3, p. 342]. The careful reader will notice that the left side of (5.6) does not include a $du(t)$ term, since $u(t) \in \Lambda^n$ implies $du(t) = 0$, nor a $||d\sigma(t)||_{H^s}$ term, since this can be absorbed into the $||\sigma(t)||_{H^{s+1}}$ term. An additional, more subtle difference is that f on the right side of (5.6) has been replaced by Δu. While these two are equivalent in the elliptic case, $\Delta u(t)$ evolves based on the initial data g while $f(t)$ is prescribed, meaning they are in general different in the parabolic setting. [2]

Proof. To simplify notation, we will often use $||\cdot||$ to mean $||\cdot||_{L^2(\Omega)}$. We adapt the proof technique of the corresponding theorem from Thomée [30, Theorem 17.2] to our setting.

Observe that (5.4) is exactly the $k = n$ case of the Hodge-Laplacian problem analyzed by Arnold, Falk and Winther [3] and the hypotheses here match their hypotheses. We can thus use a triangle inequality argument for each estimate, e.g.

$$||u(t) - u_h(t)|| \le ||u(t) - \tilde{u}_h(t)|| + ||\tilde{u}_h(t) - u_h(t)|| = ||\rho(t)|| + ||\theta(t)|| \tag{5.10}$$

The first term will be bounded using the estimates from [3] and the second by the techniques from Thomée [30]. The FEEC estimate [3, p. 342] gives immediately

$$||\rho(t)||_{L^2} \le \begin{cases} ch||\Delta u(t)||_{L^2} & \text{if } r = 0 \\ ch^{2+s}\,||\Delta u(t)||_{H^s} & \text{if } s \le r-1, \text{ for } r > 0 \end{cases} \tag{5.11}$$

Bounding $||\theta(t)||_{L^2}$ is more subtle. Set $\phi_h := \theta$ and $\omega_h := \varepsilon$ in (5.5). Adding the equations yields

$$\frac{1}{2}\frac{d}{dt}\,||\theta||^2 + ||\varepsilon||^2 = -(\rho_t, \theta), \quad t \in I \tag{5.12}$$

We use a technique from Thomée [30, p. 8] to derive an estimate for $||\theta(t)||$. Since $||\theta||$ may not be differentiable when $\theta = 0$, introduce a constant $\delta > 0$ and observe that

$$(||\theta||^2 + \delta^2)^{1/2}\frac{d}{dt}(||\theta||^2 + \delta^2)^{1/2} = \frac{1}{2}\frac{d}{dt}(||\theta||^2 + \delta^2) = \frac{1}{2}\frac{d}{dt}\,||\theta||^2 \le ||\rho_t||\,||\theta||\,,$$

the last step following by (5.12) and Cauchy-Schwarz. Since $||\theta|| \le (||\theta||^2 + \delta^2)^{1/2}$, we have that

$$\frac{d}{dt}(||\theta||^2 + \delta^2)^{1/2} \le ||\rho_t||\,.$$

Note that $\theta(0) = u_h(0) - \tilde{u}_h(0) = g_h - g_h = 0$. Thus

$$||\theta(t)|| = \lim_{\delta \to 0}\int_0^t \frac{d}{dt}(||\theta|| + \delta^2)^{1/2} \le \int_0^t ||\rho_t||\,.$$

Using the bounds on $||\rho(t)||$ from (5.11), we get

$$||\theta(t)|| \le \begin{cases} ch\displaystyle\int_0^t ||\Delta u_t(\ell)||_{L^2} d\ell & \text{if } r = 0, \\ ch^{2+s}\displaystyle\int_0^t ||\Delta u_t(\ell)||_{H^s}\, d\ell & \text{for } r > 0, \text{ if } s \le r-1. \end{cases} \tag{5.13}$$

[2] We thank one of the referees of this paper for pointing out this subtlety.

We can now assemble estimate (5.7) by collecting our results. We show the technique of the case $r = 0$ as the other case employs identical analysis.

$$
\begin{aligned}
||u_h - u||_{L^2(I,L^2\Lambda^n)} &= \left(\int_0^T ||u_h(t) - u(t)||^2 \, dt\right)^{1/2} \\
&\le \left(\int_0^T (||\rho(t)|| + ||\theta(t)||)^2 \, dt\right)^{1/2} \\
&\le ch \left(\int_0^T \left(||\Delta u(t)|| + \int_0^t ||\Delta u_t(\ell)||_{L^2} d\ell\right)^2 dt\right)^{1/2} \\
&\le ch \left(\int_0^T 2\left(||\Delta u(t)||^2 + \left(\int_0^t ||\Delta u_t(\ell)||_{L^2} d\ell\right)^2\right) dt\right)^{1/2}.
\end{aligned}
$$

Roll the 2 into the constant c and observe that the inner integral is maximal when $t = T$. Thus,

$$
\begin{aligned}
||u_h - u||_{L^2(I,L^2\Lambda^n)} &\le ch \left(\int_0^T ||\Delta u(t)||^2 + ||\Delta u_t||^2_{L^1(I,L^2)} \, dt\right)^{1/2} \\
&= ch \left(||\Delta u||^2_{L^2(I,L^2)} + T \, ||\Delta u_t||^2_{L^1(I,L^2)}\right)^{1/2} \\
&\le ch \Big(||\Delta u||^2_{L^2(I,L^2)} + T \, ||\Delta u_t||^2_{L^1(I,L^2)} + \\
&\qquad\qquad 2 \, ||\Delta u||_{L^2(I,L^2)} \sqrt{T} \, ||\Delta u_t||_{L^1(I,L^2)}\Big)^{1/2} \\
&= ch \left(||\Delta u||_{L^2(I,L^2)} + \sqrt{T} \, ||\Delta u_t||_{L^1(I,L^2)}\right).
\end{aligned}
$$

We now turn to (5.8), i.e. an error bound for the approximation of σ. We use the same technique of bounding $||\sigma(t) - \tilde{\sigma}_h(t)||$ by the corresponding FEEC estimate and $||\tilde{\sigma}_h(t) - \sigma_h(t)||$ ($= ||\varepsilon(t)||$) by a modification of (5.5). First, observe that the FEEC estimate [3, p. 342] gives

$$
||\sigma(t) - \tilde{\sigma}_h(t)|| \le ch^{1+s} \, ||\Delta u(t)||_{H^s}, \text{ if } \begin{cases} s \le r+1, & \Lambda_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ s \le r, & \Lambda_h^{n-1} = \mathcal{P}^-_{r+1}\Lambda^{n-1}(\mathcal{T}) \end{cases}. \tag{5.14}
$$

To bound $||\epsilon(t)||$, differentiate the second equation of (5.5) with respect to t and set $\phi_h := 2\theta_t$, $\omega_h := 2\varepsilon$, yielding

$$
\begin{aligned}
(\theta_t, 2\theta_t) - (\operatorname{div} \varepsilon, 2\theta_t) &= -(\rho_t, 2\theta_t), \\
(\varepsilon_t, 2\varepsilon) + (\theta_t, \operatorname{div} \varepsilon) &= 0.
\end{aligned}
$$

Adding the equations and converting to norms, we have the bound

$$
\frac{d}{dt} ||\varepsilon||^2 + 2 \, ||\theta_t||^2 = -2(\rho_t, \theta_t) \le ||\rho_t||^2 + ||\theta_t||^2,
$$

by Cauchy-Schwarz and the AM-GM inequality. Note that since $\theta(0) = 0$, we have $\varepsilon(0) = 0$ by the second equation of (5.5). Thus

$$
||\varepsilon(t)||^2 = \int_0^t \frac{d}{ds} ||\varepsilon(s)||^2 \, ds \le \int_0^t \frac{d}{ds} ||\varepsilon(s)||^2 + ||\theta_t||^2 \, ds \le \int_0^t ||\rho_t||^2 \, ds.
$$

As before, we use (5.11) to derive

$$||\varepsilon(t)|| \leq \begin{cases} ch\left(\int_0^t ||\Delta u_t(\ell)||_{L^2}^2 d\ell\right)^{1/2} & \text{if } r=0, \\ ch^{2+s}\left(\int_0^t ||\Delta u_t(\ell)||_{H^s}^2\, d\ell\right)^{1/2} & \text{for } r>0, \text{ if } s \leq r-1. \end{cases} \tag{5.15}$$

$$\begin{aligned} ||\sigma_h - \sigma||_{L^2(I,L^2\Lambda^{n-1})} &= \left(\int_0^T ||\sigma_h(t)-\sigma(t)||^2\, dt\right)^{1/2} \\ &\leq \left(\int_0^T (||\sigma(t)-\tilde{\sigma}_h(t)|| + ||\varepsilon(t)||)^2\, dt\right)^{1/2} \\ &\leq c\left(\int_0^T \left(h^{1+s}\,||\Delta u(t)||_{H^s} + h\left(\int_0^t ||\Delta u_t(\ell)||_{L^2}^2 d\ell\right)^{1/2}\right)^2 dt\right)^{1/2} \\ &\leq ch\left(\int_0^T 2\left(h^{2s}\,||\Delta u(t)||_{H^s}^2 + \int_0^t ||\Delta u_t(\ell)||_{L^2}^2 d\ell\right) dt\right)^{1/2}. \end{aligned}$$

Rolling the 2 into the constant c and again noting that the inner integral is maximal when $t=T$, we recover the first two estimates of (5.8):

$$\begin{aligned} ||\sigma_h - \sigma||_{L^2(I,L^2\Lambda^{n-1})} &\leq ch\left(\int_0^T h^{2s}\,||\Delta u(t)||_{H^s)}^2 + ||\Delta u_t||_{L^2(I,L^2)}^2\, dt\right)^{1/2} \\ &= ch\left(h^{2s}\,||\Delta u||_{L^2(I,H^s)}^2 + T\,||\Delta u_t||_{L^2(I,L^2)}^2\, dt\right)^{1/2} \\ &\leq ch\left(h^s\,||\Delta u||_{L^2(I,H^s)} + \sqrt{T}\,||\Delta u_t||_{L^2(I,L^2)}\right). \end{aligned}$$

When $r>0$, (5.14) requires $s \leq r$ or $s \leq r+1$ to obtain optimal convergence rates on the first term of the right side while (5.15) requires $s \leq r-1$ to obtain optimal rates on the second term. Thus the hypothesis $s \leq r-1$ implies both (5.14) and (5.15). The last estimate of (5.8) then follows by identical analysis to the first two cases.

Finally, we turn to estimate (5.9) and follow the same technique. Since div is the exterior operator $d : \Lambda^{n-1} \to \Lambda^n$, we have from FEEC [3, p. 342] that

$$||\text{div}(\sigma(t) - \tilde{\sigma}_h(t))|| \leq ch^s\,||\Delta u(t)||_{H^s}\,, \text{ if } s \leq r+1. \tag{5.16}$$

To bound $||\text{div}\,\varepsilon||$ set $w_h := \varepsilon$ in (5.5) and take the derivative with respect to t. This yields

$$\frac{d}{dt}\,||\varepsilon||^2 + (\theta_t, \text{div}\,\varepsilon) = 0.$$

Note that div $\varepsilon \in \Lambda_h^n$ since the discrete spaces are chosen to satisfy the relationship $\text{div}\Lambda_h^{n-1} \subset \Lambda_h^n$. Thus, we can set $\phi_h :=$ div ε in in (5.5) and substitute to get

$$\frac{d}{dt}\,||\varepsilon||^2 + ||\text{div}\,\varepsilon||^2 = (\rho_t, \text{div}\,\varepsilon).$$

By Cauchy-Schwarz, we have $||\text{div}\,\varepsilon||^2 \leq ||\rho_t||\,||\text{div}\,\varepsilon||$ and hence

$$||\text{div}\,\varepsilon|| \leq ||\rho_t||\,.$$

Again, we use (5.11) to get

$$||\text{div}\,\varepsilon(t)|| \le \begin{cases} ch||\Delta u_t||_{L^2} & \text{if } r = 0 \\ ch^{2+s}\,||\Delta u_t||_{H^s} & \text{if } s \le r-1, \text{ for } r > 0 \end{cases}. \tag{5.17}$$

The estimate (5.9) follows by combining this with (5.16). □

6. *A Priori* ERROR ESTIMATES FOR HYPERBOLIC PROBLEMS

We now analyze hyperbolic problems using the same L^2 Bochner spaces $L^2(I, L^2\Lambda^k)$ introduced in Section 4. Let $\Omega \subset \mathbb{R}^n$ and suppose that the kernel of div : $H\Lambda^{n-1} \to H\Lambda^n$ is trivial (see footnote 1). Define the **Bochner velocity-stress mixed weak formulation**: Given $f \in L^1(I, L^2)$, $u_0 \in H^1$, and $u_1 \in L^2$, find $(\mu, \sigma) : I \to H\Lambda^n \times H\Lambda^{n-1}$ such that

$$\begin{aligned} (\mu_t, \phi) - (\text{div}\,\sigma, \phi) &= (f, \phi), \quad \forall\, \phi \in H\Lambda^n, \qquad t \in I, \\ (\sigma_t, \omega) + (\mu, \text{div}\,\omega) &= 0, \qquad\quad \forall\, \omega \in H\Lambda^{n-1}, \quad t \in I, \\ \mu(0) &= u_1, \\ \sigma(0) &= \nabla u_0, \end{aligned} \tag{6.1}$$

where $\mu = u_t$ as in (2.8). Observe that (6.1) is the mixed form of (4.11) with the introduction of the variable σ defined by div $\sigma = \mu$ in a weak sense. As in the parabolic case, the trivial kernel hypothesis and the discussion at the end of Section 4 imply that (6.1) has a unique solution pair (u, σ) in the space $C(I, H\Lambda^n \times H\Lambda^{n-1}) \cap C^1(I, \Lambda^n \times \Lambda^{n-1})$. Therefore, it makes sense to look for discrete approximations of (μ, σ) as functionals on finite dimensional subsets of $H\Lambda^n \times H\Lambda^{n-1}$, e.g. finite element spaces. The **semi-discrete Bochner hyperbolic problem** is thus: Find $(\mu_h, \sigma_h) : I \to \Lambda_h^n \times \Lambda_h^{n-1}$ such that

$$\begin{aligned} (\mu_{h,t}, \phi_h) - (\text{div}\,\sigma_h, \phi_h) &= (f, \phi_h), \quad \forall\, \phi_h \in \Lambda_h^n, \qquad t \in I, \\ (\sigma_{h,t}, \omega_h) + (\mu_h, \text{div}\,\omega_h) &= 0, \qquad\quad \forall\, \omega_h \in \Lambda_h^{n-1}, \quad t \in I, \\ \mu_h(0) &= u_{1,h}, \\ \sigma(0) &= (\nabla u_0)_h. \end{aligned} \tag{6.2}$$

We now generalize the results of Geveci [13] and others into the language of FEEC. We first prove a very simple proposition explaining the approximation properties of the π_h^k operators in this context.

Proposition 6.1. *Choose finite element spaces*

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \textit{or} \\ \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \mathcal{P}_{r+1}^-\Lambda^n(\mathcal{T}) \quad (= \mathcal{P}_r\Lambda^n(\mathcal{T}))\,.$$

The smoothed projection operators from (3.1) have the approximation properties

$$\begin{aligned} \left|\left|\pi_h^{n-1}\omega - \omega\right|\right|_{L^2\Lambda^{n-1}} \le c\, h^s\, ||\omega||_{H^s\Lambda^{n-1}}\,, & \\ \textit{for } 0 \le s \le r+2, \quad \textit{if } \Lambda_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}), \textit{ or} & \\ \textit{for } 0 \le s \le r+1, \quad \textit{if } \Lambda_h^{n-1} = \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}), & \end{aligned} \tag{6.3}$$

$$||\pi_h^n\phi - \phi||_{L^2\Lambda^n} \le c\, h^s\, ||\phi||_{H^s\Lambda^n}\,, \quad \textit{for } 0 \le s \le r+1. \tag{6.4}$$

Proof. Estimate (6.3) follows directly from Theorem 3.1 (*i.*). Note that Theorem 3.1 (*i.*) is stated for the case $\mathcal{P}_r\Lambda^{n-1}$ while here we have $\mathcal{P}_{r+1}\Lambda^{n-1}$, thereby allowing for the higher bound on s in this case. Finally, since $\pi_h^n\mu - \mu = \partial_t(\pi_h^n u - u)$, Theorem 3.1 (*i.*) also implies (6.4). $\square$

Theorem 6.2. *Fix $\Omega \subset \mathbb{R}^n$ such that the kernel of* div $: H\Lambda^{n-1} \to H\Lambda^n$ *is trivial (see footnote 1) and fix $I := [0,T]$. Choose finite element spaces*

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \textit{or} \\ \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \mathcal{P}_{r+1}^-\Lambda^n(\mathcal{T}) \quad (= \mathcal{P}_r\Lambda^n(\mathcal{T}))\,.$$

Then for (μ_h, σ_h) the solution to (6.2), the following error estimate holds:

$$||\mu_h - \mu||_{L^2(I,L^2\Lambda^n)} + ||\sigma_h - \sigma||_{L^2(I,L^2\Lambda^{n-1})} \le c\left(\sqrt{T}E_1 + h^s\left(\sqrt{T}E_2 + E_3\right)\right), \tag{6.5}$$

where

$$\begin{aligned} E_1 &= ||u_1 - u_{1,h}||_{L^2} + \\ &\qquad ||(\nabla u_0) - (\nabla u_0)_h||_{L^2} && \textit{(error due to discretization of initial data)} \\ E_2 &= ||u_1||_{H^s} + ||\nabla u_0||_{H^s} && \textit{(regularity of initial data)} \\ E_3 &= ||u_t||_{L^2(I,H^s)} + ||\sigma||_{L^2(I,H^s)} && \textit{(regularity of continuous solution to (6.1))} \end{aligned}$$

Remark 6.3. This theorem strengthens and generalizes the result by Geveci [13] for $n = 2$ where L^2 projection is used instead of the smoothed projection operators π_h^k. An article by Makridakis [21] extended Geveci's results to $n = 3$ in the context of linear elastodynamics, however both papers had to assume the existence of finite element spaces and projections to them with certain properties. Our result here makes clear what these spaces and projections should be in the unified language of FEEC. Moreover, the fact that the π_h^k operators are not the L^2 projection and hence not self-adjoint requires a revised proof technique that ultimately allows the removal of the error term $||u_{tt}||_{L^2(I,H^s)} + ||\sigma_t||_{L^2(I,H^s)}$ appearing in prior error bounds.

Proof. Define $\Psi := \Lambda^n \times \Lambda^{n-1}$ with finite dimensional subspace $\Psi_h := \Lambda_h^n \times \Lambda_h^{n-1}$. Denote the components of an element $\psi_i \in \Psi$ by $\{\phi_i, \omega_i\}$. The L^2 inner product and norm on Ψ are

$$(\psi_1, \psi_2)_\Psi := (\phi_1, \phi_2)_{L^2} + (\omega_1, \omega_2)_{L^2} \quad \text{and} \quad ||\psi||_\Psi := \sqrt{(\psi, \psi)_\Psi}.$$

Define a skew-symmetric bilinear form $a : \Psi \times \Psi \to \mathbb{R}$ by

$$a(\psi_1, \psi_2) := -(\operatorname{div}\omega_1, \phi_2)_{L^2} + (\phi_1, \operatorname{div}\omega_2)_{L^2}.$$

Let $\xi := (\mu, \sigma) \in \Psi$ be the solution to (6.1) and let $\psi := (\phi, \omega) \in \Psi$ be arbitrary. Then adding the equations of (6.1) yields

$$(\xi_t, \psi)_\Psi + a(\xi, \psi) = (f, \phi)_{L^2} \qquad \forall\, \psi \in \Psi. \tag{6.6}$$

Similarly, from (6.2) we get

$$(\xi_{h,t}, \psi_h)_\Psi + a(\xi_h, \psi_h) = (f, \phi_h)_{L^2} \qquad \forall\, \psi_h \in \Psi_h. \tag{6.7}$$

Define a projection operator $\pi_h : \Psi \to \Psi_h$ using the bounded cochain projections from (3.1) via $\pi_h\psi := \left\{\pi_h^n\phi, \pi_h^{n-1}\omega\right\}$. Since π_h only affects the spatial variables, it commutes with the time derivative operator, i.e. $(\partial_t \pi_h \xi, \psi_h)_\Psi = (\pi_h \partial_t \xi, \psi_h)_\Psi$. Using this and (6.6), and letting $\mathbb{I}$ denote the identity operator, we derive

$$\begin{aligned}
a(\pi_h \xi, \psi_h) &= a(\xi, \psi_h) + a((\pi_h - \mathbb{I})\xi, \psi_h) \\
(\partial_t \pi_h \xi, \psi_h)_\Psi + a(\pi_h \xi, \psi_h) &= a(\xi, \psi_h) + (\pi_h \partial_t \xi, \psi_h)_\Psi + a((\pi_h - \mathbb{I})\xi, \psi_h) \\
(\partial_t \pi_h \xi, \psi_h)_\Psi + a(\pi_h \xi, \psi_h) &= (f, \phi_h)_{L^2} + ((\pi_h - \mathbb{I})\partial_t \xi, \psi_h)_\Psi + a((\pi_h - \mathbb{I})\xi, \psi_h),
\end{aligned} \tag{6.8}$$

which holds for all $\psi_h \in \Psi_h \subset \Psi$. Now define the error function

$$\varepsilon_h(t) := \pi_h \xi(t) - \xi_h(t).$$

The derivation of a good bound for $||\varepsilon_h(t)||_\Psi$ constitutes the bulk of the remainder of the proof. Subtracting (6.7) from (6.8) yields

$$(\partial_t \varepsilon_h(t), \psi_h)_\Psi + a(\varepsilon_h(t), \psi_h) = ((\pi_h - \mathbb{I})\partial_t \xi(t), \psi_h)_\Psi + a((\pi_h - \mathbb{I})\xi(t), \psi_h), \quad \forall\, \psi_h \in \Psi_h. \tag{6.9}$$

Define the skew-adjoint linear operator $L_h : \Psi_h \to \Psi_h$ by

$$(L_h \psi_1, \psi_2)_\Psi := a(\psi_1, \psi_2) \qquad \forall\, \psi_1, \psi_2 \in \Psi_h.$$

We can thus re-write (6.9) as an equation of functionals on Ψ_h:

$$\partial_t \varepsilon_h(t) + L_h \varepsilon_h(t) = (\pi_h - \mathbb{I})\partial_t \xi(t) + L_h(\pi_h - \mathbb{I})\xi(t) \tag{6.10}$$

To ease notation, set $Q(t) := (\pi_h - \mathbb{I})\partial_t \xi(t)$ and $R(t) := (\pi_h - \mathbb{I})\xi(t)$, yielding

$$\partial_t \varepsilon_h(t) + L_h \varepsilon_h(t) = Q(t) + L_h R(t) \tag{6.11}$$

We will use some basic results from the theory of semigroups of linear operators as can be found, for instance, in [24]. For any fixed $\tau \in \mathbb{R}$, the product rule in this context yields

$$\begin{aligned}
\partial_t(e^{(t-\tau)L_h}(\varepsilon_h(t) - R(t)) &= L_h e^{(t-\tau)L_h}(\varepsilon_h(t) - R(t)) + e^{(t-\tau)L_h}\partial_t(\varepsilon_h(t) - R(t)) \\
&= e^{(t-\tau)L_h}\left(\partial_t \varepsilon_h(t) + L_h \varepsilon_h(t) - (L_h + \partial_t)R(t)\right)
\end{aligned}$$

Note that we used the fact that $e^{(t-\tau)L_h}$ commutes with L_h, a standard result [24, Corollary 1.4]. Swapping the roles of t and τ, we re-write the above as

$$\partial_\tau(e^{-(t-\tau)L_h}(\varepsilon_h(\tau) - R(\tau))) = e^{-(t-\tau)L_h}\left(\partial_\tau \varepsilon_h(\tau) + L_h \varepsilon_h(\tau) - (L_h + \partial_\tau)R(\tau)\right) \tag{6.12}$$

Now we integrate in such a way that (6.11) and (6.12) will give us an expression for $\varepsilon_h(t)$. First observe that $\partial_\tau R(\tau) = Q(\tau)$ since ∂_τ commutes with π_h and $\mathbb{I}$. Thus,

$$\begin{aligned}
0 &= \int_0^t e^{-(t-\tau)L_h}\left(Q(\tau) - \partial_\tau R(\tau)\right)\, d\tau \\
&= \int_0^t e^{-(t-\tau)L_h}\left(Q(\tau) + L_h R(\tau) - (L_h + \partial_\tau)R(\tau)\right)\, d\tau && \text{by } \pm L_h R(\tau) \\
&= \int_0^t e^{-(t-\tau)L_h}\left(\partial_\tau \varepsilon_h(\tau) + L_h \varepsilon_h(\tau) - (L_h + \partial_\tau)R(\tau)\right)\, d\tau && \text{by (6.11)} \\
&= \int_0^t \partial_\tau(e^{-(t-\tau)L_h}(\varepsilon_h(\tau) - R(\tau))\, d\tau && \text{by (6.12)} \\
&= \varepsilon_h(t) - R(t) - e^{-tL_h}\varepsilon_h(0) + R(0) && \text{fund. thm. calculus}
\end{aligned}$$

Rewriting the above chain of equalities, we see that

$$\varepsilon_h(t) = e^{-tL_h}\varepsilon_h(0) + R(t) - R(0) \tag{6.13}$$

Observe that e^{-tL_h} is unitary meaning it preserves Ψ-norm, i.e. $\left|\left|e^{-tL_h}\psi\right|\right|_\Psi = ||\psi||_\Psi$ for all $\psi \in \Psi$. This follows from the fact that L_h is a real, skew self-adjoint operator, meaning iL_h is self-adjoint, which is equivalent to saying e^{-tL_h} is unitary [24, Theorem 10.8]. Thus, taking the $||\cdot||_\Psi$ norm of (6.13), the triangle inequality gives

$$\begin{aligned} ||\varepsilon_h(t)||_\Psi &\leq ||\varepsilon_h(0)||_\Psi + ||R(t)||_\Psi + ||R(0)||_\Psi \\ &= ||\varepsilon_h(0)||_\Psi + ||(\pi_h - \mathbb{I})\xi(t)||_\Psi + ||(\pi_h - \mathbb{I})\xi(0)||_\Psi \,, \end{aligned} \tag{6.14}$$

Unpacking the notation lets us characterize this bound in terms of the errors defined in the theorem statement. Recall that u_0 and u_1 are given initial data functions and should not be confused with u_h or u_t. We will use $f \lesssim g$ to mean $f \leq cg$ where c is some constant independent of h and T. We have

$$||\varepsilon_h(t)|| \lesssim ||\pi_h^n u_1 - u_{1,h}|| + \left|\left|\pi_h^{n-1}(\nabla u_0) - (\nabla u_0)_h\right|\right| + ||(\pi_h - \mathbb{I})\xi(t)|| + ||(\pi_h - \mathbb{I})\xi(0)|| \tag{6.15}$$

To bound the first term on the right, use (6.4) from Proposition 6.1 to get

$$||\pi_h^n u_1 - u_{1,h}|| \leq ||\pi_h^n u_1 - u_1|| + ||u_1 - u_{1,h}|| \lesssim h^s ||u_1||_{H^s} + ||u_1 - u_{1,h}||$$

Using (6.3) likewise for the second term, we have

$$||\pi_h^n u_1 - u_{1,h}|| + \left|\left|\pi_h^{n-1}(\nabla u_0) - (\nabla u_0)_h\right|\right| \lesssim E_1 + h^s E_2 \tag{6.16}$$

Also by (6.3) and (6.4), we have the bounds

$$||(\pi_h - \mathbb{I})\xi(t)|| \lesssim h^s(||\sigma(t)||_{H^s\Lambda^{n-1}} + ||u_t(t)||_{H^s\Lambda^n}) \tag{6.17}$$

$$||(\pi_h - \mathbb{I})\xi(0)|| \lesssim h^s(||\nabla u_0||_{H^s\Lambda^{n-1}} + ||u_1||_{H^s\Lambda^n}) \tag{6.18}$$

Using (6.15) in conjunction with (6.16), (6.17), and (6.18), we derive

$$\begin{aligned} \int_0^T ||\varepsilon_h(t)||^2 \, dt &\lesssim TE_1^2 + h^{2s}TE_2^2 + h^{2s}\int_0^T ||\sigma(t)||_{H^s}^2 + ||u_t(t)||_{H^s}^2 + E_2^2 \, dt \\ &\lesssim TE_1^2 + h^{2s}TE_2^2 + h^{2s}E_3 + h^{2s}TE_2^2 \end{aligned} \tag{6.19}$$

We now start building up the main result.

$$\begin{aligned} &\Big(||\mu_h - \mu||_{L^2(I,L^2\Lambda^n)} + ||\sigma_h - \sigma||_{L^2(I,L^2\Lambda^{n-1})}\Big)^2 \\ &\quad\lesssim \int_0^T \left(||\mu_h - \pi_h^n\mu||_{L^2} + \left|\left|\sigma_h - \pi_h^{n-1}\sigma\right|\right|_{L^2}\right)^2 \\ &\qquad\qquad + ||\pi_h^n\mu - \mu||_{L^2}^2 + \left|\left|\pi_h^{n-1}\sigma - \sigma\right|\right|_{L^2}^2 dt \\ &\quad= \int_0^T ||\varepsilon_h(t)||_{L^2}^2 \, dt + ||\pi_h^n\mu - \mu||_{L^2(I,L^2\Lambda^n)}^2 + \left|\left|\pi_h^{n-1}\sigma - \sigma\right|\right|_{L^2(I,L^2\Lambda^{n-1})}^2 dt \\ &\quad\lesssim \int_0^T ||\varepsilon_h(t)||_{L^2}^2 + h^{2s}||\mu(t)||_{H^s}^2 + h^{2s}||\sigma(t)||_{H^s}^2 \, dt \\ &\quad\lesssim \int_0^T ||\varepsilon_h(t)||_{L^2}^2 \, dt + h^{2s}E_3^2. \end{aligned} \tag{6.20}$$

Combining (6.20) and (6.19) yields

$$\begin{aligned} \Big(||\mu_h - \mu||_{L^2(I,L^2\Lambda^n)} + ||\sigma_h - \sigma||_{L^2(I,L^2\Lambda^{n-1})}\Big)^2 &\lesssim \left(TE_1^2 + 2h^{2s}TE_2^2 + 2h^{2s}E_3^2\right) \\ &\lesssim \left(\sqrt{T}E_1 + h^s\left(\sqrt{T}E_2 + E_3\right)\right)^2 \end{aligned}$$

Taking the square root of both sides completes the proof. $\square$

7. SEMI-LINEAR EVOLUTION PROBLEMS

We now show how the techniques developed above can be extended to certain types of non-linear evolution problems. Consider the **semi-linear heat equation**: Find $u(x,t)$ such that

$$\begin{aligned} u_t - \Delta u + F(u) = f \quad & \text{in } \Omega, \quad \text{for } t > 0 \\ u = 0 \quad & \text{on } \partial\Omega, \quad \text{for } t > 0 \quad \text{with } u(\cdot, 0) = g \text{ in } \Omega, \end{aligned} \tag{7.1}$$

where F is some non-linear operator on $L^2(\Omega)$. The existence and uniqueness of solutions to instances of this problem have been studied extensively [16, 14, 27, 33] as have finite element methods for the approximation of its solution [31, 9, 28, 30, 20].

We focus here on the case where F satisfies a Lipschitz condition

$$||F(v) - F(w)||_{L^2} \leq C\,||v - w||_{L^2}, \quad \forall\, v, w \in L^2(\Omega). \tag{7.2}$$

This condition, or a weaker locally Lipschitz condition [5], is assumed by Holst and Stern [18] in their recent extension of the FEEC error estimates to semi-linear elliptic problems. Hence, it serves as an obvious assumption for extending our evolution results to the semi-linear case.

For $\Omega \subset \mathbb{R}^n$, define the **Bochner semi-linear mixed weak form parabolic**: Given f and g, find $(u, \sigma) : I \to H\Lambda^n \times H\Lambda^{n-1}$ such that

$$\begin{aligned} (u_t, \phi) - (\text{div}\,\sigma, \phi) + (F(u), \phi) &= (f, \phi), \quad \forall\, \phi \in H\Lambda^n, \quad t \in I, \\ (\sigma, \omega) + (u, \text{div}\,\omega) &= 0, \qquad \forall\, \omega \in H\Lambda^{n-1}, \quad t \in I, \\ u(0) &= g. \end{aligned} \tag{7.3}$$

The **semi-linear semi-discrete Bochner parabolic problem** is thus: Find $(u_h, \sigma_h) : I \to \Lambda_h^n \times \Lambda_h^{n-1}$ such that

$$\begin{aligned} (u_{h,t}, \phi_h) - (\text{div}\,\sigma_h, \phi_h) + (F(u_h), \phi_h) &= (f, \phi_h), \quad \forall\, \phi_h \in \Lambda_h^n, \quad t \in I, \\ (\sigma_h, \omega_h) + (u_h, \text{div}\,\omega_h) &= 0, \qquad \forall\, \omega_h \in \Lambda_h^{n-1}, \quad t \in I, \\ u_h(0) &= g_h, \end{aligned} \tag{7.4}$$

where $g_h \in \Lambda_h^n$ is an approximation of g. Analogously to the linear case, for any $t_0 \in I$, define the **time-ignorant linear discrete elliptic problem**: find $(\tilde{u}_h, \tilde{\sigma}_h) \in \Lambda_h^n \times \Lambda_h^{n-1}$ such that

$$\begin{aligned} (\text{div}\,\tilde{\sigma}_h, \phi_h) - (\Delta u(t_0), \phi_h) &= 0, \quad \forall\, \phi_h \in \Lambda_h^n, \quad t \in I, \\ (\tilde{\sigma}_h, \omega_h) + (\tilde{u}_h, \text{div}\,\omega_h) &= 0, \quad \forall\, \omega_h \in \Lambda_h^{n-1}, \quad t \in I, \\ \tilde{u}_h(0) &= g_h, \end{aligned} \tag{7.5}$$

where now u is the solution to the continuous semi-linear problem (7.3). Similarly, define

$$\begin{aligned} \rho(t) &:= \tilde{u}_h(t) - u(t), \\ \theta(t) &:= u_h(t) - \tilde{u}_h(t), \\ \varepsilon(t) &:= \sigma_h(t) - \tilde{\sigma}_h(t), \end{aligned}$$

where u, σ and their discrete counterparts are now solutions to the corresponding semi-linear problems. We have an analogous lemma.

Lemma 7.1. *The semi-linear error functions satisfy the semi-discrete formulation, i.e.*

$$\begin{aligned} (\theta_t, \phi_h) - (\text{div}\,\varepsilon, \phi_h) &= -(\rho_t, \phi_h), \quad \forall\, \phi_h \in \Lambda_h^n, \quad t \in I, \\ (\varepsilon, \omega_h) + (\theta, \text{div}\,\omega_h) &= 0, \qquad \forall\, \omega_h \in \Lambda_h^{n-1}, \quad t \in I. \end{aligned} \tag{7.6}$$

Proof. The second equation is immediate from the second equations in (7.4) and (7.5). The first equation can be written out as

$$(u_{h,t}, \phi_h) - (\text{div}\ \sigma_h, \phi_h) + (\text{div}\ \tilde{\sigma}_h, \phi_h) - (\tilde{u}_{h,t}, \phi_h) = (u_t, \phi_h) - (\tilde{u}_{h,t}, \phi_h)$$

which is reduced as follows:

$$\begin{aligned} (u_{h,t}, \phi_h) - (\text{div}\ \sigma_h, \phi_h) + (\text{div}\ \tilde{\sigma}_h, \phi_h) &= (u_t, \phi_h) && \text{cancel like terms} \\ (u_{h,t}, \phi_h) - (\text{div}\ \sigma_h, \phi_h) &= -(\Delta u, \phi_h) + (u_t, \phi_h) && \text{by (7.5)} \\ (f, \phi_h) - (F(u_h), \phi_h) &= -(\Delta u, \phi_h) + (u_t, \phi_h) && \text{by (7.4)} \end{aligned}$$

This says that the continuous problem $u_t - \Delta u + F(u_h) = f$ should hold in a weak sense when tested against any of the functions in Λ_h^n. This is guaranteed to be true since we chose $\Lambda_h^n \subset \Lambda^n = L^2$. Thus, the error equations hold as stated. □

(see footnote 1)

Theorem 7.2. *Fix $\Omega \subset \mathbb{R}^n$ such that the kernel of* div : $H\Lambda^{n-1} \to H\Lambda^n$ *is trivial (see footnote 1) and fix $I := [0, T]$. Suppose (u, σ) is the solution to (7.3) such that the regularity estimate*

$$||u(t)||_{H^{s+2}} + ||\sigma(t)||_{H^{s+1}} + ||d\sigma(t)||_{H^s} \leq c\,||\Delta u(t)||_{H^s} \tag{7.7}$$

holds for $0 \leq s \leq s_{\max}$ and $t \in I$. Assume that the operator F satisfies the Lipschitz assumption (7.2). Choose finite element spaces

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \textit{or} \\ \mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \mathcal{P}_{r+1}^{-}\Lambda^n(\mathcal{T}) \quad (= \mathcal{P}_r\Lambda^n(\mathcal{T}))$$

Then for $0 \leq s \leq s_{\max}$, g_h defined by (5.3), and (u_h, σ_h) the solution to (7.4), the following error estimates hold:

$$||u_h - u||_{L^2(I, L^2\Lambda^n)} \leq \begin{cases} ch\left(||\Delta u||_{L^2(I,L^2)} + \sqrt{T}\,||\Delta u_t||_{L^1(I,L^2)}\right) & \textit{if } r = 0 \\ ch^{1+s}\left(||\Delta u||_{L^2(I,H^s)} + \sqrt{T}\,||\Delta u_t||_{L^1(I,H^s)}\right) & \textit{for } r > 0, \\ & \textit{if } s \leq r - 1 \end{cases} \tag{7.8}$$

$$||\sigma_h - \sigma||_{L^2(I, L^2\Lambda^{n-1})} \leq \begin{cases} ch\left(||\Delta u||_{L^2(I,H^s)} + \sqrt{T}\,||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \textit{if } r = 0,\ s = 0,\ \Lambda_h^{n-1} = \mathcal{P}_1^{-}\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\,||\Delta u||_{L^2(I,H^s)} + h\sqrt{T}\,||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \textit{if } r = 0,\ s \leq 1,\ \Lambda_h^{n-1} = \mathcal{P}_1\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\,||\Delta u||_{L^2(I,H^s)} + h^{(3/2)+s}\sqrt{T}\,||\Delta u_t||_{L^2(I,H^s)}\right) \\ \qquad \textit{for } r > 0,\ \textit{if } s \leq r - 1 \end{cases} \tag{7.9}$$

$$
||\mathrm{div}(\sigma_h - \sigma)||_{L^2(I,L^2\Lambda^n)} \leq \begin{cases} c\left(h^s\,||\Delta u||_{L^2(I,H^s)} + h\,||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \text{if } r=0,\ s\leq 1 \\ c\left(h^s\,||\Delta u||_{L^2(I,H^s)} + h^{2+s}\,||\Delta u_t||_{L^2(I,H^s)}\right) \\ \qquad \text{for } r>0,\ \text{if } s\leq r-1 \end{cases}. \tag{7.10}
$$

Proof. The proof is very similar to that of Theorem 5.2. Equation (7.5) is the $k = n$ case of the discrete mixed variational problem examined by Holst and Stern in [18, Equation (9)]. Therefore, we can use the same type of triangle inequality from (5.10) to recover the estimates. By [18, Theorem 4.2], we have the estimates

$$
||\rho(t)||_{L^2} \leq \begin{cases} ch||\Delta u(t)||_{L^2} & \text{if } r=0 \\ ch^{2+s}\,||\Delta u(t)||_{H^s} & \text{if } s\leq r-1, \text{ for } r>0 \end{cases} \tag{7.11}
$$

$$
||\sigma(t) - \tilde{\sigma}_h(t)||_{L^2} \leq ch^{1+s}\,||\Delta u(t)||_{H^s}\,,\ \text{if } \begin{cases} s\leq r+1, & \Lambda_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ s\leq r, & \Lambda_h^{n-1} = \mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}) \end{cases} \tag{7.12}
$$

$$
||\mathrm{div}(\sigma(t) - \tilde{\sigma}_h(t))||_{L^2} \leq ch^s\,||\Delta u(t)||_{H^s}\,,\ \text{if } s\leq r+1. \tag{7.13}
$$

An explanation of how these estimates are derived from the results of [18] is given in Appendix A. Note that these estimates are exactly the same as the corresponding estimates (5.11), (5.14) and (5.16) from the linear case. The proof then proceeds exactly as before since the rest of the argument does not appeal to the linearity of the problem at all. □

This approach seems likely to extend to semi-linear hyperbolic problems as well. Since the well-posedness of such problems is a significant issue in its own right, however, we do not consider such an approach in the present work.

8. Concluding Remarks

In this article, we have extended the Finite Element Exterior Calculus of Arnold, Falk, and Winther [3, 2] for linear mixed variational problems to linear and semi-linear parabolic and hyperbolic evolution systems. Both the parabolic and hyperbolic cases make strong use of the smoothed projection operators π_h^k, which are one of the most elaborate and delicate constructions in the FEEC framework. In the parabolic case, the use of the π_h^k operators was hidden somewhat by the use of elliptic projection error estimates, proofs of which rely on properties of these operators. In the hyperbolic case, the proof techniques use these properties more explicitly. In any case, the formal treatment and generalization of these operators by Arnold, Falk and Winther can now be seen as a useful tool for the analysis of evolution problems as well as elliptic PDE.

We have also seen in this article how the recent generalizations of the FEEC by Holst and Stern [17, 18] for semi-linear elliptic PDE can be extended to evolution PDE as well, both parabolic and hyperbolic types. We also anticipate that the basic approach to analyzing variational crimes in [17, 18] for the linear and semilinar elliptic cases will also work in the case of evolution problems; we will explore the question of variational crimes in a subsequent article, with the target being the analysis of surface finite element methods for evolution problems.

Appendix A. Explanation of Semi-Linear Error Estimates

In this appendix, we explain why estimates (7.11), (7.12), and (7.13) follow from [18, Theorem 4.2]. We will focus just on the $r > 0$ case of (7.11) as it requires the sharpening of a special case of an estimate appearing in [18, Theorem 4.2]. The other cases work out along similar lines by a direct application of the Holst and Stern estimates.

First, we recall some notation from [3] used in [18]. If (W, d) is a Hilbert complex with associated domain complex (V, d) and parametrized subcomplex family (V_h, d), denote the best approximation in W-norm by

$$E(w) = \inf_{v \in V_h^k} ||w - v||_W \, , \quad w \in W^k.$$

The relevant result from Holst and Stern [18, Theorem 4.2] is stated as

$$\begin{aligned} ||u - \tilde{u}_h||_V + ||p - p_h||_W \leq c(&E(u) + E(du) + E(p) \\ &+ \eta[E(\sigma) + E(d\sigma)] + (\delta + \mu)E(d\sigma) + \mu E(P_{\mathfrak{B}} u)), \end{aligned} \tag{A.1}$$

where η, δ, and μ are coefficients defined as the norms of certain abstract operators, $u \in W_k$, and p is a harmonic k-form with discrete counterpart p_h introduced to make the abstract Hodge-Laplacian problem well-posed.

Casting this into the context of the deRham complex, we have

$$(W, d) = (L^2\Lambda, d) \quad \text{and} \quad (V, d) = (H\Lambda, d).$$

Since we are interested here only in the case $k = n$, there are no harmonic k-forms so that $p = p_h = 0$. Further, $du = 0$ since $d\Lambda^n = 0$, whereby $||u - \tilde{u}_h||_V = ||u - \tilde{u}_h||_W = ||u - \tilde{u}_h||_{L^2}$. This eliminates the error terms in p and du, giving us the reduced estimate

$$||u - \tilde{u}_h||_{L^2} \leq c(E(u) + \eta[E(\sigma) + E(d\sigma)] + (\delta + \mu)E(d\sigma) + \mu E(P_{\mathfrak{B}} u)).$$

Crucially, this estimate can be reduced further when $k = n$. The derivation of (A.1) uses the estimate

$$||d(u - \tilde{u}_h)||_W \leq c(E(du) + \eta[E(d\sigma) + E(p)]$$

from [3, Theorem 2.11] which is unnecessary here since the left side is always zero. Since this is the only part of the derivation that requires the term $\eta E(d\sigma)$, we can drop it, yielding

$$||u - \tilde{u}_h||_{L^2} \leq c(E(u) + \eta E(\sigma) + (\delta + \mu)E(d\sigma) + \mu E(P_{\mathfrak{B}} u)). \tag{A.2}$$

We now give bounds on each of the terms in (A.2). The coefficients appearing in the abstract estimates can be stated in terms of powers of h in the deRham context. These appear in [3, p. 312] as

$$\eta = O(h), \quad \delta = O(h^{\min(2, r+1)}), \quad \text{and } \mu = O(h^{r+1}).$$

To bound the error terms, Arnold, Falk and Winther define smooth projection operators $\pi_h^k : L^2\Lambda^k(\Omega) \to \Lambda_h^k$ satisfying optimal convergence rates as stated precisely in [3, Theorem 5.9]. For instance, if Λ_h^k is one of $\mathcal{P}_{r+1}^- \Lambda^k(\mathcal{T}_h)$ or, if $r \geq 1$, $\mathcal{P}_r \Lambda^k(\mathcal{T}_h)$ then

$$\left|\left|w - \pi_h^k w\right|\right|_{L^2\Lambda^k(\Omega)} \leq ch^s \, ||w||_{H^s\Lambda^k(\Omega)} \, , \quad \text{for } w \in H^s\Lambda^k(\Omega), \quad 0 \leq s \leq r + 1.$$

These types of results bound $E(w)$ in terms of $||w||_{H^s\Lambda^k}$, which is in turn bounded in terms of $||\Delta u||_{H^s\Lambda^k}$ by the regularity hypothesis (7.7). Summarizing these results, we have

$$\begin{aligned} E(u) &\leq ch^{s+2}\,||\Delta u||_{H^s} \\ E(\sigma) &\leq ch^{s+1}\,||\Delta u||_{H^s} \\ E(d\sigma) &\leq ch^{s}\,||\Delta u||_{H^s} \\ E(P_{\mathfrak{B}}u) &\leq ch^{s+2}\,||\Delta u||_{H^s} \end{aligned}$$

We can now prove (7.11) by collecting results and applying them to (A.2), yielding

$$||\rho(t)||_{L^2} \leq c(h^{s+2} + h(h^{s+1}) + (h^{\min(2,r+1)} + h^{r+1})h^s + h^{r+1}h^{s+2})\,||\Delta u(t)||_{H^s}$$

The greatest common factor from the above expression is h^{s+2} hence this is the overall order estimate that can be inferred, as was claimed.

Appendix B. Acknowledgments

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