A 3D Finite Element Solver for the Initial Value Problem

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We describe a new finite element code which solves the fully general initial-value problem using the York conformal decomposition formalism. The finite element approach allows for domains with complex topology, natural representation of boundary surfaces, and provides a solid theoretical framework for analyzing the accuracy of the resulting numerical approximations. The code employs adaptive mesh refinement based on robust *a posteriori* error estimation and simplex bisection, coupled with CG-like methods, multilevel methods, and Gummel/Newton methods. The extremely efficient placement of nodes by the adaptive refinement procedure along with the nearly optimal complexity of the multilevel methods allows these computations to be performed on workstations rather than supercomputers. As an example we compute the initial data corresponding to two "stars" in an approximately circular orbit in the presence of a gravitational wave and on a slice on which tr$K$ is non-zero.

1 The General Initial-Value Problem

We work in the ADM formalism and use the standard York conformal decomposition of the 3–metric and extrinsic curvature [1]. The initial value equations in this form appear as four coupled quasi-linear elliptic equations for the conformal factor ϕ and a vector potential W^a. In the standard notation and with the usual conformal weighting these are

$$\hat{\gamma}^{ab}\hat{D}_a\hat{D}_b\phi = \frac{1}{8}\hat{R}\phi - \frac{1}{8}\phi^{-7}(\hat{A}^*_{ab} + (\hat{l}W)_{ab})^2 + \frac{1}{12}(\mathrm{tr}K)^2\phi^5 - 2\pi\hat{\rho}\phi^{-3}, \quad (1)$$

$$\hat{D}_b(\hat{l}W)^{ab} = \frac{2}{3}\phi^6\hat{D}^a\mathrm{tr}K + 8\pi\hat{j}^a, \quad (2)$$

$$(\hat{l}W)^{ab} = \hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{3}\hat{\gamma}^{ab}\hat{D}_cW^c. \quad (3)$$

The equations have been solved numerically many times in a variety of physical settings (e.g., Mathews and Wilson [2]). However, because of the complexity of the operators and the nonlinear coupling of the equations it has been common to make various simplifying assumptions, the most frequent of which are to force $\hat{\gamma}_{ab}$ to be flat and to let the initial slice be maximal (tr$K = 0$). In addition, the majority of previous work has been done with finite differences which create well-known difficulties when the boundaries of the domain do not lie along constant coordinate surfaces or when the coordinate system has singularities.

The code which we describe here allows us to solve the equations in three dimensions without recourse to any of the above assumptions. The nature of the finite element method is such that the placement of the nodes in space is independent of the coordinate system in which the equations are written. This permits the use of a non-singular coordinate system (if one exists) which covers the entire computational domain and also makes handling boundaries with complicated geometry or topology natural and automatic. Thus, supplied with the necessary freely specifiable data and appropriate boundary conditions on ϕ and W^a, the code is, in principle, able to solve *any* initial value problem in three dimensions. (In practice, on desktop machines, we are limited to problems which require 50,000 nodes or less.)

2 Finite Element Discretization and Mesh Refinement

We employ a standard Galerkin finite element method in which the basis and test functions are taken to be piecewise polynomials with local support over disjoint polyhedral subregions of the underlying spatial domain. A weak formulation of the original strong form equations (1)–(2) is required, obtained by multiplication of the strong form equations by test functions u and v^a and integration by parts. The functions ϕ and W^a are expanded in terms of the basis, yielding linear and nonlinear algebraic equations for the expansion coefficients (matrices which occur are very sparse due to the local support property of the basis).

The underlying spatial domain is broken into disjoint polyhedra; we employ simplices (tetrahedra in this setting), and the natural piecewise-linear basis. A simplex may be divided into two (bisection) or eight (octasection) child simplices in a recursive fashion such that all progeny remain non-degenerate [3,4]. Adaptive mesh refinement consists of marking certain simplices which are deemed too large, and performing bisection or octasection to produce smaller simplices covering the same region in space. The marking procedure is driven by *a posteriori* error estimation [5]. A conforming finite element method requires a conforming mesh; a requirement for such a mesh is that it contains no hanging vertices (vertices which lie along the middle of edges of one or more simplices). Adaptive octasection always produces such hanging nodes, and must be supplemented with bisection to produce conforming meshes.

The discrete coupled nonlinear algebraic system which arises from discretizing (1)–(2) with the finite element method is solved using a damped, inexact-Newton procedure for linearization, coupled with a multilevel preconditioned conjugate gradient iteration on the normal equations for solving the resulting linear systems.

3 Numerical Results

We compute initial data for the following (admittedly artificial) problem: two spherical masses with constant density and velocity fields which put them in an approximately circular orbit. The conformal metric is an axisymmetric Brill wave with moderate amplitude and tr$K$ is a small, slightly oscillatory function which is nonzero in the neighborhood of the masses and vanishes rapidly as one moves towards the outer boundary. The masses have coordinate radius 1, central separation 8, and the outer boundary has coordinate radius 100. The following table gives results for a 200MHz Pentium Pro running Linux.

Error Indicator	Unknowns	CPU Seconds	Newton Steps	RAM
1.0e-1	516	55	18	1Mb
1.0e-2	3,312	399	24	4Mb
1.0e-3	25,032	4,385	31	19Mb
5.0e-4	48,136	13,725	43	34Mb

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