ROUGH SOLUTIONS OF THE EINSTEIN CONSTRAINT EQUATIONS ON ASYMPTOTICALLY FLAT MANIFOLDS WITHOUT NEAR-CMC CONDITIONS

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ABSTRACT. In this article we consider the conformal decomposition of Einstein's constraint equations introduced by Lichnerowicz, Choquet-Bruhat, and York, on asymptotically flat (AF) manifolds. Using the non-CMC fixed-point framework developed in 2009 by Holst, Nagy, and Tsogtgerel and by Maxwell, we combine *a priori* estimates for the individual Hamiltonian and momentum constraints, barrier constructions and fixed-point techniques for the Hamiltonian constraint, Riesz-Schauder theory for the momentum constraint, together with a topological fixed-point argument for the coupled system, to establish existence of coupled non-CMC weak solutions for AF manifolds. As was the case with the 2009 rough solution results for closed manifolds, and for the more recent 2014 results of Holst, Meier, and Tsogtgerel for rough solutions on compact manifolds with boundary, our results here avoid the near-CMC assumption by assuming that the freely specifiable part of the data given by the traceless-transverse part of the rescaled extrinsic curvature and the matter fields are sufficiently small. Using a coupled topological fixed-point argument that avoids near-CMC conditions, we establish existence of coupled non-CMC weak solutions for AF manifolds of class $W^{s,p}_{\delta}$ (or $H^{s,p}_{\delta}$) where $p \in (1, \infty)$, $s \in (1 + \frac{3}{p}, \infty)$, $-1 < \delta < 0$, with metric in the positive Yamabe class. The non-CMC rough solutions results here for AF manifolds may be viewed as an extension of the 2009 and 2014 results on rough far-from-CMC positive Yamabe solutions for closed and compact manifolds with boundary to the case of AF manifolds. Similarly, our results may be viewed as extending the recent 2014 results for AF manifolds of Dilts, Isenberg, Mazzeo and Meier; while their results are restricted to smoother background metrics and data, the results here allow the regularity to be extended down to the minimum regularity allowed by the background metric and the matter, further completing the rough solution program initiated by Maxwell and Choquet-Bruhat in 2004.

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1. Introduction

In this article, we give an analysis of the coupled Hamiltonian and momentum constraints in the Einstein equations on a 3-dimensional asymptotically flat (AF) manifold. We consider the equations with matter sources satisfying an energy condition implied by the dominant energy condition in the 4-dimensional spacetime; the unknowns are a Riemannian three-metric and a two-index symmetric tensor. The equations form an under-determined system; therefore, we focus entirely on a standard reformulation used in both mathematical and numerical general relativity, called the conformal method, introduced by Lichnerowicz [36], Choquet-Bruhat [10], and York [57, 58]. The (standard) conformal method assumes that the unknown metric is known up to a scalar field called a conformal factor, and also assumes that the trace and a term proportional to the trace-free divergence-free part of the two-index symmetric tensor is known, leaving as unknown a term proportional to the traceless symmetrized derivative of a vector. Therefore, the new unknowns are a scalar and a vector field, transforming the original under-determined system for a metric and a symmetric tensor into a (potentially) well-posed elliptic system for a scalar and a vector field, which we will refer to as the Lichnerowicz-Choquet-Bruhat-York (LCBY) system.

The LCBY equations, which are a coupled nonlinear elliptic system consisting of the scalar Hamiltonian constraint coupled to the vector momentum constraint, had been studied through 2008 almost exclusively in the setting of constant mean extrinsic curvature, known as the *CMC case*. In the CMC case the equations decouple, and it has long been known how to establish existence of solutions. The case of CMC data on closed (compact without boundary) manifolds was completely resolved by several authors over the last thirty years, with the last remaining sub-cases resolved and all the CMC sub-cases on closed manifolds summarized by Isenberg in [33]. Over the ten years that followed, other CMC cases on different types of manifolds containing various kinds of matter fields were studied and partially or completely resolved; the survey [5] gives a thorough summary of the state of the theory through about 2004. New results through 2008 included extensions of the CMC theory to AF manifolds [33, 30, 38, 12], including the first results for rough solutions [39, 41, 11]. The CMC case with interior black hole boundaries is of particular interest in numerical general relativity; solution theory for this case involves the careful mathematical treatment of trapped surface boundary conditions that model apparent horizons; this was completed by 2005 [16, 40]. Although it is the primary formulation of the constraint equations actually used in numerical relativity, the complete CMC solution theory for compact manifolds with an exterior boundary that models AF behavior, and interior trapped surface boundaries that model apparent horizons, was developed only recently [30]. Results for existence of solutions for non-constant mean extrinsic curvature, but under the assumption that the mean extrinsic curvature was nearly constant (the *near-CMC* case), began to appear in 1996 [34, 35, 12, 1]; these were essentially the only non-CMC results through 2008.

The first true *non-CMC* existence results, without any smallness requirement on τ, began to appear in 2008 [31, 32, 42]. The analysis techniques first developed and refined in [31, 32, 42] for closed manifolds were intensively studied and extended to a number of other cases over the last five years, including compact manifolds with boundary [32, 30, 17], AF manifolds without interior boundaries [18], and AF manifolds with inner trapped surface boundary conditions that model apparent horizons [29]. A variation of the fixed-point analysis from [31, 32, 42] was developed in [14], which builds on the framework to construct an associated *limit equation*; this has led to a different class of non-CMC-type results [24, 15, 19, 22, 21, 23]. One of the initially alarming implications of the topological fixed-point arguments introduced in [31, 32, 42] was the lack of uniqueness results, which had always been available in the CMC case. Rather than being a limitation in the techniques, this now appears to be *generic* when far-from-CMC data is encountered, and has even been explicitly demonstrated [43]. Moreover, analytic bifurcation analysis has now also been done for some versions of the LCBY system, and the existence of a quadratic fold with respect to certain parameterizations has now been established using those techniques [28]. One of the most important new developments in the non-CMC theory of the LCBY system has been the analysis [44] of the somewhat hidden underlying structure that is common to the primary variations of the conformal method, including the original CMC formulation [36, 10, 57, 58], the LCBY formulation [46, 47], and the conformal thin sandwich formulation [60, 49]. The analysis in [44] shows that the standard conformal method and the conformal thin sandwich method are in fact the same; in addition to allowing for the immediate transfer of known results for one method to the other method, further analysis of the structure has led to a much deeper understanding of the shortcomings of the conformal method as a parameterization of initial data [45].

In this article, our goal is to tackle one of the remaining open questions with the LCBY system: The existence of rough non-CMC solutions to the LCBY problem on AF manifolds without near-CMC assumptions. Using the overall non-CMC fixed-point framework developed for the closed case in [31, 32, 42], but now developed in the setting of the function spaces that are relevant in the AF case, we combine *a priori* estimates for the individual Hamiltonian and momentum constraints, barrier constructions and fixed-point techniques for the Hamiltonian constraint, Riesz-Schauder theory for the momentum constraint, together with a topological fixed-point argument for the coupled system, to establish existence of coupled non-CMC weak solutions for AF manifolds. As was the case for the earlier 2009 rough solutions results for closed manifolds [32, 42], and for the more recent 2014 rough solutions results of Holst, Meier, and Tsogtgerel for compact manifolds with boundary [30], our results here avoid the near-CMC assumption by assuming that the freely specifiable part of the data given by the traceless-transverse part of the rescaled extrinsic curvature and the matter fields are sufficiently small. Using a coupled topological fixed-point argument that avoids near-CMC conditions, we establish existence of coupled non-CMC weak solutions for AF manifolds of class $W^{s,p}_\delta$ (or $H^{s,p}_\delta$) where $p \in (1, \infty)$, $s \in (1+\frac{3}{p}, \infty)$, $-1 < \delta < 0$, with metric in the positive Yamabe class. The non-CMC rough solution results here for AF manifolds may be viewed as an extension of the 2009 and 2014 results on rough far-from-CMC positive Yamabe solutions for closed and compact manifolds with boundary to the case of AF manifolds. Similarly, our results may be viewed as extending the recent 2014 results for AF manifolds of Dilts, Isenberg, Mazzeo and Meier [18]; while their results are restricted to smoother background metrics and data, the results here allow the regularity to be extended down to the maximum allowed by the background metric and the matter, further completing the rough solution program initiated initiated by Maxwell in [39, 41] (see also [11]), and thus further extending the known solution theory for the Einstein constraint equations.

A Brief Remark Concerning the Results Contained in the Paper. Along the way to proving the main existence result in the paper, we will need to assemble a number of new supporting technical results; we include some of these results in the main body of the paper when needed to maintain the flow of an argument, whereas it was possible to place other supporting results into the included appendices without damaging the flow of the main arguments. One of the technical results we need, which is not available in the literature, concerns cases of multiplication properties of functions in weighted spaces. While the limited version of the result needed for this paper is included, this has led to a second project on establishing some multiplication lemmas that are not yet in the literature. These results will appear in [6]. Lastly, note that we have included the complete bootstrapping argument that has been only outlined in prior articles (including some of our own) for obtaining the higher-smoothness results from the rough results. This argument is in fact somewhat non-trivial, and we felt that it should be included somewhere in the literature on the conformal method.

Outline of the Paper. An extended outline of the remainder of the paper is as follows.

In Section 2, we give a brief overview of the conformal method, and introduce notation that we use throughout the paper. We summarize the conformal decomposition of Einstein's constraint equations introduced by Lichnerowicz and York, on an AF manifold, and describe the classical strong formulation of the resulting coupled elliptic system.

In Section 3, and we define weak formulations of the constraint equations that will allow us to develop solution theories for the constraints in the spaces with the weakest possible regularity. In particular, we focus on one of two possible weak formulations of the LCBY equations; a second alternative, which has some advantages but which we do not use in the main body of the paper, is described in Appendix F.

In Section 4, we study the momentum constraint in isolation from the Hamiltonian constraint. We develop some basic technical results for the momentum constraint operator under the weakest possible assumptions on the problem data, including existence of weak solutions to the momentum constraint, given the conformal factor as data.

In Section 5, we study the individual Hamiltonian constraint. We assume the existence of barriers (weak sub- and super-solutions) to the Hamiltonian constraint equation forming a nonempty positive bounded interval, and then derive several properties of the Hamiltonian constraint that are needed in the analysis of the coupled system. The results are established under the weakest possible assumptions on the problem data.

In Section 6, we develop a new approach for the construction of global sub- and supersolutions for the Hamiltonian constraint on AF manifolds. In particular, we give constructions for both sub- and supersolutions in the positive Yamabe case that have several key features, including: (1) they are near-CMC free; (2) they require minimal assumptions on the data in order to be used for developing rough solutions; and (3) they have appropriate asymptotic behavior to be compatible with an overall fixed-point argument for the coupled system.

Finally, in Section 7 we develop our main results for the coupled system. In particular, we clearly state and then prove the main existence result (Theorem 7.3) for rough positive Yamabe solutions to the constraint equations on AF manifolds without near-CMC assumptions.

For ease of exposition, various supporting technical results are given in several appendices as follows: Appendix §A – construction of fractional order Sobolev spaces of sections of vector bundles over AF manifolds; Appendix §B – *a priori* estimates and related results for elliptic operators on AF manifolds; Appendix §C – artificial conformal covariance of the Hamiltonian constraint on AF manifolds; Appendix §D – results on Yamabe positive metrics on AF manifolds; Appendix §E – some remarks on the alternative use of Bessel Potential spaces; and Appendix §F – an alternative weak formulation of the LCBY system on AF manifolds that makes possible additional results that are not developed in the paper.

2. Preliminary Material

We give a brief overview of the Einstein constraint equations and the conformal method. A more detailed overview can be found in [32, 5]. Let $(\mathcal{M}, g_{\mu\nu})$ be a 4-dimensional globally hyperbolic spacetime, that is, $\mathcal{M}$ is a 4-dimensional smooth manifold, $g_{\mu\nu}$ is smooth, Lorentzian metric on $\mathcal{M}$ with signature $(-,+,+,+)$ and $\mathcal{M}$ admits a Cauchy surface (so it can be foliated by a family of spacelike hypersurfaces). Let ∇_μ be the Levi-Civita connection associated with the metric $g_{\mu\nu}$. The Einstein field equation is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu},$$

where $T_{\mu\nu}$ is the stress-energy tensor, and $\kappa = 8\pi G/c^4$, with G the gravitation constant and c the speed of light. The Ricci tensor is $R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma}$ and $R = R_{\mu\nu} g^{\mu\nu}$ is the Ricci scalar, where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, that is $g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu{}^\nu$. The Riemann tensor is defined by $R_{\mu\nu\sigma}{}^{\rho} w_\rho = \big(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\big) w_\sigma$, where w_μ is any 1-form on M. The stress energy tensor $T_{\mu\nu}$ is assumed to be symmetric and to satisfy the condition $\nabla_\mu T^{\mu\nu} = 0$ and the **dominant energy condition**, that is, the vector $-T^{\mu\nu} v_\nu$ is time-like and future-directed, where v^μ is any time-like and future-directed vector field.

The Einstein field equation allows a formulation as an initial value problem. The metric is the fundamental variable and the equations involve the second derivatives of the metric. Roughly speaking, since the equation is of order two in time, in order to solve the problem, we need initial data on the metric and on a first order time derivative of the metric. In the case of a globally hyperbolic spacetime, which supports a complete foliation with space-like hypersurfaces parameterized by a scalar time function, one can pick a constant time hypersurface of the spacetime Σ and then specify the initial data $(g|_\Sigma = \hat{h}, \dfrac{\partial g}{\partial t}|_\Sigma \sim \hat{k})$ on that hypersurface [9]. The problem then becomes that one is not allowed to freely specify the initial conditions in that hypersurface; rather, the Gauss-Codazzi-Menardi equations imply that the initial data satisfy certain conditions which are known as *constraint equations* [13]. More precisely, we have the following definition:

Definition 2.1. *A triple $(M, \hat{h}, \hat{k})$ is said to be an initial data set for the Cauchy formulation of the Einstein field equations iff $(M, \hat{h})$ is a 3-dimensional smooth Riemannian manifold and $\hat{k}$ is a symmetric covariant tensor of order 2 on M such that*

$$\hat{R} - |\hat{k}|^2_{\hat{h}} + (tr_{\hat{h}} \hat{k})^2 = 2\kappa\hat{\rho}, \qquad \textit{(Hamiltonian constraint)}$$
$$\operatorname{div}_{\hat{h}} \hat{k} - d(tr_{\hat{h}} \hat{k}) = \kappa \hat{J}, \qquad \textit{(Momentum constraint)}$$

where $\hat{R}$ is the scalar curvature of $\hat{h}$, and where $\hat{\rho}$ is a non-negative scalar field and $\hat{J}$ is a 1 form on M, representing the energy and momentum densities of the matter and non-gravitational fields, respectively. κ is a constant.

The above equations are called the **Einstein constraint equations**. Using any local frame we may write the above equations as follows:

$$\hat{R} + (\hat{h}^{ab}\hat{k}_{ab})^2 - \hat{k}_{ab}\hat{k}^{ab} = 2\kappa\hat{\rho},$$
$$\hat{\nabla}^b(\hat{h}^{ac}\hat{k}_{ac}) - \hat{\nabla}_a \hat{k}^{ab} = -\kappa \hat{J}^b, \quad 1 \le b \le 3.$$

When the above equations hold, the manifold M can be embedded as a hypersurface in a 4-dimensional manifold corresponding to a solution of the Einstein field equations, and the push forward of $\hat{h}$ and $\hat{k}$ represent the first and second fundamental forms of the embedded hypersurface. This leads to the terminology **extrinsic curvature** for $\hat{k}$, and **mean extrinsic curvature** for its trace $tr_{\hat{h}}\hat{k}$. If the source terms are zero ($\hat{\rho}=0$, $\hat{J}=0$), the constraint equations are called the **vacuum** constraint equations.

A general statement of the problem we are interested in is as follows.

The Initial Data Problem in GR: *Given a* 3*-dimensional smooth manifold* M*, a scalar function* $\hat{\rho}$ *and a vector valued function (or 1 form)* $\hat{J}$*, find a Riemannian metric (a symmetric, positive definite covariant tensor* $\hat{h}$ *of order* 2*), and a symmetric covariant tensor* $\hat{k}$ *of order 2, such that the triple* $(M,\hat{h},\hat{k})$ *forms an initial data set for the Einstein constraint equations (i.e., such that* $(\hat{h},\hat{k})$ *satisfies the constraint equations).*

The constraint equations constitute an under-determined system of equations (the number of unknowns is twelve, whereas the number of equations is four). In order to produce a unique solution we must specify certain unknowns and then solve the constraint equations for the remaining unknowns. To this end, we employ a standard reformulation known as the **conformal transverse-traceless method**, introduced by Lichnerowicz, York, and O Murchadha [36, 59, 46]. In this method, the initial data on M is divided into two sets: the *Free (Conformal) Data*, and the *Determined Data*, such that given a choice of free data, the constraint equations become a **determined** system to be solved for the determined data [5]. There are several ways to do this; here we focus on the "semi-decoupling split", and examine briefly how the method works.

- **Step 1:** The original unknowns, $\hat{h}$ and $\hat{k}$, each has six distinct components, therefore we have twelve unknowns. We can decompose $\hat{k}_{ab}$ into the trace-free and the pure trace parts:

$$\hat{k}^{ab}=\hat{s}^{ab}+\frac{1}{3}(tr_{\hat{h}}\hat{k})\hat{h}^{ab}.$$

  Clearly $tr_{\hat{h}}\hat{s}=0$.

- **Step 2: Conformal rescaling.** Let

$$\hat{h}_{ab}=\phi^r h_{ab},\quad \hat{s}^{ab}=\phi^s s^{ab},\quad tr_{\hat{h}}\hat{k}=\phi^t\tau,$$

  where r, s, and t are fixed but arbitrary integers. Note that if $t=0$ then τ is the mean extrinsic curvature. We denote the Levi-Civita connection for h_{ab} by ∇_a. We will assume h_{ab} and τ are given (i.e we consider them as free data) so we are left with 7 unknowns (components of s_{ab} and ϕ).

- **Step 3: York decomposition.** We begin by first defining the *conformal Killing operator* $\mathcal{L}_h:\chi(M)\to\tau_2^0(M)$ as follows:

$$\mathcal{L}_h(W)=\nabla^b W^a+\nabla^a W^b-\frac{2}{3}(\mathrm{div}_h W)h^{ab}\quad(\mathrm{div}_h W=\nabla_c W^c).$$

  Here $\chi(M)$ denotes the collection of vector fields on M and $\tau_2^0(M)$ is the collection of contravariant tensors of order 2. The elements in the kernel of $\mathcal{L}_h$ are called *conformal Killing fields*. In the case where the background metric is clear from the context we may denote the conformal Killing operator by $\mathcal{L}$ instead of $\mathcal{L}_h$. In particular, in what follows ∇, $\mathcal{L}$ and div are all taken with respect to the metric h. For closed manifolds and AF manifolds, under mild conditions on the regularity of h, one can show that if ψ is a symmetric traceless contravariant tensor of order 2, then there exists $W\in\chi(M)$, uniquely determined up to conformal Killing fields, such that $\mathrm{div}(\mathcal{L}W)=\mathrm{div}\psi$ [33, 41]. $\mathrm{div}\mathcal{L}$ is sometimes called *vector Laplacian* and is denoted by Δ_L. Therefore, there exists $W \in \chi(M)$ such that

$$\Delta_L W := \mathrm{div}(\mathcal{L}W) = \mathrm{div}s \quad (\nabla_c(\mathcal{L}W)^{ac} = \nabla_c s^{ac}).$$

Now define $\sigma^{ab} := s^{ab} - (\mathcal{L}W)^{ab}$. Clearly, $\mathrm{div}\sigma = 0$. It is easy to check that σ is trace-free as well. So in fact σ is a *transverse-traceless* tensor.

- **Step 4:** We assume σ^{ab} is given, i.e, we will consider it as part of the free data; now we are left with four unknowns (components of the vector field W^a and the scalar function ϕ).

Therefore, the set of free (conformal) data consists of a background Riemannian metric h, a transverse-traceless symmetric tensor σ, and a function τ. The set of determined data consists of a positive function ϕ and a vector field W. The transformed system consists of the *Lichnerowicz-Choquet-Bruhat-York (LCBY) equations*. For the semi-decoupling split we set $r = 4$, $s = -10$, $t = 0$. $\hat{\rho}$ is a non-negative scalar field and $\hat{J}$ is a 1 form on M, representing When energy and momentum densities of matter and non-gravitational fields are present, one also takes $\rho = \phi^8\hat{\rho}$ and $J^b = \phi^{10}\hat{J}^b$.

The conformal formulation of the Einstein constraint equations. Applying the conformal method by following **Steps 1–4** above, one produces a coupled nonlinear elliptic system for the unknown conformal factor $\phi \in C^\infty(M)$ and $W \in \chi(M)$:

$$-8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - [\sigma_{ab} + (\mathcal{L}W)_{ab}][\sigma^{ab} + (\mathcal{L}W)^{ab}]\phi^{-7} = 2\kappa\rho\phi^{-3}, \tag{2.1}$$

$$-\nabla_a(\mathcal{L}W)^{ab} + \frac{2}{3}\phi^6\nabla^b\tau = -\kappa J^b, \tag{2.2}$$

where the first equation (2.1) is referred to as the **conformal formulation of the Hamiltonian constraint**, and the second equation (2.2) is referred to as the **conformal formulation of the momentum constraint**. In the vacuum case, the right-hands sides of both equations vanish.

In order to give a complete and well-defined mathematical formulation of the problem our study here, we begin by setting

$$F(\phi, W) = a_R\phi + a_\tau\phi^5 - a_W\phi^{-7} - a_\rho\phi^{-3}, \quad \mathbb{F}(\phi) = b_\tau\phi^6 + b_J,$$

where

$$b_\tau^b = (2/3)\nabla^b\tau, \quad b_J^b = \kappa J^b, \quad a_R = R/8, \quad a_\tau = \tau^2/12,$$
$$a_\rho = \kappa\rho/4, \quad a_W = [\sigma_{ab} + (\mathcal{L}W)_{ab}][\sigma^{ab} + (\mathcal{L}W)^{ab}]/8.$$

The **classical formulation** of the LCBY equations can be stated as follows.

Classical Formulation. *Given smooth functions τ and ρ, rank 2 transverse-traceless tensor field σ, and vector field J on the smooth 3-dimensional Riemannian manifold (M, h), find a scalar field $\phi > 0$ in $C^\infty(M)$ and a vector field W in $\chi(M)$ such that*

$$-\Delta\phi + F(\phi, W) = 0,$$
$$-\Delta_L W + \mathbb{F}(\phi) = 0.$$

As motivated clearly in the introduction, our goal in this article is to provide an answer to the question of existence of non-CMC solutions in the case of AF manifolds with very low regularity assumptions on the data. Our approach follows closely that taken in [32], and is based on the following fundamental ideas:

- **Abstract interpretation of the differential equation**: We interpret any PDE as an equation of the form $Au = f$ where A is an operator between suitable function spaces. In this view, the existence of a unique solution for all f is equivalent to A being bijective. This abstract interpretation allows one to employ a number of general results from linear and nonlinear analysis.

- **Conformal covariance of the Hamiltonian constraint**: The basic idea is that in the study of existence of solutions to the Hamiltonian constraint, we have some sort of freedom in the choice of the background metric h. Note that the coefficient a_W is the only part of the Hamiltonian constraint that depends on the solution of the momentum constraint. Let's consider the individual Hamiltonian constraint by assuming that a_W is given as data. Clearly, the Hamiltonian constraint depends on the background metric h: the differential operator in the Hamiltonian constraint is the Laplacian which is defined using h, and also the scalar curvature R is with respect to h. An important question that one may ask is "does the existence of solution depend on the background metric h"? More specifically, if h and $\tilde{h}$ are two conformally equivalent metrics, does the existence of solution for $\tilde{h}$ imply the existence of solution for h? The answer for the general "non-CMC" case is, unfortunately, a resounding "NO" [44]. However, the situation is not completely hopeless. We examine this at length in Appendix C, and we show that one can *artificially* define a_W, a_ρ and other coefficients in the Hamiltonian constraint with respect to the new conformally equivalent metric in such a way that some sort of connection is made between the two equations. This generalized type of conformal covariance is enough for our purposes here. This should not be confused with the genuine (geometric) conformal covariance that is true for the CMC case, and is discussed in [44]. For both CMC case and non-CMC case (as discussed in Appendix C), in the study of existence of solutions to the Hamiltonian constraint, one may perform a conformal transformation and use a metric in the conformal class whose scalar curvature has "nice" properties. This is exactly why the **Yamabe classes** play an important role in the study of constraint equations.

- **Fredholm alternative:** If A is a "nice" linear operator (in this context, meaning Fredholm of index zero), then uniqueness implies existence.

- **Maximum Principle:** A linear operator A satisfies the maximum principle if $Au \leq 0$ implies $u \leq 0$ in some suitable pointwise sense. If A satisfies the maximum principle then the solution of $Au = f$ (if it exists) is unique.

- **Sub- and Supersolutions and A Priori Estimates:** Consider the equation $-\Delta\phi + G(\phi) = 0$ where G is a given function. Functions ϕ_+ and ϕ_- satisfying

$$-\Delta\phi_+ + G(\phi_+) \geq 0, \quad -\Delta\phi_- + G(\phi_-) \leq 0$$

  are called a *supersolution* and *subsolution*, respectively. One can show that under certain conditions the existence of super- and subsolutions implies the existence of a solution ϕ to the PDE.

- **Fixed Point Theorems:** (in particular the contraction mapping and Schauder theorems) We may reduce the problem of existence of solutions to the problem of existence of fixed points of suitably defined operators.

- **The Implicit Function Theorem:** Although we do not use the implicit function theorem in this paper, it is important to know that the implicit function theorem can be used in several different ways to prove existence of solutions. For instance in [12] this theorem has been used to prove the existence of solutions of the coupled constraint equations near a given one. Also the "Continuity Method", which for instance is used in [8] to study the constraint equations, usually makes use of the implicit function theorem. The basic idea of the continuity method is as follows: Let $\Phi(u) = 0$ be the equation to solve. The Continuity Method consists of the following three steps [3, 25]:

- **Step 1:** Find a continuous family of functions Φ_τ with $\tau \in [0,1]$, such that $\Phi_1(u) = \Phi(u)$ and $\Phi_0(u) = 0$ is a known equation which has a solution u_0.
- **Step 2:** Prove that the set $J = \{\tau \in [0,1] : \Phi_\tau(u) = 0 \,\text{has a solution}\}$ is open. To show this, the Implicit Function Theorem is typically used.
- **Step 3:** Prove that the set J is closed.

Therefore J is a nonempty subset of $[0,1]$ that is both open and closed. This means $J = [0,1]$ and in particular $1 \in J$.

The main difficulty is in finding the appropriate function spaces as the domain and codomain of the differential operator A, and ensuring that by using those function spaces we are allowed to apply the maximum principle, Fredholm theory, fixed point theorems, and so forth. For elliptic equations on the whole space $\mathbb{R}^n$ (and also for AF manifolds), the appropriate spaces are weighted Sobolev spaces. To make for a reasonably self-contained article, a summary of the main properties of weighted Sobolev spaces, and differential operators between such spaces, has been included in Appendices A and B.

We note that although the situation that we study in this article is more complicated, the main ideas which are employed to prove the theorems, mostly follow those which have been used in [32] (non-CMC case on closed manifolds) and [38, 41] (CMC case on AF manifolds).

Notation. Throughout this paper we use the standard notations for Sobolev spaces. See Appendix §A for a summary of the standard notation we use here for norms. We use the notation $A \preceq B$ to mean $A \leq cB$ where c is a positive constant that does not depend on the non-fixed parameters appearing in A and B.

3. Weak Formulation on Asymptotically Flat Manifolds

First let us precisely define what we mean by an asymptotically flat manifold.

Definition 3.1. *Let M be an n-dimensional smooth connected manifold and let h be a metric on M for which (M,h) is complete. Let $E_r = \{x \in \mathbb{R}^n : |x| > r\}$. We say (M,h) is asymptotically flat (AF) of class $W^{s,p}_\delta$ (where $s \in \mathbb{R}$, $p \in (1,\infty)$, and $\delta < 0$) if*

(1) $h \in W^{s,p}_{loc}(M)$.

(2) There is a finite collection $\{U_i\}_{i=1}^m$ of open sets of M and diffeomorphisms $\phi_i : U_i \to E_1$ such that $M \setminus (\cup_{i=1}^m U_i)$ is compact.

(3) There exists a constant $\vartheta \geq 1$ such that for each i

$$\forall\, x \in E_1 \;\forall y \in \mathbb{R}^n \quad \vartheta^{-1}|y|^2 \leq ((\phi_i^{-1})^* h)_{rs}(x) y^r y^s \leq \vartheta |y|^2. \quad (\textit{see Remark A.10})$$

(4) There exists a positive constant ω such that for each i, $(\phi_i^{-1})^* h - \omega \bar{h} \in W^{s,p}_\delta(E_1)$, where $\bar{h}$ is the Euclidean metric.

The charts (U_i, ϕ_i) are called end charts*, and the corresponding coordinates are called* end coordinates.

Our goal is to come up with a weak formulation of LCBY conformally rescaled Einstein constraint equations in order to accommodate nonsmooth data on an AF manifold (M,h) of class $W^{s,p}_{\delta}$. There are at least two different general settings where the constraint equations are well-defined with rough data; one of them is described in this section and the other is discussed in Appendix F. In both settings it is assumed that the AF manifold is of class $W^{s,p}_{\delta}$ where $s>\frac{3}{p}$ (and of course $p\in(1,\infty)$, $\delta<0$). So by Corollary A.27, $W^{s,p}_{\delta}$ is a Banach algebra and $W^{s,p}_{\delta}\hookrightarrow C^0_{\delta}\hookrightarrow L^{\infty}_{\delta}$. The framework that is described in Appendix F (which we refer to as **Weak Formulation 2**) only works for $s\le 2$, but the framework that is described in this section (which we refer to as **Weak Formulation 1**) works for all $s>3/p$ with $p\in(1,\infty)$, even when $s>2$. (However, as we explain at some length in Appendix F, **Weak Formulation 2** is not simply a special case of **Weak Formulation 1**.)

Note that if (M,h) is a 3-dimensional AF manifold of class $W^{s,p}_{\delta}$ and if $u\in W^{s,p}_{\delta}$ is a positive function, then (M,u^4h) is not asymptotically flat of class $W^{s,p}_{\delta}$ (item (4) in Definition 3.1 is not satisfied). However, if u is a positive function such that $u-\mu\in W^{s,p}_{\delta}(M)$ for some positive constant μ, then (M,u^4h) is also AF of class $W^{s,p}_{\delta}$. Indeed,

$$u-\mu\in W^{s,p}_{\delta}\Rightarrow u-\mu\in W^{s,p}_{loc}\Rightarrow u\in W^{s,p}_{loc}\Rightarrow u^4h\in W^{s,p}_{loc}\quad (W^{s,p}_{loc}\text{ is an algebra}).$$

In addition, $u-\mu\in W^{s,p}_{\delta}$ implies that u is bounded and $\inf u>0$ (see Remark 5.2; note that u is a positive function). Thus, there exists a positive number ζ such that $\zeta^{-1}<u^4<\zeta$. Consequently for each i, $(\phi_i^{-1})^*u^4=u^4\circ\phi_i^{-1}$ is between ζ^{-1} and ζ which subsequently implies that

$$\forall\, x\in E_1\ \forall y\in\mathbb{R}^n\quad (\zeta\vartheta)^{-1}|y|^2\le((\phi_i^{-1})^*(u^4h))_{rs}(x)y^ry^s\le(\zeta\vartheta)|y|^2.$$

Finally, since (M,h) is AF of class $W^{s,p}_{\delta}$, there exists a constant ω such that $(\phi_i^{-1})^*h-\omega^4\bar{h}\in W^{s,p}_{\delta}(E_1)$; if we let $v=u-\mu$ and $f(x)=(\mu+x)^4$, then for each $1\le i\le m$ (end coordinates) we have

$$\begin{aligned}
(\phi_i^{-1})^*(u^4h)-(\mu\omega)^4\bar{h}&=(u^4\circ\phi_i^{-1})(\phi_i^{-1})^*h-(\mu\omega)^4\bar{h}\\
&=(\mu+v\circ\phi_i^{-1})^4(\phi_i^{-1})^*h-(\mu\omega)^4\bar{h}\\
&=f(v\circ\phi_i^{-1})(\phi_i^{-1})^*h-(\mu\omega)^4\bar{h}\\
&=f(v\circ\phi_i^{-1})((\phi_i^{-1})^*h-\omega^4\bar{h})+(\omega^4f(v\circ\phi_i^{-1})-(\mu\omega)^4)\bar{h}.
\end{aligned}$$

Since $(\phi_i^{-1})^*h-\omega^4\bar{h}\in W^{s,p}_{\delta}(E_1)$, by Lemma A.28 the first term on the right is in $W^{s,p}_{\delta}(E_1)$. Also as a direct consequence of Corollary A.30, the second term on the right is in $W^{s,p}_{\delta}(E_1)$.

In the LCBY equations, $\phi>0$ is the conformal factor, so assuming (M,h) is a 3-dimensional AF manifold of class $W^{s,p}_{\delta}$, by what was mentioned above, it seems reasonable to let $\phi=\psi+\mu$ (so $\psi>-\mu$) where $\psi\in W^{s,p}_{\delta}$ and μ is an arbitrary but fixed positive constant (we have freedom in choosing the constant μ).

We can write the Hamiltonian constraint in terms of ψ as:

$$-\Delta\psi+f(\psi,W)=0,$$

where

$$\begin{aligned}
f(\psi,W)&=F(\phi,W)\\
&=a_R\phi+a_{\tau}\phi^5-a_W\phi^{-7}-a_{\rho}\phi^{-3}\\
&=a_R(\psi+\mu)+a_{\tau}(\psi+\mu)^5-a_W(\psi+\mu)^{-7}-a_{\rho}(\psi+\mu)^{-3}.
\end{aligned}$$

Since $\psi \in W^{s,p}_{\delta}$, we want to be able to extend $-\Delta : C^{\infty} \to C^{\infty}$ to an operator $A_L : W^{s,p}_{\delta} \to W^{s-2,p}_{\delta-2}$. As discussed in Appendix B, since $-\Delta \in D^{s,p}_{2,\delta}$ (See Definition B.1), by the extension theorem (Theorem B.2), the only extra assumption needed to ensure the above extension is possible is $s \geq 1$. Indeed, according to Theorem B.2, we must check the following conditions (following the numbering used in Theorem B.2):

$$\begin{array}{rll} \text{(i)} & p \in (1,\infty), & \text{(true by assumption)} \\ \text{(ii)} & s \geq 2-s, & \text{(so need to assume } s \geq 1) \\ \text{(iii)} & s-2 \leq s-2, & \text{(trivially true)} \\ \text{(iv)} & s-2 < s-2+s-\frac{3}{p}, & \text{(since } s > \frac{3}{p}) \\ \text{(v)} & s-2-\frac{3}{p} \leq s-\frac{3}{p}-2, & \text{(trivially true)} \\ \text{(vi)} & s-\frac{3}{p} > 2-3-s+\frac{3}{p}. & \text{(since } s > \frac{3}{p}) \end{array}$$

Framework 1:
In this framework we look for W in $\mathbf{W}^{e,q}_{\beta}$ where $\beta < 0$. For the momentum constraint to be well-defined, we need to ensure that

$$\text{The operator: } -\Delta_L : \mathbf{C}^{\infty} \to \mathbf{C}^{\infty} \text{ can be extended to } \mathcal{A}_L : \mathbf{W}^{e,q}_{\beta} \to \mathbf{W}^{e-2,q}_{\beta-2}, \tag{3.1}$$

$$\text{It holds that: } b_{\tau}(\psi+\mu)^6 + b_J \in \mathbf{W}^{e-2,q}_{\beta-2}. \tag{3.2}$$

The vector Laplacian belongs to the class $D^{s,p}_{2,\delta}$ (See Definition B.1). Therefore, by Theorem B.2, in order to ensure that condition (3.1) holds true, it is enough to require e and q satisfy the following conditions (again, the numbering below corresponds to numbering in Theorem B.2):

$$\begin{array}{rll} \text{(i)} & q \in (1,\infty), & \\ \text{(ii)} & e \geq 2-s, & \\ \text{(iii)} & e-2 \leq \min\{e,s\}-2,\ p \leq q \text{ if } e = s \notin \mathbb{N}_0, & \text{(in particular, need } e \leq s) \\ \text{(iv)} & e-2 < e-2+s-\frac{3}{p}, & \text{(holds by assumption } s > \frac{3}{p}) \\ \text{(v)} & e-2-\frac{3}{q} \leq s-\frac{3}{p}-2, & \text{(must assume } e \leq s+\frac{3}{q}-\frac{3}{p}) \\ \text{(vi)} & e-\frac{3}{q} > 2-3-s+\frac{3}{p}. & \text{(must assume } e > -s+\frac{3}{p}-1+\frac{3}{q}) \end{array}$$

Combining these constraints, we see it is enough to have

$$\begin{aligned} & q \in (1,\infty), \\ & e \in (2-s, s] \cap (-s+\frac{3}{p}-1+\frac{3}{q}, s+\frac{3}{q}-\frac{3}{p}]. \quad (p = q \text{ if } e = s \notin \mathbb{N}_0) \end{aligned}$$

Note that in case $e = s \notin \mathbb{N}_0$ we need to assume $p \leq q$, which together with the inequality $s = e \leq s + \frac{3}{q} - \frac{3}{p}$ justifies the assumption $p = q$ in this case.

In order to ensure that condition (3.2) holds true, it is enough to make the extra assumptions that τ is given in $W^{e-1,q}_{\beta-1}$ and J is given in $\mathbf{W}^{e-2,q}_{\beta-2}$. Indeed, note that $\tau \in W^{e-1,q}_{\beta-1}$ implies $b_{\tau} \in \mathbf{W}^{e-2,q}_{\beta-2}$. Since $\psi \in W^{s,p}_{\delta}$, it follows from Lemma A.28 that $b_{\tau}(\psi+\mu)^6 \in \mathbf{W}^{e-2,q}_{\beta-2}$; Lemma A.28 can be applied since (numbering corresponds to the numbering of conditions in Lemma A.28):

$$\begin{array}{rll} \text{(i)} & e-2 \in [-s, s], & \text{(since } e \in (2-s, s]) \\ \text{(ii)} & e-2-\frac{3}{q} \leq s-\frac{3}{p}, & \text{(since } e \leq s+\frac{3}{q}-\frac{3}{p}) \\ & -3-s+\frac{3}{p} \leq e-2-\frac{3}{q}. & \text{(since } e > -s+\frac{3}{p}-1+\frac{3}{q}) \end{array}$$

In summary, for the momentum constraint to be well-defined, it is enough to make the following additional assumptions:

$$q \in (1,\infty), \quad e \in (2-s,s]\cap(-s+\frac{3}{p}-1+\frac{3}{q}, s+\frac{3}{q}-\frac{3}{p}], \quad \tau \in W^{e-1,q}_{\beta-1}, \quad J \in \mathbf{W}^{e-2,q}_{\beta-2}.$$

Of course, we let $p = q$ if $e = s \notin \mathbb{N}_0$, and the base assumptions hold as well ($s \geq 1$, $p \in (1,\infty)$, $\delta, \beta < 0$, $s > \frac{3}{p}$). Note that for $(2-s,s]$ to be nonempty, in fact we need $s > 1$.

Finally, we now consider the Hamiltonian constraint. Note that $W \in \mathbf{W}^{e,q}_{\beta}$ and so that $\mathcal{L}W \in W^{e-1,q}_{\beta-1}$. For $a_W = \frac{1}{8}|\sigma + \mathcal{L}W|^2$ to be well-defined, it is enough to assume $\sigma \in W^{e-1,q}_{\beta-1}$. Recall that A_L is a well-defined operator from $W^{s,p}_{\delta}$ to $W^{s-2,p}_{\delta-2}$. If we set $\eta = \max\{\beta,\delta\}$, then $W^{s-2,p}_{\delta-2} \hookrightarrow W^{s-2,p}_{\eta-2}$. In fact, A_L can be considered as an operator from $W^{s,p}_{\delta}$ to $W^{s-2,p}_{\eta-2}$ where $\eta = \max\{\beta,\delta\}$. Consequently, for the Hamiltonian constraint to be well-defined, we need to have

$$f(\psi, W) = a_R(\psi+\mu) + a_\tau(\psi+\mu)^5 - a_W(\psi+\mu)^{-7} - a_\rho(\psi+\mu)^{-3} \in W^{s-2,p}_{\eta-2}.$$

One way to guarantee that the above statement holds true is to ensure that

$$a_\tau, a_\rho, a_W \in W^{s-2,p}_{\beta-2}, \quad a_R \in W^{s-2,p}_{\delta-2},$$

and then show that if f is smooth on $(-\mu,\infty)$, $u \in W^{s,p}_{\delta}$, and $v \in W^{s-2,p}_{\eta-2}$ then $f(u)v \in W^{s-2,p}_{\eta-2}$. We claim that for above statements to be true it is enough to make the following extra assumptions:

$$e > 1+\frac{3}{q}, \quad e \geq s-1, \quad e \geq \frac{3}{q}+s-\frac{3}{p}-1, \quad \rho \in W^{s-2,p}_{\beta-2}.$$

The details are as follows:

(1) If f is smooth and $u \in W^{s,p}_{\delta}$, $v \in W^{s-2,p}_{\eta-2}$ then $f(u)v \in W^{s-2,p}_{\eta-2}$.
By Lemma A.28, we just need to check the following (the numbering matches that of the conditions in Lemma A.28):

(i)	$s-2 \in [-s,s]$,	(since $s \geq 1$)
(ii)	$s-2-\frac{3}{p} \in [-3-s+\frac{3}{p}, s-\frac{3}{p}]$.	(since $s > \frac{3}{p}$)

This shows that $f(u)v \in W^{s-2,p}_{\eta-2}$.

(2) $a_\tau = \frac{1}{12}\tau^2$.
We want to ensure $a_\tau \in W^{s-2,p}_{\beta-2}$. Note that $\tau \in W^{e-1,q}_{\beta-1}$; since $e-1 > \frac{3}{q}$, $W^{e-1,q}_{\beta-1} \times W^{e-1,q}_{\beta-1} \hookrightarrow W^{e-1,q}_{2\beta-2}$ (see Corollary A.26). Therefore $\tau^2 \in W^{e-1,q}_{2\beta-2}$. Thus we want to have $W^{e-1,q}_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$. We will see that because of the assumptions $e \geq s-1$ and $e \geq \frac{3}{q}+s-\frac{3}{p}-1$ this embedding holds true. We just need to check that the assumptions of Theorem A.17 are satisfied (numbering follows assumptions of Theorem A.17):

(ii)	$e-1 \geq s-2$,	(since $e \geq s-1$)
(iii)	$e-1-\frac{3}{q} \geq s-2-\frac{3}{p}$,	(since $e \geq \frac{3}{q}+s-\frac{3}{p}-1$)
(iv)	$2\beta-2 < \beta-2$.	(since $\beta < 0$)

(3) $a_R = \frac{R}{8}$.
We want to ensure $a_R \in W^{s-2,p}_{\delta-2}$. Note that h is an AF metric of class $W^{s,p}_{\delta}$ and R involves the second derivatives of h, so $R \in W^{s-2,p}_{\delta-2}$. We do not need to impose any extra restrictions for this one.

(4) $a_\rho = \kappa\rho/4$.
Clearly $a_\rho \in W^{s-2,p}_{\beta-2}$ iff $\rho \in W^{s-2,p}_{\beta-2}$.

(5) $a_W = [\sigma_{ab} + (\mathcal{L}W)_{ab}][\sigma^{ab} + (\mathcal{L}W)^{ab}]/8$.
We want to ensure that $a_W \in W^{s-2,p}_{\beta-2}$. Note that $\mathcal{L}W, \sigma \in W^{e-1,q}_{\beta-1}$ and by our restrictions on e, $W^{e-1,q}_{\beta-1} \times W^{e-1,q}_{\beta-1} \hookrightarrow W^{e-1,q}_{2\beta-2}$ and $W^{e-1,q}_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$. Thus, we have $a_W = \frac{1}{8}|\sigma + \mathcal{L}W|^2 \in W^{e-1,q}_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$.

We are finally in a position to give a well-defined weak formulation of the Einstein constraint equations on AF manifolds with rough data, through the use of **Framework 1**. (In Appendix F, we show how **Framework 2** leads to an alternative weak formulation, leading to slightly different existence results.)

Weak Formulation 1. *Let (M, h) be a 3-dimensional AF Riemannian manifold of class $W^{s,p}_\delta$ where $p \in (1,\infty)$, $\beta, \delta < 0$ and $s \in (1 + \frac{3}{p}, \infty)$. Select q and e to satisfy*

$$\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}],$$
$$e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}].$$

Let $q = p$ if $e = s \notin \mathbb{N}_0$. Fix source functions:

$$\tau \in W^{e-1,q}_{\beta-1}, \quad \sigma \in W^{e-1,q}_{\beta-1}, \quad \rho \in W^{s-2,p}_{\beta-2} (\rho \geq 0), \quad J \in \boldsymbol{W}^{e-2,q}_{\beta-2}.$$

Let $\eta = \max\{\beta, \delta\}$. Define $f : W^{s,p}_\delta \times \boldsymbol{W}^{e,q}_\beta \to W^{s-2,p}_{\eta-2}$ and $\boldsymbol{f} : W^{s,p}_\delta \to \boldsymbol{W}^{e-2,q}_{\beta-2}$ by

$$f(\psi, W) = a_R(\psi + \mu) + a_\tau(\psi + \mu)^5 - a_W(\psi + \mu)^{-7} - a_\rho(\psi + \mu)^{-3},$$
$$\boldsymbol{f}(\psi) = b_\tau(\psi + \mu)^6 + b_J.$$

Find $(\psi, W) \in W^{s,p}_\delta \times \boldsymbol{W}^{e,q}_\beta$ such that

$$A_L\psi + f(\psi, W) = 0, \tag{3.3}$$
$$\mathcal{A}_L W + \boldsymbol{f}(\psi) = 0. \tag{3.4}$$

Remark 3.2. *We make the following observations regarding **Weak Formulation 1**.*

- *Since $s \geq 1$, the condition $e > 1$ implies $e > 2 - s$. Therefore, we did not explicitly state the condition $e > 2 - s$ in the above formulation.*
- *The condition $e > \frac{3}{q} + 1$ together with $s > \frac{3}{p}$ imply that $e > -s + \frac{3}{p} - 1 + \frac{3}{q}$. Therefore, we did not explicitly state the condition $e > -s + \frac{3}{p} - 1 + \frac{3}{q}$ in the above formulation.*
- *For $(1 + \frac{3}{q}, \infty) \cap [s-1, s]$ to be nonempty we need to have $1 + \frac{3}{q} < s$. This is why we have $\frac{1}{q} \in (0, \frac{s-1}{3})$ in the weak formulation.*
- *For $(1 + \frac{3}{q}, \infty) \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$ to be nonempty we need to have $1 + \frac{3}{q} < \frac{3}{q} + s - \frac{3}{p}$. That is, $s > 1 + \frac{3}{p}$ (therefore, we did not need to explicitly state $s \geq 1$).*
- *For $[s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$ to be nonempty we need to have $\frac{1}{p} - \frac{1}{3} \leq \frac{1}{q} \leq \frac{1}{p} + \frac{1}{3}$. That is, $\frac{1}{q} \in [\frac{3-p}{3p}, \frac{3+p}{3p}]$.*

Remark 3.3. ***Our analysis in this paper is based on the weak formulation described above***. *In some of the theorems that follow, for the claimed estimates to be true in the case $e \leq 2$ we will need to restrict the admissible space of τ. In those cases we will assume $\tau \in W^{1,z}_{\beta-1}$ where $z = \frac{3q}{3+(2-e)q}$. We note that z has been chosen in this form to ensure that $W^{1,z}_{\beta-1} \hookrightarrow W^{e-1,q}_{\beta-1}$ (and so $L^z_{\beta-2} \hookrightarrow W^{e-2,q}_{\beta-2}$). Indeed, by Theorem A.16, for $W^{1,z}_{\beta-1} \hookrightarrow W^{e-1,q}_{\beta-1}$ to hold true we need to have (the numbering follows the assumptions in Theorem A.16):*

(i) $z \leq q$,
(ii) $1 \geq e - 1$, *(true for* $e \leq 2$*)*
(iii) $1 - \frac{3}{z} \geq e - 1 - \frac{3}{q}$.

Now note that if we set $z = \frac{3q}{(2-e)q+3}$*, then the first condition and the third condition are both satisfied (for* $e \leq 2$*):*

$$z \leq q \Leftrightarrow \frac{3q}{(2-e)q+3} \leq q \Leftrightarrow \frac{3}{(2-e)q+3} \leq 1 \Leftrightarrow 2 - e \geq 0,$$

$$1 - \frac{3}{z} \geq e - 1 - \frac{3}{q} \Leftrightarrow \frac{3}{z} \leq (2-e) + \frac{3}{q} \Leftrightarrow z \geq \frac{3q}{(2-e)q+3}.$$

4. Results for the Momentum Constraint

We now develop the main results will need for the momentum constraint operator on AF manifolds with rough data.

Theorem 4.1. *Let* (M, h) *be a 3-dimensional, smooth, AF Riemannian manifold of class* $W^{s,p}_\delta$ *with* $p \in (1, \infty)$*,* $\delta < 0$ *and* $s \in (\frac{3}{p}, \infty) \cap (1, \infty)$*. Select* q, e *to satisfy:*

$$q \in (1, \infty), \quad e \in (2-s, s] \cap (-s + \frac{3}{p} - 1 + \frac{3}{q}, s - \frac{3}{p} + \frac{3}{q}]. \tag{4.1}$$

In case $e = s \notin \mathbb{N}_0$*, assume* $q = p$*. In case* $e = s \in \mathbb{N}_0, p > 2, q < 2$*, assume* $e > \frac{3}{q} - \frac{1}{2}$*. In case* $e = s - \frac{3}{p} + \frac{3}{q}, p < 2, q > 2$ *assume* $e > \frac{1}{2}$*. Suppose* $\beta \in (-1, 0)$ *and* b_J *and* b_τ *and* ψ *are such that* $\boldsymbol{f}(\psi) \in W^{e-2,q}_{\beta-2}$ *(in particular, we know that if we fix the source terms* b_J *and* b_τ *in* $\mathbf{W}^{e-2,q}_{\beta-2}$ *and* $\psi \in W^{s,p}_\delta$ *then* $\boldsymbol{f}(\psi) \in \mathbf{W}^{e-2,q}_{\beta-2}$*). Then* $\mathcal{A}_L : \mathbf{W}^{e,q}_\beta \to \mathbf{W}^{e-2,q}_{\beta-2}$ *is Fredholm of index zero. Moreover if* h *has no nontrivial conformal Killing fields, then the momentum constraint* $\mathcal{A}_L W + \boldsymbol{f}(\psi) = 0$ *has a unique solution* $W \in \mathbf{W}^{e,q}_\beta$ *with*

$$\|W\|_{\mathbf{W}^{e,q}_\beta} \leq C \|\boldsymbol{f}(\psi)\|_{\mathbf{W}^{e-2,q}_{\beta-2}},$$

where $C > 0$ *is a constant.*

Remark 4.2. *In the above theorem the ranges for* e *and* q *are chosen so that the momentum constraint is well-defined. Also note that for* $(2-s, s]$ *to be a nonempty interval we had to assume that s is* ***strictly*** *larger than* 1.

Remark 4.3. *There are important cases where the assumption that "*h *has no nontrivial conformal Killing fields" is automatically satisfied. For instance in* [41] *it is proved that if* (M, h) *is AF of class* $W^{s,2}_\delta$ *with* $s > \frac{3}{2}$ *(and of course* $\delta < 0$*) and if* $X \in W^{s,2}_\rho$ *with* $\rho < 0$ *is a conformal Killing field, then* X *vanishes identically. We do not pursue this issue here, but interested readers may find more information in* [41] *and* [40].

Proof. **(Theorem 4.1)** The proof will involve three main steps.

- **Step 1: Establish that $\mathcal{A}_L$ is Fredholm of index zero.**
  $\mathcal{A}_L$ is of class $D^{s,p}_{2,\delta}$. Therefore by Proposition B.7, $\mathcal{A}_L : \mathbf{W}^{e,q}_\beta \to \mathbf{W}^{e-2,q}_{\beta-2}$ is semi-Fredholm (this is exactly why it is assumed $\beta \in (-1, 0)$). On the other hand, vector Laplacian of the rough metric can be approximated by the vector Laplacian of smooth metrics and it is well known that vector Laplacian of a smooth metric is Fredholm of index zero. Therefore since the index of a semi-Fredholm map is locally constant, it follows that $\mathcal{A}_L : \mathbf{W}^{e,q}_\beta \to \mathbf{W}^{e-2,q}_{\beta-2}$ is Fredholm with index 0.

- **Step 2: Show that if $Ker\mathcal{L} = \{0\}$, then $Ker\mathcal{A}_L = \{0\}$.**

  The proof of this step involves considering six distinct cases. In each case, we denote the operator $\mathcal{A}_L$ acting on $\mathbf{W}^{e,q}_{\beta}$ by $(\mathcal{A}_L)_{e,q,\beta}$. In order to best organize these arguments for these six cases, we make the following definitions:

  nice triple: A triple (e, q, β) where $-1 < \beta < 0$ and e, q satisfy (4.1).
  super nice triple: A ***nice*** triple (e, q, β) where $e \neq s$ and $e \neq s - \frac{3}{p} + \frac{3}{q}$.

  We now make three observations about relationships between these definitions.

  - **Observation 1**: For any $-1 < \beta < 0$, $(e = 1, q = 2, \beta)$ is ***super nice*** and $(e = s, q = p, \beta)$ is ***nice***. Indeed,

$$\begin{array}{ll} 1 \in (2-s, s), & \text{(since } s > 1) \\ 1 > -s + \frac{3}{p} - 1 + \frac{3}{2}, & \text{(since } s > \frac{3}{p}) \\ 1 < s - \frac{3}{p} + \frac{3}{2}, & \text{(since } s > \frac{3}{p}) \\ s \in (2-s, s], & \text{(trivially true; note } s > 1) \\ s > -s + \frac{3}{p} - 1 + \frac{3}{p}, & \text{(since } s > \frac{3}{p}) \\ s \leq s - \frac{3}{p} + \frac{3}{p}. & \text{(trivially true)} \end{array}$$

  - **Observation 2**: If (e, q, β) is ***super nice***, then $(2 - e, q', -1 - \beta)$ is also ***super nice***. Indeed,

$$\begin{array}{l} q \in (1, \infty) \Rightarrow q' \in (1, \infty), \\ \beta \in (-1, 0) \Rightarrow -1 - \beta \in (-1, 0), \\ e \in (2-s, s) \Rightarrow 2 - e \in (2-s, s), \\ e < s - \frac{3}{p} + \frac{3}{q} \Rightarrow 2 - e > 2 - s + \frac{3}{p} - \frac{3}{q} = -s + \frac{3}{p} - 1 + \frac{3}{q'}, \\ e > -s + \frac{3}{p} - 1 + \frac{3}{q} \Rightarrow 2 - e < 2 + s - \frac{3}{p} + 1 - \frac{3}{q} = s - \frac{3}{p} + \frac{3}{q'}. \end{array}$$

  - **Observation 3**: Suppose (e_1, q_1, β_1) and (e_2, q_2, β_2) are ***nice*** triples. If we have $\mathbf{W}^{e_2,q_2}_{\beta_2} \hookrightarrow \mathbf{W}^{e_1,q_1}_{\beta_1}$, then $(\mathcal{A}_L)_{e_2,q_2,\beta_2}$ is the restriction of $(\mathcal{A}_L)_{e_1,q_1,\beta_1}$ to $\mathbf{W}^{e_2,q_2}_{\beta_2}$ and so $Ker(\mathcal{A}_L)_{e_2,q_2,\beta_2} \subseteq Ker(\mathcal{A}_L)_{e_1,q_1,\beta_1}$. In particular, if $Ker(\mathcal{A}_L)_{e_1,q_1,\beta_1} = \{0\}$ holds, then $Ker(\mathcal{A}_L)_{e_2,q_2,\beta_2} = \{0\}$.

  Now let (e, q, β) be a ***nice*** triple. We consider now the following six cases:

  - **Case 1:** $e = 1, q = 2$

    In order to prove the claim first we show that if $\beta' \in (-1, \frac{-1}{2})$ then

$$\forall X, Y \in \mathbf{W}^{1,2}_{\beta'} \quad \langle \mathcal{A}_L X, Y \rangle_{(M,h)} = \frac{1}{2} \langle \mathcal{L}X, \mathcal{L}Y \rangle_{L^2}.$$

    First let us ensure that both sides are well-defined. Note that $\mathcal{A}_L : \mathbf{W}^{1,2}_{\beta'} \to \mathbf{W}^{-1,2}_{\beta'-2}$ and so $\mathcal{A}_L X \in \mathbf{W}^{-1,2}_{\beta'-2}$. According to our discussion on duality pairing in Appendix B, we know that the duality pairing of $\mathbf{W}^{-1,2}_{\beta'-2}$ and $\mathbf{W}^{1,2}_{-1-\beta'}$ is well-defined. So for the LHS to be well-defined, we just need to ensure that $Y \in \mathbf{W}^{1,2}_{-1-\beta'}$, that is we need to have $\mathbf{W}^{1,2}_{\beta'} \hookrightarrow \mathbf{W}^{1,2}_{-1-\beta'}$. But clearly this is true because by assumption $\beta' < \frac{-1}{2}$. Also note that $\mathcal{L}X, \mathcal{L}Y \in L^2_{\beta'-1}$; since $\beta' - 1 < \frac{-3}{2}$, by Remark A.1 we have $L^2_{\beta'-1} \hookrightarrow L^2$ and so the RHS makes sense. Now, it is well known that the claimed equality holds true for $X, Y \in C^\infty_c$ and so by density it holds true for $X, Y \in \mathbf{W}^{1,2}_{\beta'}$.

    Let $X \in Ker(\mathcal{A}_L)_{e=1,q=2,\beta}$. Since $(e = 1, q = 2, \beta)$ is a ***nice*** triple, by Lemma B.9 there exists $\beta' \in (-1, \frac{-1}{2})$ such that $X \in \mathbf{W}^{1,2}_{\beta'}$. So by what was proved above we can conclude that $\langle \mathcal{L}X, \mathcal{L}X\rangle_{L^2} = 0$ which implies that X is a conformal Killing field and so $X = 0$.

- **Case 2:** $e \neq 1, q = 2$

  If $e > 1$, then $\mathbf{W}^{e,q}_{\beta} \hookrightarrow \mathbf{W}^{1,q}_{\beta}$ and hence the claim follows from Observation 3. Suppose $e < 1$. So in particular $e \neq s$ and $e \neq s - \frac{3}{p} + \frac{3}{2}$ (because both s and $s - \frac{3}{p} + \frac{3}{2}$ are larger than 1) and therefore $(e, q = 2, \beta)$ is ***super nice***. Consequently $(2 - e, q' = 2, -1 - \beta)$ is also ***super nice***. Since $2 - e > 1$ we know that $Ker(\mathcal{A}_L)_{2-e,q'=2,-1-\beta} = \{0\}$. But $\mathcal{A}_L$ is formally self adjoint and so $Ker((\mathcal{A}_L)_{e,q=2,\beta})^* = Ker(\mathcal{A}_L)_{2-e,q'=2,-1-\beta} = \{0\}$. Finally $(\mathcal{A}_L)_{e,q=2,\beta}$ is Fredholm of index zero, so $Ker(\mathcal{A}_L)_{e,q=2,\beta} = \{0\}$.

- **Case 3:** ($p \leq 2, q < 2$) **or** ($e > \frac{3}{q} - \frac{1}{2}, q < 2$)

  It is enough to show that there exists $\tilde{e}$ such that $\mathbf{W}^{e,q}_{\beta} \hookrightarrow \mathbf{W}^{\tilde{e},2}_{\beta}$ where $(\tilde{e}, 2, \beta)$ is ***nice***. That is, we need to find $\tilde{e}$ that satisfies

$$\begin{aligned} &e \geq \tilde{e}, \\ &e - \frac{3}{q} \geq \tilde{e} - \frac{3}{2} \quad (\Leftrightarrow \tilde{e} \leq e + \frac{3}{2} - \frac{3}{q}), \\ &\tilde{e} \in (2 - s, s], \\ &\tilde{e} \in (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}]. \end{aligned}$$

  Since $\frac{3}{q} > \frac{3}{2}$, the second condition is stronger than the first condition. Also $s > \frac{3}{p}$ so $(-s + \frac{3}{p} + \frac{1}{2}, s - \frac{3}{p} + \frac{3}{2}]$ is nonempty. Now note that

$$\text{If } p \leq 2 \text{ then} \quad (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}] \subseteq (2 - s, s],$$

$$\text{If } p > 2 \text{ then} \quad (2 - s, s] \subseteq (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}].$$

  Therefore in order to ensure such an $\tilde{e}$ exists it is enough to have

$$2 - s < e + \frac{3}{2} - \frac{3}{q}, \quad -s + \frac{3}{p} + \frac{1}{2} < e + \frac{3}{2} - \frac{3}{q}.$$

  The second inequality is true because (e, q, β) is a ***nice*** triple. Now if $p \leq 2$, then

$$e > -s + \frac{3}{p} - 1 + \frac{3}{q} \geq -s + \frac{3}{q} + \frac{1}{2},$$

  so the first inequality also holds true. Moreover, for all values of p, if $e > \frac{3}{q} - \frac{1}{2}$, then the first inequality holds true (note that $s > 1$).

- **Case 4:** ($p \geq 2, q > 2$) **or** ($e > \frac{1}{2}, q > 2$)

  Let $\beta' \in (\beta, 0)$. By Theorem A.17, $\mathbf{W}^{e,q}_{\beta} \hookrightarrow \mathbf{W}^{e,2}_{\beta'}$. So it is enough to show that under the assumption of this case, $(e, 2, \beta')$ is a ***nice*** triple. Since (e, q, β) is ***nice***, we know $e \in (2 - s, s]$. Therefore we just need to check that $e \in (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}]$.

$$q > 2 \Rightarrow \frac{3}{q} < \frac{3}{2} \Rightarrow e \leq s - \frac{3}{p} + \frac{3}{q} < s - \frac{3}{p} + \frac{3}{2}.$$

Also if $p \geq 2$, then $-s + \frac{3}{p} + \frac{1}{2} \leq -s + 2 < e$. Moreover, for all values of p, if $e > \frac{1}{2}$, then $e > -s + \frac{3}{p} + \frac{1}{2}$.

- **Case 5:** $p < 2, q > 2$
  First note that e cannot be equal to s. Otherwise we would have $s = e \leq s - \frac{3}{p} + \frac{3}{q}$ and so $p \geq q$ which contradicts the assumption of this case.
  If $e = s - \frac{3}{p} + \frac{3}{q}$, then the claim follows from case 4 (because by assumption $e > \frac{1}{2}$). So WLOG we can assume that $(e, q > 2, \beta)$ is a ***super nice*** triple. This implies that $(2-e, q' < 2, -1-\beta)$ is also a ***super nice*** triple. Since $q' < 2$ by what was proved in case 3, $Ker(\mathcal{A}_L)_{2-e,q',-1-\beta} = \{0\}$. So by an argument exactly the same as the one given in case 2 we can conclude that $Ker(\mathcal{A}_L)_{e,q,\beta} = \{0\}$.

- **Case 6:** $p > 2, q < 2$
  First note that e cannot be equal to $s - \frac{3}{p} + \frac{3}{q}$. Otherwise we would have $s - \frac{3}{p} + \frac{3}{q} = e \leq s$ and so $p \leq q$ which contradicts the assumption of this case.
  If $e = s \in \mathbb{N}_0$, then the claim follows from case 3 (because by assumption $e > \frac{3}{q} - \frac{1}{2}$). So WLOG we can assume $(e, q < 2, \beta)$ is a ***super nice*** triple. Therefore $(2-e, q' > 2, -1-\beta)$ is also a ***super nice*** triple. Since $q' > 2$ by what was proved in case 4, $Ker(\mathcal{A}_L)_{2-e,q',-1-\beta} = \{0\}$. So by an argument exactly the same as the one given in case 2 we can conclude that $Ker(\mathcal{A}_L)_{e,q,\beta} = \{0\}$.

- **Step 3: Show that if $Ker\mathcal{A}_L = \{0\}$, then $\mathcal{A}_L$ is an isomorphism.**
  By the previous steps we know that $\mathcal{A}_L$ is Fredholm of index zero and also it is injective. It follows that $\mathcal{A}_L$ is a bijective continuous operator and so according to the open mapping theorem it is an isomorphism. In particular $(\mathcal{A}_L)^{-1}$ is continuous and so $\|W\|_{\mathbf{W}^{e,q}_{\beta}} \leq C\|\mathbf{f}(\psi)\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$.

$\square$

Corollary 4.4. *Let the following assumptions hold:*

- (M, h) *is a 3-dimensional, smooth, AF Riemannian manifold of class* $W^{s,p}_{\delta}$.
- $p \in (1, \infty)$, $s \in (1 + \frac{3}{p}, \infty)$, $\delta < 0$.
- $q \in (3, \infty)$, $e \in (1, s] \cap (1 + \frac{3}{q}, s - \frac{3}{p} + \frac{3}{q}] \cap (1, 2]$. *($q = p$ if $e = s \notin \mathbb{N}_0$)*
- $-1 < \beta < 0$, $z = \frac{3q}{3+(2-e)q}$, $b_\tau \in L^z_{\beta-2}$.
- h *has no conformal Killing fields.*
- $W \in \boldsymbol{W}^{e,q}_{\beta}$ *uniquely solves the momentum constraint with source* $\psi \in W^{s,p}_{\delta}$.

Then:

$$\|\mathcal{L}W\|_{L^\infty_{\beta-1}} \preceq \|b_\tau(\mu+\psi)^6\|_{L^z_{\beta-2}} + \|b_J\|_{\boldsymbol{W}^{e-2,q}_{\beta-2}} \preceq (\mu + \|\psi\|_{L^\infty_\delta})^6 \|b_\tau\|_{L^z_{\beta-2}} + \|b_J\|_{\boldsymbol{W}^{e-2,q}_{\beta-2}}.$$

Moreover, $\|W\|_{\boldsymbol{W}^{e,q}_{\beta}}$ *can be bounded by the same expressions. The implicit constants in the above inequalities do not depend on* μ, W, *or* ψ.

Remark 4.5. *In this theorem, the restrictions on* e *and* q *serve the following purposes:*
- $L^z_{\beta-2} \hookrightarrow W^{e-2,q}_{\beta-2}$. *(note that* $e \leq 2$*)*
- $\mathcal{A}_L : \boldsymbol{W}^{e,q}_{\beta} \to \boldsymbol{W}^{e-2,q}_{\beta-2}$ *is well-defined.*
- $e > 1 + \frac{3}{q}$ *and so* $W^{e,q}_{\beta} \hookrightarrow L^\infty_\beta$ *and also* $W^{e-1,q}_{\beta-1} \hookrightarrow L^\infty_{\beta-1}$.

Also note that
- *If* $e > 1 + \frac{3}{q}$ *then* $e > -s + \frac{3}{p} - 1 + \frac{3}{q}$ *is automatically satisfied.*

- *For* $(1+\frac{3}{q}, s-\frac{3}{p}+\frac{3}{q}]$ *to be nonempty we must have* $s > 1+\frac{3}{p}$.
- *If* $s>1$ *then* $e>2-s$ *follows from* $e>1$.
- $2 \geq e > 1+\frac{3}{q}$ *and so we must have* $q>3$.

Proof. **(Corollary 4.4)** First note that $e>1+\frac{3}{q}$ and so $W^{e,q}_{\beta} \hookrightarrow L^{\infty}_{\beta}$ and also $W^{e-1,q}_{\beta-1} \hookrightarrow L^{\infty}_{\beta-1}$. That is, $W^{e,q}_{\beta} \hookrightarrow W^{1,\infty}_{\beta}$. Also $\mathcal{L}: W^{1,\infty}_{\beta} \to L^{\infty}_{\beta-1}$ is continuous ($\mathcal{L}$ is a differential operator of order 1) and so we have

$$\begin{aligned}
\|\mathcal{L}W\|_{L^{\infty}_{\beta-1}} &\preceq \|W\|_{\mathbf{W}^{1,\infty}_{\beta}} \preceq \|W\|_{\mathbf{W}^{e,q}_{\beta}} \preceq \|\mathbf{f}(\psi)\|_{\mathbf{W}^{e-2,q}_{\beta-2}} \\
&= \|b_\tau(\mu+\psi)^6 + b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}} \leq \|b_\tau(\mu+\psi)^6\|_{\mathbf{W}^{e-2,q}_{\beta-2}} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}} \\
&\preceq \|b_\tau(\mu+\psi)^6\|_{L^{z}_{\beta-2}} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}} \quad (\text{note that} \quad L^{z}_{\beta-2} \hookrightarrow W^{e-2,q}_{\beta-2})
\end{aligned}$$

Now note that

$$\begin{aligned}
\|b_\tau(\mu+\psi)^6\|_{L^{z}_{\beta-2}} &= \|b_\tau \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k}\psi^k\|_{L^{z}_{\beta-2}} \leq \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k} \|b_\tau \psi^k\|_{L^{z}_{\beta-2}} \\
&\preceq \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k} \|b_\tau \psi^k\|_{L^{z}_{\beta+\delta-2}} \preceq \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k} \|\psi^k\|_{L^{\infty}_{\delta}} \|b_\tau\|_{L^{z}_{\beta-2}} \\
&\qquad (\text{note that} \quad L^{\infty}_{\delta} \times L^{z}_{\beta-2} \hookrightarrow L^{z}_{\beta+\delta-2}) \\
&\preceq \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k} \|\psi\|^k_{L^{\infty}_{\delta}} \|b_\tau\|_{L^{z}_{\beta-2}} \\
&\qquad (\text{note that} \quad L^{\infty}_{\delta} \times L^{\infty}_{\delta} \hookrightarrow L^{\infty}_{2\delta} \hookrightarrow L^{\infty}_{\delta}) \\
&= (\mu + \|\psi\|_{L^{\infty}_{\delta}})^6 \|b_\tau\|_{L^{z}_{\beta-2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathcal{L}W\|_{L^{\infty}_{\beta-1}} &\preceq \|W\|_{\mathbf{W}^{e,q}_{\beta}} \\
&\preceq \|b_\tau(\mu+\psi)^6\|_{L^{z}_{\beta-2}} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}} \\
&\preceq (\mu + \|\psi\|_{L^{\infty}_{\delta}})^6 \|b_\tau\|_{L^{z}_{\beta-2}} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}.
\end{aligned}$$

□

Lemma 4.6. *All the assumptions in corollary 4.4 hold. In particular,* W *is the solution to the momentum constraint with source* ψ. *Then*

$$a_W \preceq r^{2\beta-2}(k_1\|\mu+\psi\|^{12}_{\infty} + k_2)$$

where

(1) $r = (1+|x|^2)^{\frac{1}{2}}$ *and* $|x|$ *is the geodesic distance from a fixed point* O *in the compact core (see Remark A.9),*

(2) $k_1 = \|b_\tau\|^2_{L^{z}_{\beta-2}}$, $\quad k_2 = \|\sigma\|^2_{L^{\infty}_{\beta-1}} + \|b_J\|^2_{\mathbf{W}^{e-2,q}_{\beta-2}}$.

The implicit constant in the above inequality does not depend on μ, W *or* ψ.

Proof. **(Lemma 4.6)** By Corollary 4.4 we have

$$\begin{aligned}
\|\mathcal{L}W\|_{L^{\infty}_{\beta-1}} &\preceq \|b_\tau(\mu+\psi)^6\|_{L^{z}_{\beta-2}} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}} \\
&\leq \|b_\tau\|_{L^{z}_{\beta-2}} \|\mu+\psi\|^6_{\infty} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}, \quad (\text{here we used Remark A.1})
\end{aligned}$$

and considering Remark A.5 we get the following pointwise bound for $\mathcal{L}W$:

$$|\mathcal{L}W| \preceq r^{\beta-1}(\|b_\tau\|_{L^z_{\beta-2}}\|\mu+\psi\|^6_\infty + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}).$$

Note that $\mathcal{L}W$ has a continuous version and so the above inequality holds everywhere (not just "almost everywhere"). Now we can write

$$\begin{aligned}
a_W = \frac{1}{8}|\sigma+\mathcal{L}W|^2 &\preceq |\sigma|^2 + |\mathcal{L}W|^2 \\
&\preceq r^{2\beta-2}\|\sigma\|^2_{L^\infty_{\beta-1}} + |\mathcal{L}W|^2 \\
&\quad \text{(here we used Remark A.5; note } \sigma \in W^{e-1,q}_{\beta-1} \hookrightarrow C^0_{\beta-1} \hookrightarrow L^\infty_{\beta-1}) \\
&\preceq r^{2\beta-2}\|\sigma\|^2_{L^\infty_{\beta-1}} + r^{2\beta-2}\big(\|b_\tau\|_{L^z_{\beta-2}}\|\mu+\psi\|^6_\infty + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}\big)^2 \\
&\preceq r^{2\beta-2}(k_1\|\mu+\psi\|^{12}_\infty + k_2).
\end{aligned}$$

□

Remark 4.7. *We make the following important remark concerning notation. Consider the space $W^{\alpha,\gamma}_\delta(M)$ where $\alpha\gamma > 3$. An order on $W^{\alpha,\gamma}_\delta(M)$ can be defined as follows: the functions $\chi_1, \chi_2 \in W^{\alpha,\gamma}_\delta(M)$ satisfy $\chi_2 \geq \chi_1$ if and only if the continuous versions of χ_1, χ_2 satisfy $\chi_2(x) \geq \chi_1(x)$ for all $x \in M$ (clearly this definition agrees with the one that is described in Remark B.4). Equipped with this order, $W^{\alpha,\gamma}_\delta(M)$ becomes an ordered Banach space. By the interval $[\chi_1, \chi_2]_{\alpha,\gamma,\delta}$ we mean the set of all functions $\chi \in W^{\alpha,\gamma}_\delta(M)$ such that $\chi_1 \leq \chi \leq \chi_2$.*

Lemma 4.8. *Let the following assumptions hold:*

- *All the assumptions in corollary 4.4 hold.*
- *$\tilde{s} \in (\frac{3}{p}, s]$ and $\tilde{\delta} \in [\delta, 0)$ are such that $W^{e-2,q}_{\beta-2} \times W^{\tilde{s},p}_{\tilde{\delta}} \hookrightarrow W^{e-2,q}_{\beta+\tilde{\delta}-2} \hookrightarrow W^{e-2,q}_{\beta-2}$. For example, using multiplication lemma, one can easily check that for $\tilde{s} = s$ and $\tilde{\delta} = \delta$ these inclusions hold true.*
- *$\psi_-, \psi_+ \in W^{s,p}_\delta$, $\psi_+ \geq \psi_- > -\mu$, $\psi_1, \psi_2 \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$.*
- *W_1 and W_2 are solutions to the momentum constraint corresponding to ψ_1 and ψ_2, respectively.*

Then:

$$\|W_1 - W_2\|_{e,q,\beta} \preceq \big(1 + \max\{\|\psi_-\|_{L^\infty_{\tilde{\delta}}}, \|\psi_+\|_{L^\infty_{\tilde{\delta}}}\}\big)^5 \|b_\tau\|_{L^z_{\beta-2}}\|\psi_2 - \psi_1\|_{\tilde{s},p,\tilde{\delta}}.$$

The implicit constant in the above inequality depends on μ but it is independent of ψ_1, ψ_2, W_1, W_2.

Proof. **(Lemma 4.8)** The momentum equation is linear and so $W_1 - W_2$ is the solution to the momentum constraint with right hand side $\mathbf{f}(\psi_1) - \mathbf{f}(\psi_2)$.

$$\begin{aligned}
\|W_1 - W_2\|_{e,q,\beta} &\preceq \|\mathbf{f}(\psi_1) - \mathbf{f}(\psi_2)\|_{e-2,q,\beta-2} = \|b_\tau[(\mu+\psi_1)^6 - (\mu+\psi_2)^6]\|_{e-2,q,\beta-2} \\
&= \|b_\tau \sum_{j=0}^{5} (\mu+\psi_2)^j(\mu+\psi_1)^{5-j}(\psi_2-\psi_1)\|_{e-2,q,\beta-2} \\
&\leq \sum_{j=0}^{5} \|b_\tau(\mu+\psi_2)^j(\mu+\psi_1)^{5-j}(\psi_2-\psi_1)\|_{e-2,q,\beta-2} \\
&\preceq \sum_{j=0}^{5} \|b_\tau(\mu+\psi_2)^j(\mu+\psi_1)^{5-j}\|_{e-2,q,\beta-2}\|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\qquad (\text{by assumption } W^{e-2,q}_{\beta-2} \times W^{\tilde{s},p}_{\tilde{\delta}} \hookrightarrow W^{e-2,q}_{\beta-2}) \\
&\preceq \sum_{j=0}^{5} \|b_\tau(\mu+\psi_2)^j(\mu+\psi_1)^{5-j}\|_{L^z_{\beta-2}}\|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\qquad (\text{using } L^z_{\beta-2} \hookrightarrow W^{e-2,q}_{\beta-2}) \\
&\preceq \sum_{j=0}^{5} \|b_\tau \sum_{m=0}^{j} \binom{j}{m} \psi_2^m \sum_{l=0}^{5-j} \binom{5-j}{l} \psi_1^l \|_{L^z_{\beta-2}}\|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\leq \sum_{j=0}^{5}\sum_{m=0}^{j}\sum_{l=0}^{5-j} \binom{j}{m}\binom{5-j}{l} \|b_\tau \psi_2^m \psi_1^l\|_{L^z_{\beta-2}}\|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\leq \sum_{j=0}^{5}\sum_{m=0}^{j}\sum_{l=0}^{5-j} \binom{j}{m}\binom{5-j}{l} \|b_\tau \psi_2^m \psi_1^l\|_{L^z_{\beta+2\tilde{\delta}-2}}\|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\qquad (\text{since } \tilde{\delta} < 0) \\
&\preceq \sum_{j=0}^{5}\sum_{m=0}^{j}\sum_{l=0}^{5-j} \binom{j}{m}\binom{5-j}{l} \|b_\tau\|_{L^z_{\beta-2}} \|\psi_2^m\|_{L^\infty_{\tilde{\delta}}} \|\psi_1^l\|_{L^\infty_{\tilde{\delta}}} \|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\qquad (\text{using } W^{\tilde{s},p}_{\tilde{\delta}} \hookrightarrow L^\infty_{\tilde{\delta}}, \quad L^\infty_{\tilde{\delta}} \times L^z_{\beta-2} \hookrightarrow L^z_{\beta+\tilde{\delta}-2}) \\
&\leq \sum_{j=0}^{5}\sum_{m=0}^{j}\sum_{l=0}^{5-j} \binom{j}{m}\binom{5-j}{l} \|b_\tau\|_{L^z_{\beta-2}} \|\psi_2\|^m_{L^\infty_{\tilde{\delta}}} \|\psi_1\|^l_{L^\infty_{\tilde{\delta}}} \|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\qquad (\text{due to } L^\infty_{\tilde{\delta}} \times L^\infty_{\tilde{\delta}} \hookrightarrow L^\infty_{2\tilde{\delta}} \hookrightarrow L^\infty_{\tilde{\delta}}) \\
&= \sum_{j=0}^{5} \|b_\tau\|_{L^z_{\beta-2}} (1+\|\psi_2\|_{L^\infty_{\tilde{\delta}}})^j (1+\|\psi_1\|_{L^\infty_{\tilde{\delta}}})^{5-j} \|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&\leq \sum_{j=0}^{5} \|b_\tau\|_{L^z_{\beta-2}} (1+\max\{\|\psi_-\|_{L^\infty_{\tilde{\delta}}}, \|\psi_+\|_{L^\infty_{\tilde{\delta}}}\})^5 \|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}} \\
&= 6(1+\max\{\|\psi_-\|_{L^\infty_{\tilde{\delta}}}, \|\psi_+\|_{L^\infty_{\tilde{\delta}}}\})^5 \|b_\tau\|_{L^z_{\beta-2}} \|(\psi_2-\psi_1)\|_{\tilde{s},p,\tilde{\delta}},
\end{aligned}$$

where we have used ($|\psi_i| \leq \max\{|\psi_+|, |\psi_-|\}$, so that $\|\psi_i\|_{L^\infty_{\tilde{\delta}}} \leq \max\{\|\psi_-\|_{L^\infty_{\tilde{\delta}}}, \|\psi_+\|_{L^\infty_{\tilde{\delta}}}\}$). □

5. RESULTS FOR THE HAMILTONIAN CONSTRAINT

We now develop the main results will need for the Hamiltonian constraint on AF manifolds with rough data. We study primarily the "shifted" Hamiltonian constraint; the reason for introducing a shift (the function a_s in the following lemma) is briefly discussed in Remark 5.5.

Lemma 5.1. *Let the following assumptions hold:*

- (M, h) *is a 3-dimensional, smooth, AF Riemannian manifold of class* $W^{s,p}_{\delta}$.
- $p \in (\frac{3}{2}, \infty)$, $s \in (\frac{3}{p}, \infty) \cap [1, 3]$.
- $\beta < 0$, $-1 < \delta < 0$, *and* $\eta = \max\{\delta, \beta\}$.
- $a_\tau, a_\rho, a_W \in W^{s-2,p}_{\beta-2}$, $a_R \in W^{s-2,p}_{\delta-2}$.
- $a_0 \in W^{s-2,p}_{\eta-2}$, $a_0 \neq 0$, *and* $a_0 \geq 0$ *(see Remark B.4).*
- $\tilde{s} \in (\frac{3}{p}, s] \cap [1, 1+\frac{3}{p})$, $\delta \leq \tilde{\delta} < 0$.
- $t \in (\frac{3}{p}, \tilde{s}] \cap [1, 1+\frac{3}{p})$, $\tilde{\delta} \leq \gamma < 0$.
- $\psi_-, \psi_+ \in W^{\tilde{s},p}_{\tilde{\delta}}$ *and* $-\mu < \psi_- \leq \psi_+$.
- $V \in W^{\tilde{s},p}_{loc}$, $V > 0$ *is such that* $a_W V \in W^{s-2,p}_{\eta-2}$, *and* $\|a_W V\|_{s-2,p,\eta-2} \preceq C(\psi_+, \psi_-)\|a_W\|_{s-2,p,\eta-2}$ *where* $C(\psi_+, \psi_-)$ *is a constant independent of* V.
- $a_s = a_0 + a_W V \in W^{s-2,p}_{\eta-2}$.
- $A^{shifted}_L : W^{s,p}_{\delta} \to W^{s-2,p}_{\eta-2}$ *is defined by* $A^{shifted}_L \psi = A_L\psi + a_s\psi$.
- $\boldsymbol{f}^{shifted}_W : [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \to W^{s-2,p}_{\eta-2}$ *is defined by* $\boldsymbol{f}^{shifted}_W(\psi) = \boldsymbol{f}_W(\psi) - a_s\psi$ *where*

$$\boldsymbol{f}_W(\psi) = a_\tau(\mu+\psi)^5 + a_R(\mu+\psi) - a_\rho(\mu+\psi)^{-3} - a_W(\mu+\psi)^{-7}.$$

Then:

(1) Suppose $A^{shifted}_L : W^{s,p}_{\delta} \to W^{s-2,p}_{\eta-2}$ *is an isomorphism. If we define* $T^{shifted} : [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \times W^{s-2,p}_{\beta-2} \to W^{s,p}_{\delta}$ *by* $T^{shifted}(\psi, a_W) = -(A^{shifted}_L)^{-1}\boldsymbol{f}^{shifted}_W(\psi)$, *then* $T^{shifted}$ *is continuous in both arguments and moreover*

$$\|T^{shifted}(\psi, a_W)\|_{s,p,\delta} \preceq (1 + \|a_W\|_{s-2,p,\eta-2})(1 + \|\psi\|_{t,p,\gamma}).$$

The implicit constant in the above inequality depends on μ *but it is independent of* ψ *and* a_W.

(2) If $\beta \leq \delta$, *(that is if* $\eta = \delta$*), then* $A^{shifted}_L : W^{s,p}_{\delta} \to W^{s-2,p}_{\eta-2}$ *is an isomorphism.*

Proof. **(Lemma 5.1)** The proof will involve six main steps.

- **Step 1:** We first check that the assumptions actually make sense. To this end, we need to check that both $A^{shifted}_L$ and $\mathbf{f}^{shifted}_W(\psi)$ are well-defined.

  We first verify that $A^{shifted}_L$ is well-defined, that is it sends elements of $W^{s,p}_{\delta}$ to elements in $W^{s-2,p}_{\eta-2}$. Since we know this is true for A_L, we just need to show that if $\psi \in W^{s,p}_{\delta} \hookrightarrow W^{t,p}_{\gamma}$ then $a_s\psi \in W^{s-2,p}_{\eta-2}$ (note that $a_s \in W^{s-2,p}_{\eta-2}$). To this end we use the multiplication lemma (Lemma A.24) to prove that $W^{s-2,p}_{\eta-2} \times W^{t,p}_{\gamma} \hookrightarrow W^{s-2,p}_{\eta-2}$. To use the lemma, we need the following conditions (the numbering follows the numbering in Lemma A.24):

(i) $s-2 \geq s-2$, (trivially true)
$t \geq s-2$, (since $t \geq 1 \geq s-2$)
(ii) $s-2+t \geq 0$, (since $s-2 \geq -1,\ t \geq 1$)
(iii) $(s-2)-(s-2) \geq 3(\frac{1}{p}-\frac{1}{p})$, (trivially true)
$t-(s-2) \geq 3(\frac{1}{p}-\frac{1}{p})$, (since $t \geq 1 \geq s-2$)
(iv) $(s-2)+t-(s-2) > 3(\frac{1}{p}+\frac{1}{p}-\frac{1}{p})$, (since $t > \frac{3}{p}$)
(v) Case $s-2<0$: $(s-2)+t > 3(\frac{1}{p}+\frac{1}{p}-1)$,

where the last case holds since $(s-2)+t > \frac{3}{p}-2+\frac{3}{p} > 3(\frac{1}{p}+\frac{1}{p}-1)$. Therefore, we can conclude that $W^{s-2,p}_{\eta-2} \times W^{t,p}_{\gamma} \hookrightarrow W^{s-2,p}_{\gamma+\eta-2} \hookrightarrow W^{s-2,p}_{\eta-2}$.

We now confirm that $\mathbf{f}^{shifted}_W(\psi)$ is well-defined. To this end, we just need to show $\mathbf{f}_W$ sends $W^{\tilde{s},p}_{\tilde{\delta}}$ to $W^{s-2,p}_{\eta-2}$. Note that in previous sections by using Lemma A.28 we showed that $\mathbf{f}_W$ sends $W^{s,p}_{\delta}$ to $W^{s-2,p}_{\eta-2}$. By the same argument the above claim can be proved.

- **Step 2:** As a direct consequence of Lemma A.28 and the multiplication lemma, $\mathbf{f}^{shifted}_W$ is a continuous function from $W^{\tilde{s},p}_{\tilde{\delta}}$ to $W^{s-2,p}_{\eta-2}$ (note that a_W, a_τ, a_R, a_ρ are fixed). The continuity of $a_W \to \mathbf{f}_W(\psi)$ for a fixed $\psi \in W^{\tilde{s},p}_{\tilde{\delta}}$ also follows from Lemma A.28.

- **Step 3:** According to Step 2 and the assumption that $A^{shifted}_L$ is an isomorphism, $T^{shifted}$ is a composition of continuous maps with respect to each of its arguments. Therefore $T^{shifted}$ is continuous in both arguments.

- **Step 4:** Let $\theta = \frac{1}{p} - \frac{t-1}{3}$; note that by assumption $t < 1 + \frac{3}{p}$ and so $\theta > 0$. We claim that $\frac{1}{p} \in (\frac{s-1}{2}\theta, 1-\frac{3-s}{2}\theta)$. Indeed, $\frac{1}{p} < 1 - \frac{3-s}{2}\theta$ because
$$t > \frac{3}{p} \Rightarrow \theta = \frac{1}{3}+\frac{1}{p}-\frac{t}{3} < \frac{1}{3},$$
$$s \geq 1 \Rightarrow \frac{3-s}{2} \leq 1 \Rightarrow 1-\frac{3-s}{2}\theta > 1-\theta > \frac{2}{3}.$$
Consequently, since $p > \frac{3}{2}$, we have $\frac{1}{p} < \frac{2}{3} < 1-\frac{3-s}{2}\theta$. It remains to show that $\frac{1}{p} > \frac{s-1}{2}\theta$. Note that
$$\frac{s-1}{2}\theta = \frac{s-1}{2}(\frac{1}{p}-\frac{t-1}{3}) = \frac{s-1}{2p} - \frac{(s-1)(t-1)}{6},$$
and so
$$\frac{1}{p} > \frac{s-1}{2}\theta \Leftrightarrow \frac{(s-1)(t-1)}{6} > \frac{s-1}{2p} - \frac{1}{p} = \frac{s-3}{2p}.$$
The latter inequality is obviously true: if $s=1$ then LHS is zero but RHS is negative. If $s>1$ then LHS is positive but RHS is less than or equal to zero (recall that by assumption $s \leq 3$).

- **Step 5:** Since $s-2 \in [-1,1]$ and $\frac{1}{p} \in (\frac{s-1}{2}\theta, 1-\frac{3-s}{2}\theta)$, we may use Lemma A.31 to estimate $\|\mathbf{f}^{shifted}_W(\psi)\|_{s-2,p,\eta-2}$. (Lemma A.31 is used for estimating similar quantities in later arguments as well, so we give the justification for use of Lemma A.31 as Remark 5.2 following this proof.)

For all $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$ we have (note that $W^{\tilde{s},p}_{\tilde{\delta}} \hookrightarrow W^{t,p}_{\gamma}$ so $\psi \in W^{t,p}_{\gamma}$)

$$
\begin{aligned}
&\|\mathbf{f}^{shifted}_W(\psi)\|_{s-2,p,\eta-2} \\
&\quad = \|a_\tau(\mu+\psi)^5 + a_R(\mu+\psi) - a_\rho(\mu+\psi)^{-3} - a_W(\mu+\psi)^{-7} - a_s\psi\|_{s-2,p,\eta-2} \\
&\quad \leq \|a_\tau(\mu+\psi)^5\|_{s-2,p,\eta-2} + \|a_R(\mu+\psi)\|_{s-2,p,\eta-2} + \|a_\rho(\mu+\psi)^{-3}\|_{s-2,p,\eta-2} \\
&\qquad + \|a_W(\mu+\psi)^{-7}\|_{s-2,p,\eta-2} + \|a_s\psi\|_{s-2,p,\eta-2} \\
&\quad \preceq \|a_\tau\|_{s-2,p,\eta-2}(\|(\mu+\psi)^5\|_{L^\infty} + \|5(\mu+\psi)^4\|_{L^\infty}\|\psi\|_{t,p,\gamma}) \\
&\qquad + \|a_R\|_{s-2,p,\eta-2}(\|\mu+\psi\|_{L^\infty} + \|1\|_{L^\infty}\|\psi\|_{t,p,\gamma}) \\
&\qquad + \|a_0\psi\|_{s-2,p,\eta-2} + \|a_W V\psi\|_{s-2,p,\eta-2} \\
&\qquad + \|a_\rho\|_{s-2,p,\eta-2}(\|(\mu+\psi)^{-3}\|_{L^\infty} + \|-3(\mu+\psi)^{-4}\|_{L^\infty}\|\psi\|_{t,p,\gamma}) \\
&\qquad + \|a_W\|_{s-2,p,\eta-2}(\|(\mu+\psi)^{-7}\|_{L^\infty} + \|-7(\mu+\psi)^{-8}\|_{L^\infty}\|\psi\|_{t,p,\gamma}) \\
&\quad \preceq \|a_\tau\|_{s-2,p,\eta-2}(\|(\mu+\psi)^4\|_{L^\infty}\|\mu+\psi\|_{L^\infty} + \|5(\mu+\psi)^4\|_{L^\infty}\|\psi\|_{t,p,\gamma}) \\
&\qquad + \|a_R\|_{s-2,p,\eta-2}(\|\mu+\psi\|_{L^\infty} + \|\psi\|_{t,p,\gamma}) \\
&\qquad + \|a_0\|_{s-2,p,\eta-2}\|\psi\|_{t,p,\gamma} + \|a_W V\|_{s-2,p,\eta-2}\|\psi\|_{t,p,\gamma} \\
&\qquad\quad (\text{note that } a_W V \in W^{s-2,p}_{\eta-2},\ \ W^{s-2,p}_{\eta-2} \times W^{t,p}_{\gamma} \hookrightarrow W^{s-2,p}_{\eta-2}) \\
&\qquad + \|a_\rho\|_{s-2,p,\eta-2}(\|(\mu+\psi)^{-4}\|_{L^\infty}\|\mu+\psi\|_{L^\infty} + \|-3(\mu+\psi)^{-4}\|_{L^\infty}\|\psi\|_{t,p,\gamma}) \\
&\qquad + \|a_W\|_{s-2,p,\eta-2}(\|(\mu+\psi)^{-8}\|_{L^\infty}\|\mu+\psi\|_{L^\infty} + \|-7(\mu+\psi)^{-8}\|_{L^\infty}\|\psi\|_{t,p,\gamma}).
\end{aligned}
$$

Now note that $W^{t,p}_{\gamma} \hookrightarrow L^\infty_\gamma \hookrightarrow L^\infty$, so

$$
\|\mu+\psi\|_{L^\infty} \leq \mu + \|\psi\|_{L^\infty} \preceq \mu + \|\psi\|_{t,p,\gamma} \preceq 1 + \|\psi\|_{t,p,\gamma}.
$$

Hence

$$
\begin{aligned}
\|\mathbf{f}^{shifted}_W(\psi)\|_{s-2,p,\eta-2} \preceq & \|a_\tau\|_{s-2,p,\eta-2}\|(\mu+\psi)^4\|_{L^\infty}(1+\|\psi\|_{t,p,\gamma}) \\
& + \|a_R\|_{s-2,p,\eta-2}(1+\|\psi\|_{t,p,\gamma}) \\
& + \|a_0\|_{s-2,p,\eta-2}(1+\|\psi\|_{t,p,\gamma}) \\
& + \|a_W\|_{s-2,p,\eta-2}C(\psi_+,\psi_-)(1+\|\psi\|_{t,p,\gamma}) \\
& + \|a_\rho\|_{s-2,p,\eta-2}\|(\mu+\psi)^{-4}\|_{L^\infty}(1+\|\psi\|_{t,p,\gamma}) \\
& + \|a_W\|_{s-2,p,\eta-2}\|(\mu+\psi)^{-8}\|_{L^\infty}(1+\|\psi\|_{t,p,\gamma}).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\|\mathbf{f}^{shifted}_W(\psi)\|_{s-2,p,\eta-2} \\
&\quad \preceq \big[\|a_\tau\|_{s-2,p,\eta-2}\|(\mu+\psi_+)^4\|_{L^\infty} + \|a_R\|_{s-2,p,\eta-2} \\
&\qquad + \|a_\rho\|_{s-2,p,\eta-2}\|(\mu+\psi_-)^{-4}\|_{L^\infty} + \|a_0\|_{s-2,p,\eta-2} \\
&\qquad + \|a_W\|_{s-2,p,\eta-2}(\|(\mu+\psi_-)^{-8}\|_{L^\infty} + C(\psi_+,\psi_-))\big](1+\|\psi\|_{t,p,\gamma}) \\
&\quad \preceq [1 + \|a_W\|_{s-2,p,\eta-2}](1+\|\psi\|_{t,p,\gamma}).
\end{aligned}
$$

Finally note that by assumption $(A_L^{shifted})^{-1} : W^{s-2,p}_{\eta-2} \to W^{s,p}_{\delta}$ is continuous and therefore

$$\begin{aligned}\|T^{shifted}(\psi, a_W)\|_{s,p,\delta} &= \| - (A_L^{shifted})^{-1}\mathbf{f}_W^{shifted}(\psi)\|_{s,p,\delta}\\ &\preceq \|\mathbf{f}_W^{shifted}(\psi)\|_{s-2,p,\eta-2}\\ &\preceq [1 + \|a_W\|_{s-2,p,\eta-2}](1 + \|\psi\|_{t,p,\gamma}).\end{aligned}$$

- **Step 6:** In this step we prove the second claim. By the last item in Lemma B.12, $A_L : W^{s,p}_{\delta} \to W^{s-2,p}_{\delta-2}$ is Fredholm of index zero. By Lemma B.13, $A_L^{shifted} : W^{s,p}_{\delta} \to W^{s-2,p}_{\delta-2}$ is a compact perturbation of A_L. Since A_L is Fredholm of index zero we can conclude that $A_L^{shifted}$ is also Fredholm of index zero. Now maximum principle (Lemma B.11) implies that the kernel of $A_L^{shifted} : W^{s,p}_{\delta} \to W^{s-2,p}_{\delta-2}$ is trivial. An injective operator of index zero is surjective as well. Consequently $A_L^{shifted} : W^{s,p}_{\delta} \to W^{s-2,p}_{\delta-2}$ is a continuous bijective operator. Therefore by the open mapping theorem, $A_L^{shifted} : W^{s,p}_{\delta} \to W^{s-2,p}_{\delta-2}$ is an isomorphism of Banach spaces. In particular the inverse is continuous and so $\|u\|_{s,p,\delta} \preceq \|A_L^{shifted} u\|_{s-2,p,\delta-2}$.

□

Remark 5.2. *In the above proof we used Lemma A.31 to estimate $\|\boldsymbol{f}_W^{shifted}(\psi)\|_{s-2,p,\eta-2}$. Note that since $\psi \in W^{\tilde{s},p}_{\tilde{\delta}} \hookrightarrow C^0_{\tilde{\delta}}$, and $\tilde{\delta} < 0$ we can conclude that $\psi \to 0$ as $|x| \to \infty$ (in the asymptotic ends). Therefore there exists a compact set B such that outside of B, $|\psi| < \frac{\mu}{2}$. On the compact set B, the continuous function ψ attains its minimum which by assumption must be larger than $-\mu$. Consequently $\inf \psi > \min\{-\frac{\mu}{2}, \min_{x\in B} \psi(x)\} > -\mu$. Because of this functions of the form $f(x) = (\mu + x)^{-m}$ where $m \in \mathbb{N}$ are smooth on $[\inf \psi, \sup \psi]$ as it is required by Lemma A.31.*

Lemma 5.3. *In addition to the conditions of Lemma 5.1 (including $\beta \leq \delta$), assume $a_R \geq 0$ (see Remark B.4) and define the shift function a_s by*

$$a_s = a_R + 3\frac{(\mu+\psi_+)^2}{(\mu+\psi_-)^6}a_\rho + 5(\mu+\psi_+)^4 a_\tau + 7\frac{(\mu+\psi_+)^6}{(\mu+\psi_-)^{14}}a_W.$$

Then for any fixed $a_W \in W^{s-2,p}_{\beta-2}$, the map $T^{shifted} : [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \to W^{s,p}_{\delta}$ is monotone increasing.

Proof. **(Lemma 5.3)** First note that the above definition of a_s satisfies the assumptions that we had for a_s in Lemma 5.1. Note that

$$\begin{aligned}a_0 &= a_R + 3\frac{(\mu+\psi_+)^2}{(\mu+\psi_-)^6}a_\rho + 5(\mu+\psi_+)^4 a_\tau,\\ V &= 7\frac{(\mu+\psi_+)^6}{(\mu+\psi_-)^{14}}.\end{aligned}$$

We first must check $a_0 \in W^{s-2,p}_{\eta-2}$ and $\|a_W V\|_{s-2,p,\eta-2} \preceq C(\psi_+, \psi_-)\|a_W\|_{s-2,p,\eta-2}$.

We first check that $a_0 \in W^{s-2,p}_{\eta-2}$. By assumption $a_R \in W^{s-2,p}_{\delta-2} = W^{s-2,p}_{\eta-2}$. The fact that $\frac{(\mu+\psi_+)^2}{(\mu+\psi_-)^6}a_\rho$ and $(\mu+\psi_+)^4 a_\tau$ are in $W^{s-2,p}_{\eta-2}$ follows directly from Lemma A.28. Therefore $a_0 \in W^{s-2,p}_{\eta-2}$.

We now check that $\|a_W V\|_{s-2,p,\eta-2} \preceq C(\psi_+,\psi_-)\|a_W\|_{s-2,p,\eta-2}$. By Lemma A.31 we have

$$\begin{aligned}\|a_W(\mu+\psi_+)^6\|_{s-2,p,\eta-2} &\preceq \|a_W\|_{s-2,p,\eta-2}(\|(\mu+\psi_+)^6\|_{L^\infty} + \|6(\mu+\psi_+)^5\|_{L^\infty}\|\psi_+\|_{\tilde{s},p,\tilde{\delta}}) \\ &= C_1(\psi_+)\|a_W\|_{s-2,p,\eta-2},\end{aligned}$$

and so (recall Remark 5.2)

$$\begin{aligned}\|a_W V\|_{s-2,p,\eta-2} &\preceq \|a_W(\mu+\psi_+)^6(\mu+\psi_-)^{-14}\|_{s-2,p,\eta-2} \\ &\preceq \|a_W(\mu+\psi_+)^6\|_{s-2,p,\eta-2}(\|(\mu+\psi_-)^{-14}\|_{L^\infty} \\ &\quad + \|-14(\mu+\psi_-)^{-15}\|_{L^\infty}\|\psi_-\|_{\tilde{s},p,\tilde{\delta}}) \\ &= C_2(\psi_-)\|a_W(\mu+\psi_+)^6\|_{s-2,p,\eta-2} \\ &\preceq C_1(\psi_+)C_2(\psi_-)\|a_W\|_{s-2,p,\eta-2} \\ &= C(\psi_+,\psi_-)\|a_W\|_{s-2,p,\eta-2}.\end{aligned}$$

Now that we have confirmed the two conditions we can proceed. For all $\psi_1,\psi_2 \in [\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}$ with $\psi_1 \le \psi_2$ we have

$$\begin{aligned}\mathbf{f}_W^{shifted}(\psi_2) - \mathbf{f}_W^{shifted}(\psi_1) &= \mathbf{f}_W(\psi_2) - \mathbf{f}_W(\psi_1) - a_s(\psi_2-\psi_1) \\ &= a_\tau[(\mu+\psi_2)^5 - (\mu+\psi_1)^5] + a_R[\psi_2-\psi_1] \\ &\quad - a_\rho[(\mu+\psi_2)^{-3} - (\mu+\psi_1)^{-3}] \\ &\quad - a_W[(\mu+\psi_2)^{-7} - (\mu+\psi_1)^{-7}] - a_s(\psi_2-\psi_1).\end{aligned}$$

Note that for all $m \in \mathbb{N}$

$$\begin{aligned}(\mu+\psi_2)^m - (\mu+\psi_1)^m &= \Big(\sum_{j=0}^{m-1}(\mu+\psi_2)^j(\mu+\psi_1)^{m-1-j}\Big)(\psi_2-\psi_1) \\ &\le m(\mu+\psi_+)^{m-1}(\psi_2-\psi_1) - [(\mu+\psi_2)^{-m} - (\mu+\psi_1)^{-m}] \\ &= \frac{(\mu+\psi_2)^m - (\mu+\psi_1)^m}{[(\mu+\psi_2)(\mu+\psi_1)]^m} \\ &\le m\frac{(\mu+\psi_+)^{m-1}}{(\mu+\psi_-)^{2m}}(\psi_2-\psi_1).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{f}_W^{shifted}(\psi_2) - \mathbf{f}_W^{shifted}(\psi_1) &\le [5(\mu+\psi_+)^4 a_\tau + a_R + 3\frac{(\mu+\psi_+)^2}{(\mu+\psi_-)^6}a_\rho \\ &\qquad + 7\frac{(\mu+\psi_+)^6}{(\mu+\psi_-)^{14}}a_W - a_s](\psi_2-\psi_1) \\ &= 0.\end{aligned}$$

So $\mathbf{f}_W^{shifted}$ is decreasing over $[\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}$. Also $A_L^{shifted}: W_\delta^{s,p} \to W_{\eta-2}^{s-2,p}$ satisfies the maximum principle, hence the inverse $(A_L^{shifted})^{-1}$ is monotone increasing [32]. Consequently $T^{shifted}: [\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}} \to W_\delta^{s,p}$ defined by $-(A_L^{shifted})^{-1}\mathbf{f}_W^{shifted}$ is monotone increasing. □

Lemma 5.4. *Let the conditions of Lemma 5.3 hold, with ψ_- and ψ_+ sub- and supersolutions of the Hamiltonian constraint (equation (3.3)), respectively (with a_W as* *source). Then, we have $T^{shifted}(\psi_+, a_W) \leq \psi_+$ and $T^{shifted}(\psi_-, a_W) \geq \psi_-$. In particular, since $T^{shifted}$ is monotone increasing in its first variable, $T^{shifted}$ is invariant on $U = [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$, that is, if $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$, then $T^{shifted}(\psi, a_W) \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$.*

Proof. **(Lemma 5.4)** Since ψ_+ is a supersolution, by definition (which can be found in the next section), $A_L\psi_+ + \mathbf{f}_W(\psi_+) \geq 0$ with respect to the order of $W^{s-2,p}_{\delta-2}$ (see Remark B.4). We have

$$\begin{aligned}\psi_+ - T^{shifted}(\psi_+, a_W) &= (A_L^{shifted})^{-1}[A_L^{shifted}\psi_+ + \mathbf{f}_W^{shifted}(\psi_+)] \\ &= (A_L^{shifted})^{-1}[A_L\psi_+ + \mathbf{f}_W(\psi_+)],\end{aligned}$$

which is nonnegative since ψ_+ is supersolution and $(A_L^{shifted})^{-1}$ is linear and monotone increasing. The proof of the other inequality is completely analogous. □

Remark 5.5. *As seen in the proof of the above lemmas, the introduction of the shift function a_s into $\boldsymbol{f}_W^{shifted}$ ensures it is a decreasing function on $[\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$, which subsequently implies that $T^{shifted}$ is invariant on $U = [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$. This property of $T^{shifted}$ plays an important role in the fixed point framework we use for our existence theorem for the coupled system, following closely the approach taken in* [32].

6. GLOBAL SUB- AND SUPERSOLUTION CONSTRUCTIONS

In this section, based on a combination of ideas employed in [32, 41, 18], we introduce a new method for constructing global sub- and supersolutions for the Hamiltonian constraint on AF manifolds. We begin with giving the precise definitions of local and global sub- and supersolutions.

Consider the Hamiltonian constraint (equation 3.3):

$$A_L\psi + f(\psi, W) = 0.$$

- A **local subsolution** of (3.3) is a function $\psi_- \in W^{s,p}_\delta$, $\psi_- > -\mu$ such that
$$A_L\psi_- + f(\psi_-, W) \leq 0$$
for at least one $W \in \mathbf{W}^{e,q}_\beta$. Note that the inequality is with respect to the order of $W^{s-2,p}_{\delta-2}$ (see Remark B.4).
- A **local supersolution** of (3.3) is a function $\psi_+ \in W^{s,p}_\delta$, $\psi_+ > -\mu$ such that
$$A_L\psi_+ + f(\psi_+, W) \geq 0$$
for at least one $W \in \mathbf{W}^{e,q}_\beta$.
- A **global subsolution** of (3.3) is a function $\psi_- \in W^{s,p}_\delta$, $\psi_- > -\mu$ such that
$$A_L\psi_- + f(\psi_-, W_\psi) \leq 0$$
for all vector fields W_ψ solution of (3.4) (momentum constraint) with source $\psi \in W^{s,p}_\delta$ and $\psi \geq \psi_-$.
- A **global supersolution** of (3.3) is a function $\psi_+ \in W^{s,p}_\delta$, $\psi_+ > -\mu$ such that
$$A_L\psi_+ + f(\psi_+, W_\psi) \geq 0$$
for all vector fields W_ψ solution of (3.4) (momentum constraint) with source $\psi \in W^{s,p}_\delta$ and $-\mu < \psi \leq \psi_+$.
- We say a pair of a subsolution and a supersolution, ψ_- and ψ_+, is **compatible** if $-\mu < \psi_- \leq \psi_+ < \infty$ (so $[\psi_-, \psi_+]_{s,p,\delta}$ is nonempty).

For our main existence theorem we need to have compatible global subsolution and supersolution. The goal of this section is to prove the existence of such compatible global barriers. In what follows we use the following notation: Given any scalar function $v \in L^\infty$, let $v^\wedge = \operatorname{ess\,sup}_M v$, and $v^\vee = \operatorname{ess\,inf}_M v$.

Proposition 6.1. *Assume all the conditions of **Weak Formulation 1** and Corollary 4.4 hold true. Additionally assume that h belongs to the positive Yamabe class, $-1 < \beta \leq \delta < 0$, and $\|\sigma\|_{L^\infty_{\beta-1}}$, $\|\rho\|_{L^\infty_{2\beta-2}}$, $\|J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$ are sufficiently small. Moreover, suppose that there exists a positive continuous function $\Lambda \in W^{s-2,p}_{\delta-2}$ and a number $\delta' \in (2\beta, \delta)$ such that $\Lambda \sim r^{\delta'-2}$ (that is, $r^{\delta'-2} \preceq \Lambda \preceq r^{\delta'-2}$) for sufficiently large $r = (1+|x|^2)^{\frac{1}{2}}$ (see Remark 6.2). If $\mu > 0$ is chosen to be sufficiently small, then there exists a global supersolution $\psi_+ \in W^{s,p}_\delta$ to the Hamiltonian constraint.*

Proof. **(Proposition 6.1)** Since h belongs to the positive Yamabe class, there exists a function $\xi \in W^{s,p}_\delta$, $\xi > -1$ such that if we set $\tilde{h} = (1+\xi)^4 h$, then $R_{\tilde{h}} = 0$. Let $H(\psi, a_W, a_\tau, a_\rho)$ and $\tilde{H}(\psi, a_W, a_\tau, a_\rho)$ be as in Appendix C. In what follows we will show that there exists $\tilde{\psi}_+ > 0$ such that

$$\forall\, \varphi \in (-\mu, (\xi+1)\tilde{\psi}_+ + \mu\,\xi]_{s,p,\delta} \quad \tilde{H}(\tilde{\psi}_+, a_{W_\varphi}, a_\tau, a_\rho) \geq 0. \tag{6.1}$$

Here W_φ is the solution of the momentum constraint with source φ. Let's assume we find such a function. Then if we define $\psi_+ = (\xi+1)\tilde{\psi}_+ + \mu\,\xi$, we have $\psi_+ \in W^{s,p}_\delta$, $\psi_+ > -\mu$ and it follows from Corollary C.2 that

$$\forall\, \varphi \in (-\mu, \psi_+]_{s,p,\delta} \quad H(\psi_+, a_{W_\varphi}, a_\tau, a_\rho) \geq 0$$

which precisely means that ψ_+ is a global supersolution of the Hamiltonian constraint. So it is enough to prove the existence of $\tilde{\psi}_+$.

Let $\Lambda \in W^{s-2,p}_{\delta-2}$ be a positive continuous function such that $\Lambda \sim r^{\delta'-2}$ for sufficiently large $|x|$; here δ' is a fixed but arbitrary number in the interval $(2\beta, \delta)$. By Lemma B.12 there exists a unique function $u \in W^{s,p}_\delta$ such that $-\Delta_{\tilde{h}} u = \Lambda$. By the maximum principle (Lemma B.11) u is positive ($u > 0$). Recall that μ is a fixed nonzero number but we have freedom in choosing μ. We claim that if $\mu > 0$ is sufficiently small, then $\tilde{\psi}_+ := \mu\, u$ satisfies (6.1). Indeed, for all $\varphi \in (-\mu, (\xi+1)\tilde{\psi}_+ + \mu\,\xi]_{s,p,\delta}$ we have

$$\begin{aligned}
\tilde{H}(\tilde{\psi}_+, a_{W_\varphi}, a_\tau, a_\rho) &= -\Delta_{\tilde{h}}\tilde{\psi}_+ + a_\tau(\tilde{\psi}_+ + \mu)^5 - (1+\xi)^{-12} a_{W_\varphi}(\tilde{\psi}_+ + \mu)^{-7} \\
&\quad - (1+\xi)^{-8} a_\rho(\tilde{\psi}_+ + \mu)^{-3} \quad (R_{\tilde{h}} = 0) \\
&= \mu\,\Lambda + a_\tau(\tilde{\psi}_+ + \mu)^5 - (1+\xi)^{-12} a_{W_\varphi}(\tilde{\psi}_+ + \mu)^{-7} \\
&\quad - (1+\xi)^{-8} a_\rho(\tilde{\psi}_+ + \mu)^{-3} \\
&\geq \mu\,\Lambda - (1+\xi)^{-12} a_{W_\varphi}(\tilde{\psi}_+ + \mu)^{-7} - (1+\xi)^{-8} a_\rho(\tilde{\psi}_+ + \mu)^{-3}.
\end{aligned}$$

The argument in Remark 5.2 shows that $(\inf \xi) > -1$ and so $\inf(1+\xi) > 0$. Therefore if we let $\tilde{C} = \max\{((1+\xi)^\vee)^{-12}, ((1+\xi)^\vee)^{-8}\}$, then

$$\begin{aligned}
\tilde{H}(\tilde{\psi}_+, a_{W_\varphi}, a_\tau, a_\rho) &\geq \mu\,\Lambda - \tilde{C} a_{W_\varphi}(\tilde{\psi}_+ + \mu)^{-7} - \tilde{C} a_\rho(\tilde{\psi}_+ + \mu)^{-3} \\
&= \mu\,\Lambda - \tilde{C}\mu^{-7} a_{W_\varphi}(u+1)^{-7} - \tilde{C}\mu^{-3} a_\rho(u+1)^{-3} \\
&\geq \mu\,\Lambda - C\mu^{-7} r^{2\beta-2}(k_1\|\mu+\varphi\|_\infty^{12} + k_2)(u+1)^{-7} \\
&\quad - \tilde{C}\mu^{-3} a_\rho(u+1)^{-3},
\end{aligned}$$

where we have used Lemma 4.6. Recall that C (the implicit constant in Lemma 4.6) does not depend on μ. Now note that $\forall\, \varphi \in (-\mu, (\xi+1)\tilde{\psi}_+ + \mu\,\xi]_{s,p,\delta}$ we have $0 \leq \mu + \varphi \leq (\xi+1)(\mu+\tilde{\psi}_+)$ and so

$$\|\mu+\varphi\|_\infty^{12} \leq [(\xi+1)^\wedge]^{12}[(\mu+\tilde{\psi}_+)^\wedge]^{12}.$$

Let $k_3 = (\xi+1)^\wedge \frac{(1+u)^\wedge}{(1+u)^\vee}$. We can write

$$\begin{aligned}\|\mu+\varphi\|_\infty^{12} &\leq [(\xi+1)^\wedge]^{12}[(\mu+\tilde{\psi}_+)^\wedge]^{12} = [(\xi+1)^\wedge]^{12}\mu^{12}[(1+u)^\wedge]^{12}\\ &= k_3^{12}\mu^{12}[(1+u)^\vee]^{12} \leq k_3^{12}\mu^{12}(u+1)^{12}.\end{aligned}$$

Consequently

$$\begin{aligned}\tilde{H}(\tilde{\psi}_+, a_{W_\varphi}, a_\tau, a_\rho) &\geq \mu\,\Lambda - C\mu^{-7}r^{2\beta-2}(k_1k_3^{12}\mu^{12}(u+1)^{12} + k_2)(u+1)^{-7}\\ &\quad - \tilde{C}\mu^{-3}a_\rho(u+1)^{-3}\\ &= \mu\,\Lambda - C\mu^5 r^{2\beta-2}k_1k_3^{12}(u+1)^5 - C\mu^{-7}r^{2\beta-2}k_2(u+1)^{-7}\\ &\quad - \tilde{C}\mu^{-3}a_\rho(u+1)^{-3}\\ &\geq \mu\,\Lambda - C\mu^5 r^{2\beta-2}k_1k_3^{12}((u+1)^\wedge)^5 - C\mu^{-7}r^{2\beta-2}k_2((u+1)^\vee)^{-7}\\ &\quad - \tilde{C}\mu^{-3}a_\rho((u+1)^\vee)^{-3}.\end{aligned}$$

Note that $\Lambda \sim r^{\delta'-2}$ for sufficiently large r and $2\beta-2 < \delta'-2 < 0$. We claim that this allows one to choose μ small enough so that

$$\frac{\Lambda}{2} > C\mu^4 r^{2\beta-2}k_1k_3^{12}((u+1)^\wedge)^5. \tag{6.2}$$

The justification of this claim is as follows. There exists a constant C_1 and a number $r_1 > 0$ such that for $r > r_1$, we have $\Lambda \geq C_1 r^{\delta'-2}$. Also since $2\beta-2 < \delta'-2 < 0$, there exists $r_2 > 0$ such that for all $r > r_2$

$$\frac{C_1}{2}r^{\delta'-2} > r^{2\beta-2}[Ck_1k_3^{12}((u+1)^\wedge)^5].$$

Consequently for all $0 < \mu \leq 1$ and $r > \max\{r_1, r_2\}$

$$\frac{\Lambda}{2} > r^{2\beta-2}[Ck_1k_3^{12}((u+1)^\wedge)^5] \geq C\mu^4 r^{2\beta-2}k_1k_3^{12}((u+1)^\wedge)^5.$$

Also the positive continuous function Λ attains its minimum $\Lambda^\vee > 0$ on the compact set $r \leq \max\{r_1, r_2\}$. We choose $\mu \leq 1$ small enough such that

$$\frac{\Lambda^\vee}{2} > C\mu^4 k_1k_3^{12}((u+1)^\wedge)^5. \tag{6.3}$$

Since $r^{2\beta-2} \leq 1$ the above inequality implies that (6.2) holds even if $r \leq \max\{r_1, r_2\}$. (Note that on the entire M, $\Lambda^\vee = 0$, so we could not use (6.3) on whole M to determine μ; this is exactly why first we needed to study what happens for large r.) For such μ by requiring that $\|\sigma\|_{L^\infty_{\beta-1}}$, $\|\rho\|_{L^\infty_{2\beta-2}}$, $\|J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$ are sufficiently small (note that according to Remark A.5, $a_\rho \leq r^{2\beta-2}\|a_\rho\|_{L^\infty_{2\beta-2}}$a.e.) we can ensure that

$$\frac{\Lambda}{2} \geq C\mu^{-8}r^{2\beta-2}k_2((u+1)^\vee)^{-7} + \tilde{C}\mu^{-4}a_\rho((u+1)^\vee)^{-3}.$$

□

Remark 6.2. *Pick an arbitrary number $\delta' \in (2\beta, \delta)$. If $s \leq 2$, then $\Lambda = r^{\delta'-2}$ satisfies the desired conditions: clearly Λ is positive, continuous, and $\Lambda \sim r^{\delta'-2}$. Also obviously $\Lambda \in L^{\infty}_{\delta'-2}$ and*

$$L^{\infty}_{\delta'-2} \hookrightarrow L^{p}_{\delta-2} \hookrightarrow W^{s-2,p}_{\delta-2} \quad (\Longrightarrow \Lambda \in W^{s-2,p}_{\delta-2})$$

The first inclusion is true because $\delta'$is strictly less than δ; the second inclusion is true because $s-2 \leq 0$.

Proposition 6.3. *Assume all the conditions of **Weak Formulation 1**. Additionally assume that h belongs to the positive Yamabe class and $-1 < \beta \leq \delta < 0$. If $\mu > 0$ is chosen to be sufficiently small, then there exists a global subsolution $\psi_- \in W^{s,p}_{\delta}$ to the Hamiltonian constraint which is compatible with the global supersolution that was constructed in Proposition 6.1 (provided the extra assumptions of that proposition hold true).*

Proof. **(Proposition 6.3)** Since h belongs to the positive Yamabe class, there exists a function $\xi \in W^{s,p}_{\delta}$, $\xi > -1$ such that if we set $\tilde{h} = (1+\xi)^4 h$, then $R_{\tilde{h}} = 0$. Let $H(\psi, a_W, a_\tau, a_\rho)$ and $\tilde{H}(\psi, a_W, a_\tau, a_\rho)$ be as in Appendix C. In what follows we will show that there exists $-\mu < \tilde{\psi}_- < 0$ such that

$$\forall\, \varphi \in W^{s,p}_{\delta}, \quad \tilde{H}(\tilde{\psi}_-, a_{W_\varphi}, a_\tau, a_\rho) \leq 0. \tag{6.4}$$

Here W_φ is the solution of the momentum constraint with source φ. Note that since $\tilde{\psi}_+ > 0$ ($\tilde{\psi}_+$ is the function that was introduced in the proof of the previous proposition), clearly $\tilde{\psi}_- \leq 0 < \tilde{\psi}_+$. Let's assume we find such a function. Then if we define $\psi_- = (\xi+1)\tilde{\psi}_- + \mu\,\xi$, we have $\psi_- \in W^{s,p}_{\delta}$, and

$$\begin{aligned}
&\tilde{\psi}_- > -\mu \Longrightarrow (\xi+1)(\tilde{\psi}_- + \mu) > 0 \Longrightarrow (\xi+1)\tilde{\psi}_- + \mu\,\xi > -\mu \Longrightarrow \psi_- > -\mu \\
&\tilde{\psi}_- \leq \tilde{\psi}_+ \Longrightarrow \psi_- \leq \psi_+
\end{aligned}$$

Moreover, it follows from Corollary C.2 that

$$\forall\, \varphi \in W^{s,p}_{\delta}, \quad H(\psi_-, a_{W_\varphi}, a_\tau, a_\rho) \leq 0,$$

which clearly implies that ψ_- is a global subsolution of the Hamiltonian constraint. So it is enough to prove the existence of $\tilde{\psi}_-$.

We may consider two cases:

Case 1: $a_\tau \equiv 0$

In this case $\tilde{\psi}_- \equiv 0$ satisfies the desired conditions; Indeed,

$$\tilde{H}(\tilde{\psi}_- \equiv 0, a_{W_\varphi}, a_\tau, a_\rho) = -(1+\xi)^{-12} a_{W_\varphi} \mu^{-7} - (1+\xi)^{-8} a_\rho \mu^{-3} \leq 0.$$

Case 2: $a_\tau \not\equiv 0$

By Lemma B.12 there exists a unique function $u \in W^{s,p}_{\delta}$ such that $-\Delta_{\tilde{h}} u = -a_\tau$. By the maximum principle (Lemma B.11) $u \leq 0$ and clearly $u \not\equiv 0$ (because $a_\tau \not\equiv 0$). Note that $W^{s,p}_{\delta} \hookrightarrow L^{\infty}_{\delta} \hookrightarrow L^{\infty}$ (the latter embedding is true because $\delta < 0$). Let $m = \|u\|_\infty + 1$; so in particular $-m < \inf u < 0$. Recall that we have freedom in choosing the fixed number μ as small as we want. We claim that if $\mu > 0$ is sufficiently small, then $\tilde{\psi}_- := \frac{1}{m}\mu\, u$ satisfies (6.4). Clearly $\tilde{\psi}_- \leq 0$; also

$$u > -m \Longrightarrow \mu(u+m) > 0 \Longrightarrow \mu(\frac{u+m}{m}) > 0 \Longrightarrow \frac{1}{m}\mu u > -\mu \Longrightarrow \tilde{\psi}_- > -\mu.$$

Moreover, for all $\varphi \in W^{s,p}_\delta$ we have

$$\begin{aligned}\tilde{H}(\tilde{\psi}_-, a_{W_\varphi}, a_\tau, a_\rho) &= -\Delta_{\tilde{h}}\tilde{\psi}_- + a_\tau(\tilde{\psi}_- + \mu)^5 - (1+\xi)^{-12}a_{W_\varphi}(\tilde{\psi}_- + \mu)^{-7}\\ &\qquad - (1+\xi)^{-8}a_\rho(\tilde{\psi}_- + \mu)^{-3}\\ &\leq -\Delta_{\tilde{h}}(\frac{1}{m}\mu u) + a_\tau(\frac{1}{m}\mu u + \mu)^5\\ &= -\frac{1}{m}\mu a_\tau + \mu^5 a_\tau(\frac{1}{m}u + 1)^5\\ &= \mu a_\tau\big[-\frac{1}{m} + \mu^4(\frac{1}{m}u+1)^5\big].\end{aligned}$$

Now note that $-m < u < m$ and so $0 < 1 + \frac{1}{m}u < 2$, therefore

$$\tilde{H}(\tilde{\psi}_-, a_{W_\varphi}, a_\tau, a_\rho) \leq \mu a_\tau\big[-\frac{1}{m} + 32\mu^4\big].$$

Thus if we choose μ so that $\mu^4 < \frac{1}{32m}$, then $\tilde{H}(\tilde{\psi}_-, a_{W_\varphi}, a_\tau, a_\rho) \leq 0$. □

Remark 6.4. *The compatible global barrier constructions in* [18] *and* [32] *both make critical use of the fact that the conformal factor ϕ, which is the primary unknown in their formulations, is positive. When the subsolution and supersolution are both positive, then one can scale the subsolution to make it smaller than the supersolution. In the formulation presented in this paper, which is designed to allow very low regularity assumptions on the data on AF manifolds, the primary unknown is a shifted version of the conformal factor (ψ). ψ can be negative and so in particular the scaling argument cannot be directly applied here. Due to the nonlinear nature of the Hamiltonian constraint, this situation cannot be resolved simply by finding compatible barriers for the original positive unknown ϕ and then shifting those to obtain compatible barriers for ψ.*

7. The Main Existence Result for Rough non-CMC Solutions

We now establish existence of coupled non-CMC weak solutions for AF manifolds by combining the results for the individual Hamiltonian and momentum constraints developed in Sections 5 and 4, the barrier constructions developed in Section 6, together with the following topological fixed-point theorem for the coupled system from [32]:

Theorem 7.1 (Coupled Schauder Theorem)**.** [32] *Let X and Y be Banach spaces, and let Z be an ordered Banach space with compact embedding $X \hookrightarrow Z$. Let $[\psi_-, \psi_+] \subset Z$ be a non-empty interval, and set $U = [\psi_-, \psi_+] \cap \bar{B}_M \subset Z$ where $\bar{B}_M$ is a closed ball of finite radius $M > 0$ in Z. Assume U is nonempty and let $S : U \to \mathcal{R}(S) \subset Y$ and $T : U \times \mathcal{R}(S) \to U \cap X$ be continuous maps. Then, there exist $w \in \mathcal{R}(S)$ and $\psi \in U \cap X$ such that*

$$\psi = T(\psi, w) \quad \textit{and} \quad w = S(\psi).$$

Remark 7.2. *In* [32] *the above theorem is stated with the extra assumption that $\bar{B}_M$ is a ball of radius M* ***about the origin*** *but the same proof works even if $\bar{B}_M$ is not centered at the origin.*

With all of the supporting results we need now in place, we state and prove our main result.

Theorem 7.3. *Let (M, h) be a 3-dimensional AF Riemannian manifold of class $W^{s,p}_\delta$ where $p \in (1,\infty)$, $s \in (1 + \dfrac{3}{p}, \infty)$ and $-1 < \delta < 0$ are given. Suppose h admits no* *nontrivial conformal Killing field (see Remark 4.3) and is in the positive Yamabe class. Let $\beta \in (-1, \delta]$. Select q and e to satisfy:*

$$\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}],$$
$$e \in (1+\frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q}+s-\frac{3}{p}-1, \frac{3}{q}+s-\frac{3}{p}].$$

Let $q = p$ if $e = s \notin \mathbb{N}_0$. Moreover if $s > 2$, $s \notin \mathbb{N}_0$, assume $e < s$.

Assume that the data satisfies:

- *$\tau \in W^{e-1,q}_{\beta-1}$ if $e \geq 2$ and $\tau \in W^{1,z}_{\beta-1}$ otherwise, where $z = \dfrac{3q}{3+(2-e)q}$ (note that if $e = 2$, then $W^{e-1,q}_{\beta-1} = W^{1,z}_{\beta-1}$),*
- *$\sigma \in W^{e-1,q}_{\beta-1}$,*
- *$\rho \in W^{s-2,p}_{\beta-2} \cap L^{\infty}_{2\beta-2}$, $\rho \geq 0$ (ρ can be identically zero),*
- *$J \in \mathbf{W}^{e-2,q}_{\beta-2}$.*

Recall that we have freedom in choosing the positive constant μ in equations (3.3) and (3.4). If μ is chosen to be sufficiently small and if $\|\sigma\|_{L^{\infty}_{\beta-1}}$, $\|\rho\|_{L^{\infty}_{2\beta-2}}$, and $\|J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$ are sufficiently small, then there exists $\psi \in W^{s,p}_{\delta}$ with $\psi > -\mu$ and $W \in \mathbf{W}^{e,q}_{\beta}$ solving (3.3) and (3.4).

Remark 7.4. *As discussed in Appendix E, the assumptions "$p = q$ if $e = s \notin \mathbb{N}_0$" and "$e < s$ if $s > 2$, $s \notin \mathbb{N}_0$" can be removed if we replace weighted Sobolev-Slobodeckij spaces with weighted Bessel potential spaces.*

Proof. **(Theorem 7.3)** First we prove the claim for the case $s \leq 2$ and then we extend the proof for $s > 2$ by bootstrapping.

Case 1: $s \leq 2$

Note that by assumption $e \leq s$, so e is also less than or equal to 2. Also since $2 \geq s > 1 + \frac{3}{p}$, p is larger than 3.

Since $s \leq 2$, it follows from Proposition 6.1, Remark 6.2 and Proposition 6.3 that if μ is chosen to be sufficiently small, then for $\|\sigma\|_{L^{\infty}_{\beta-1}}$, $\|\rho\|_{L^{\infty}_{2\beta-2}}$, and $\|J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$ sufficiently small, there exists a compatible pair of global subsolution and supersolution. We fix such μ and assume that $\|\sigma\|_{L^{\infty}_{\beta-1}}$, $\|\rho\|_{L^{\infty}_{2\beta-2}}$, and $\|J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$ are sufficiently small (according to Proposition 6.1).

Step 1: The choice of function spaces.

- $X = W^{s,p}_{\delta}$, with s and p as given in the theorem statement.
- $Y = \mathbf{W}^{e,q}_{\beta}$, with e, q as given in the theorem statement.
- $Z = W^{\tilde{s},p}_{\tilde{\delta}}$, $\tilde{s} \in (1, 1+\dfrac{3}{p})$ and $\tilde{\delta} > \delta$, so that $X = W^{s,p}_{\delta} \hookrightarrow W^{\tilde{s},p}_{\tilde{\delta}} = Z$ is compact. Note that $\tilde{s} \in (1, 1+\dfrac{3}{p})$ implies that $\tilde{s} \in (\frac{3}{p}, s)$ (because $p > 3$ and $s > 1 + \frac{3}{p}$).
- $U = [\psi_-, \psi_+]_{W^{\tilde{s},p}_{\tilde{\delta}}} \cap \bar{B}_M \subset W^{\tilde{s},p}_{\tilde{\delta}} = Z$, with ψ_- and ψ_+ compatible global barriers constructed in the previous section and with sufficiently large M to be determined below.

Step 2: Construction of the mapping S. Using Lemma A.28, it can be easily checked that for any $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$, $\mathbf{f}(\psi) = b_\tau(\psi+\mu)^6 + b_J \in \mathbf{W}^{e-2,q}_{\beta-2}$. Therefore, since the metric admits no nontrivial conformal Killing field, by Theorem 4.1, the momentum constraint is uniquely solvable for any "source" $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$ (it is easy to see that the assumptions of Theorem 4.1 are satisfied; see Remark 3.2) . The ranges for the exponents ensure that the momentum constraint solution map

$$S:[\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}\to \mathbf{W}^{e,q}_{\beta}=Y,\quad S(\psi)=-\mathcal{A}_L^{-1}\mathbf{f}(\psi)$$

is continuous. Indeed, by Lemma A.28, $\psi\to\mathbf{f}(\psi)$ is a continuous map from $W^{\tilde{s},p}_{\tilde{\delta}}$ to $\mathbf{W}^{e-2,q}_{\beta-2}$ and by Theorem 4.1, $\mathcal{A}_L^{-1}:\mathbf{W}^{e-2,q}_{\beta-2}\to\mathbf{W}^{e,q}_{\beta}$ is continuous.
Step 3: Construction of the mapping T. By assumption h belongs to the positive Yamabe class. In particular, there exists $\xi\in W^{s,p}_{\delta}$, $\xi>-1$ such that $R_{\tilde{h}}=0$ where $\tilde{h}=(1+\xi)^4h$. Let $\tilde{\psi}_+$ and $\tilde{\psi}_-$ be the functions that were constructed in the proofs of Proposition 6.1 and Proposition 6.3. Also let

$$\tilde{a}_\tau:=a_\tau,\ \tilde{a}_\rho:=(1+\xi)^{-8}a_\rho,\ \tilde{a}_W:=(1+\xi)^{-12}a_W,\ \tilde{a}_R:=a_{R_{\tilde{h}}}=0.$$

Notice that the above notations agree with the ones that are introduced in Appendix C. Using Lemma A.28 it is easy to see that $\tilde{a}_\rho$, $\tilde{a}_W$ remain in $W^{s-2,p}_{\beta-2}$. So we may use $(\tilde{h},\tilde{a}_\tau,\tilde{a}_\rho,\tilde{a}_W,\tilde{a}_R=0,\tilde{\psi}_+,\tilde{\psi}_-)$ as data in Lemmas 5.1, 5.3, and 5.4. That is, if we define

$$\tilde{a}_s:=\tilde{a}_R+3\frac{(\mu+\tilde{\psi}_+)^2}{(\mu+\tilde{\psi}_-)^6}\tilde{a}_\rho+5(\mu+\tilde{\psi}_+)^4\tilde{a}_\tau+7\frac{(\mu+\tilde{\psi}_+)^6}{(\mu+\tilde{\psi}_-)^{14}}\tilde{a}_W,$$
$$\tilde{A}_L^{shifted}:W^{s,p}_{\delta}\to W^{s-2,p}_{\delta-2},\quad \tilde{A}_L^{shifted}\psi=-\Delta_{\tilde{h}}\psi+\tilde{a}_s\psi,$$
$$\tilde{\mathbf{f}}_W^{shifted}(\psi)=\tilde{a}_\tau(\mu+\psi)^5+\tilde{a}_R(\mu+\psi)-\tilde{a}_\rho(\mu+\psi)^{-3}-\tilde{a}_W(\mu+\psi)^{-7}-\tilde{a}_s\psi,$$
$$\tilde{T}^{shifted}:[\tilde{\psi}_-,\tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}\times W^{s-2,p}_{\beta-2}\to W^{s,p}_{\delta},\quad \tilde{T}^{shifted}(\psi,\tilde{a}_W)=-(\tilde{A}_L^{shifted})^{-1}\tilde{\mathbf{f}}_W^{shifted}(\psi),$$

then, according to the aforementioned lemmas, $\tilde{T}^{shifted}$ is continuous with respect to both of its arguments and it is invariant on $[\tilde{\psi}_-,\tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}$. Notice that if we define the **scaled Hamiltonian constraint** as in Appendix C, that is, if we let

$$\tilde{H}(\psi,a_W,a_\tau,a_\rho)=-\Delta_{\tilde{h}}\psi+\tilde{a}_R(\psi+\mu)+\tilde{a}_\tau(\psi+\mu)^5-\tilde{a}_W(\psi+\mu)^{-7}-\tilde{a}_\rho(\psi+\mu)^{-3}$$

then $\tilde{\psi}_-$ and $\tilde{\psi}_+$ are subsolution and supersolution of $\tilde{H}=0$ and moreover

$$\tilde{H}(\psi,a_W,a_\tau,a_\rho)=0\Longleftrightarrow\tilde{T}^{shifted}(\psi,\tilde{a}_W)=\psi.$$

Now we define the mapping $T:[\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}\times\mathbf{W}^{e,q}_{\beta}\to W^{s,p}_{\delta}$ as follows:

$$T(\psi,W)=(\xi+1)\tilde{T}^{shifted}(\frac{\psi-\mu\,\xi}{\xi+1},(\xi+1)^{-12}a_W)+\mu\,\xi.$$

Here ψ_+ and ψ_- are the supersolution and subsolution that were constructed in the proofs of Proposition 6.1 and Proposition 6.3. Recall that by our construction

$$\tilde{\psi}_-=\frac{\psi_--\mu\,\xi}{\xi+1},\quad \tilde{\psi}_+=\frac{\psi_+-\mu\,\xi}{\xi+1}$$

so for $\psi\in[\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}$, we have $\tilde{\psi}_-\leq\frac{\psi-\mu\,\xi}{\xi+1}\leq\tilde{\psi}_+$. In fact using Lemma A.28 one can easily show that T is well-defined. That is, $\frac{\psi-\mu\,\xi}{\xi+1}$ is in $[\tilde{\psi}_-,\tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}$ and $(\xi+1)\tilde{T}^{shifted}(.,.)+\mu\,\xi$ is in $W^{s,p}_{\delta}$. Continuity of T follows from the continuity of $\tilde{T}^{shifted}$ and Lemma A.28.

Considering the coupled Schauder theorem, in order to complete the proof for the case $s\leq 2$, it is enough to prove the following claim:

***Claim**: There exists $M>0$ such that if we set $U=[\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}\cap\bar{B}_M(\mu\,\xi)$, then U is nonempty and*

$$(\psi,W)\in U\times S(U)\Longrightarrow T(\psi,a_W)\in U,$$

where $\bar{B}_M(\mu\xi)$ is the ball of radius M in $W^{\tilde{s},p}_{\tilde{\delta}}$ centered at $\mu\xi \in W^{s,p}_{\delta} \hookrightarrow W^{\tilde{s},p}_{\tilde{\delta}}$.

Proof of Claim. First, as mentioned above, note that $T(\psi, a_W)$ certainly belongs to $X = W^{s,p}_{\delta}$, so instead of $T(\psi, a_W) \in U$ on the right hand side we could write $T(\psi, a_W) \in U \cap X$. We now prove that if $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$, then for all $a_W \in W^{s-2,p}_{\beta-2}$ (and so for all $W \in \mathbf{W}^{e,q}_{\beta}$), $T(\psi, W) \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$:

$$\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \Longrightarrow \frac{\psi - \mu\xi}{\xi + 1} \in [\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}.$$

But we know that $\tilde{T}^{shifted}$ is invariant on $[\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}$ and so

$$\forall\, a_W \in W^{s-2,p}_{\beta-2} \quad \tilde{T}^{shifted}(\frac{\psi - \mu\xi}{\xi + 1}, (1+\xi)^{-12} a_W) \in [\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}.$$

Therefore for all $W \in \mathbf{W}^{e,q}_{\beta}$

$$(1+\xi)\tilde{\psi}_- + \mu\xi \le (1+\xi)\tilde{T}^{shifted}(\frac{\psi - \mu\xi}{\xi + 1}, (1+\xi)^{-12} a_W) + \mu\xi \le (1+\xi)\tilde{\psi}_+ + \mu\xi$$
$$\psi_- \le T(\psi, W) \le \psi_+$$

Thus $T(\psi, W) \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$ (note that as it was already mentioned $T(\psi, W) \in W^{s,p}_{\delta} \hookrightarrow W^{\tilde{s},p}_{\tilde{\delta}}$).

Now to complete the proof of the claim above, it is enough to show that the following auxiliary claim holds true:

***Auxiliary Claim**: There exists $\hat{M} > 0$ such that for all $M \ge \hat{M}$ the following holds:*

$$\textit{If } \psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \cap \bar{B}_M(\mu\xi)$$

$$\textit{Then } \forall\, W \in S([\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}), \quad T(\psi, a_W) \in \bar{B}_M(\mu\xi). \tag{7.1}$$

Remark 7.5. *We make two remarks before we continue.*

1. *In order to prove the main claim, it is enough to prove the auxiliary claim for $W \in S([\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \cap \bar{B}_M(\mu\xi))$ not $W \in S([\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}})$. So what we will prove here is slightly stronger than what we need.*
2. *Since we will prove (7.1) is true for **all** $M \ge \hat{M}$, we can certainly choose an M such that $[\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \cap \bar{B}_M(\mu\xi)) \ne \emptyset$.*

Proof of Auxiliary Claim. We will rely on two supporting results (Lemma 8.1 and 8.2), which will be stated and proved following the completion of our proof of the main result here.

To begin, let $t \in (\frac{3}{p}, \tilde{s}) \cap [1, 1 + \frac{3}{p})$ and let $\gamma \in (\tilde{\delta}, 0)$; also for all $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$ let $\tilde{\psi} := \frac{\psi - \mu\xi}{\xi+1}$. It follows from Lemma 5.1 that there exists $K > 0$ such that for all $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$ and for all $W \in S([\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}})$

$$\|\tilde{T}^{shifted}(\tilde{\psi}, \tilde{a}_W)\|_{\tilde{s},p,\tilde{\delta}} \le \tilde{\tilde{C}} \|\tilde{T}^{shifted}(\tilde{\psi}, \tilde{a}_W)\|_{s,p,\delta} \le K[1 + \|\tilde{a}_W\|_{s-2,p,\delta-2}](1 + \|\tilde{\psi}\|_{t,p,\gamma}).$$

Now note that $W^{\tilde{s},p}_{\tilde{\delta}} \hookrightarrow W^{t,p}_{\gamma}$ is compact and $W^{t,p}_{\gamma} \hookrightarrow L^p_{\gamma}$ is continuous. Therefore by Ehrling's lemma (Lemma B.14) for any $\epsilon > 0$ there exists $\tilde{C}(\epsilon) > 0$ such that

$$\|\tilde{\psi}\|_{t,p,\gamma} \le \epsilon \|\tilde{\psi}\|_{\tilde{s},p,\tilde{\delta}} + \tilde{C}(\epsilon) \|\tilde{\psi}\|_{L^p_{\gamma}}.$$

Since $-\mu < \tilde{\psi}_- \leq \tilde{\psi} \leq \tilde{\psi}_+$, $\|\tilde{\psi}\|_{L^p_\gamma}$ is bounded uniformly with a constant P which we absorb into $\tilde{C}(\epsilon)$. Making use of Lemma 8.2 below, we have

$$\|\tilde{T}^{shifted}(\tilde{\psi},\tilde{a}_W)\|_{\tilde{s},p,\tilde{\delta}} \leq K[1+C](1+\epsilon\|\tilde{\psi}\|_{\tilde{s},p,\tilde{\delta}}+\tilde{C}(\epsilon))$$

Therefore we can write $\forall\,\psi \in [\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}$ and $\forall\,W \in S([\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}})$,

$$\begin{aligned}
\|T(\psi,W)-\mu\,\xi\|_{\tilde{s},p,\tilde{\delta}} &= \|(1+\xi)\tilde{T}^{shifted}(\tilde{\psi},\tilde{a}_W)\|_{\tilde{s},p,\tilde{\delta}}\\
&= \|\tilde{T}^{shifted}(\tilde{\psi},\tilde{a}_W)\|_{\tilde{s},p,\tilde{\delta}} + \|\xi\tilde{T}^{shifted}(\tilde{\psi},\tilde{a}_W)\|_{\tilde{s},p,\tilde{\delta}}\\
&\leq C_4(\|\xi\|_{s,p,\delta}+1)\|\tilde{T}^{shifted}(\tilde{\psi},\tilde{a}_W)\|_{\tilde{s},p,\tilde{\delta}}\\
&\qquad \text{(note that } W^{s,p}_\delta \times W^{s,p}_\delta \hookrightarrow W^{s,p}_\delta \hookrightarrow W^{\tilde{s},p}_{\tilde{\delta}}\text{)}\\
&\leq C_4(\|\xi\|_{s,p,\delta}+1)K[1+C](1+\epsilon\|\tilde{\psi}\|_{\tilde{s},p,\tilde{\delta}}+\tilde{C}(\epsilon)).
\end{aligned}$$

Now let $A := C_4(\|\xi\|_{s,p,\delta}+1)K[1+C]$, so for all $\psi \in [\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}$ and for all $W \in S([\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}})$

$$\|T(\psi,W)-\mu\,\xi\|_{\tilde{s},p,\tilde{\delta}} \leq A(1+\epsilon\|\frac{\psi-\mu\,\xi}{\xi+1}\|_{\tilde{s},p,\tilde{\delta}}+\tilde{C}(\epsilon))$$

Using the argument in Lemma 8.1 below, one can show that for $f \in W^{\tilde{s},p}_{\tilde{\delta}}$

$$\|\frac{1}{\xi+1}f\|_{\tilde{s},p,\tilde{\delta}} \leq C_5(1+\|\frac{\xi}{\xi+1}\|_{s,p,\delta})\|f\|_{\tilde{s},p,\tilde{\delta}},$$

so if we let $\alpha := C_5(1+\|\frac{\xi}{\xi+1}\|_{s,p,\delta})$, then $\|\frac{\psi-\mu\,\xi}{\xi+1}\|_{\tilde{s},p,\tilde{\delta}} \leq \alpha\|\psi-\mu\,\xi\|_{\tilde{s},p,\tilde{\delta}}$ and therefore

$$\|T(\psi,W)-\mu\,\xi\|_{\tilde{s},p,\tilde{\delta}} \leq A(1+\epsilon\alpha\|\psi-\mu\,\xi\|_{\tilde{s},p,\tilde{\delta}}+\tilde{C}(\epsilon)).$$

Let $\epsilon = \frac{1}{2\alpha A}$ and define $\hat{M} := 2A + 2A\tilde{C}(\epsilon)$. For all $M \geq \hat{M}$ we have

$$\begin{aligned}
&\forall\,\psi \in [\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}} \cap \bar{B}_M(\mu\,\xi) \quad \forall\,W \in S([\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}})\\
&\quad \|T(\psi,W)-\mu\,\xi\|_{\tilde{s},p,\tilde{\delta}} \leq A(1+\epsilon\alpha M+\tilde{C}(\epsilon)) \quad \text{(note that } \|\psi-\mu\,\xi\|_{\tilde{s},p,\tilde{\delta}} \leq M\text{)}\\
&\qquad = A + (\epsilon\alpha A)M + A\tilde{C}(\epsilon)\\
&\qquad = A + \frac{1}{2}M + \frac{\hat{M}-2A}{2}\\
&\qquad = \frac{1}{2}M + \frac{1}{2}\hat{M} \leq M.
\end{aligned}$$

Therefore $T(\psi,W) \in \bar{B}_M(\mu\,\xi)$. This completes the proof of the auxiliary claim. Clearly the claim of the theorem now follows from the coupled Schauder theorem.

Case 2: $s > 2$

We say the 10-tuple $A = (s,p,e,q,\delta,\beta,\tau,\sigma,\rho,J)$ is **beautiful** if it satisfies the hypotheses of the theorem, that is, if

$$\begin{aligned}
&p \in (1,\infty), \quad s \in (1+\frac{3}{p},\infty), \quad -1 < \beta \leq \delta < 0,\\
&\frac{1}{q} \in (0,1), \quad e \in (1+\frac{3}{q},\infty) \cap [s-1,s] \cap [\frac{3}{q}+s-\frac{3}{p}-1, \frac{3}{q}+s-\frac{3}{p}]\\
&p = q \text{ if } e = s \notin \mathbb{N}_0, \quad e < s \text{ if } s > 2 \text{ and } s \notin \mathbb{N}_0
\end{aligned}$$

and

- $\tau \in W^{e-1,q}_{\beta-1}$ if $e \geq 2$ and $\tau \in W^{1,z}_{\beta-1}$ otherwise, where $z = \dfrac{3q}{3+(2-e)q}$,
- $\sigma \in W^{e-1,q}_{\beta-1}$,
- $\rho \in W^{s-2,p}_{\beta-2} \cap L^{\infty}_{2\beta-2}$, $\rho \geq 0$,
- $J \in \mathbf{W}^{e-2,q}_{\beta-2}$.

Note that the condition $\frac{1}{q} \in \cap(0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$ in the statement of the theorem was to ensure that the intersection for the admissible intervals for e is nonempty. Here since we start by the assumption that e exists, we do not need to explicitly state that condition.

We say that a 10-tuple $\tilde{A} = (\tilde{s}, \tilde{p}, \tilde{e}, \tilde{q}, \tilde{\delta}, \tilde{\beta}, \tilde{\tau}, \tilde{\sigma}, \tilde{\rho}, \tilde{J})$ is **faithful** to the 10-tuple $A = (s, p, e, q, \delta, \beta, \tau, \sigma, \rho, J)$ if

$$\begin{aligned}
&\tilde{\delta} = \delta + \frac{|\delta|}{2}, \quad \tilde{\beta} = \beta + \frac{|\delta|}{2}, \quad \tilde{\tau} = \tau, \quad \tilde{\sigma} = \sigma, \quad \tilde{\rho} = \rho, \quad \tilde{J} = J, \\
&\tilde{e} = \max\{2, e-2\}, \quad \tilde{s} = \max\{2, s-2\}, \\
&\frac{1}{\tilde{p}} \leq \frac{1}{p}, \quad 1 < \tilde{e} - \frac{3}{\tilde{q}} \leq e - \frac{3}{q}, \quad 1 < \tilde{s} - \frac{3}{\tilde{p}} \leq s - \frac{3}{p}.
\end{aligned}$$

We say that $\tilde{A}$ is **extremely faithful** to A if $\tilde{A}$ is both **beautiful** and **faithful** to A.

Remark 7.6. *Note that if $\tilde{A}$ is **faithful** to A, then*

$$L^{\infty}_{\beta-1} \hookrightarrow L^{\infty}_{\tilde{\beta}-1}, \quad L^{\infty}_{2\beta-2} \hookrightarrow L^{\infty}_{2\tilde{\beta}-2}, \quad \mathbf{W}^{e-2,q}_{\beta-2} \hookrightarrow \mathbf{W}^{\tilde{e}-2,\tilde{q}}_{\tilde{\beta}-2}.$$

So, in particular, $\|\cdot\|_{L^{\infty}_{\tilde{\beta}-1}}$, $\|\cdot\|_{L^{\infty}_{2\tilde{\beta}-2}}$, $\|\cdot\|_{\mathbf{W}^{\tilde{e}-2,\tilde{q}}_{\tilde{\beta}-2}}$ can be controlled by $\|\cdot\|_{L^{\infty}_{\beta-1}}$, $\|\cdot\|_{L^{\infty}_{2\beta-2}}$, $\|\cdot\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$, respectively.

We now complete the proof of the theorem for $s > 2$, under the condition that the following two claims hold. We will then proceed to prove both claims.

Claim 1: *Suppose the 10-tuple $\tilde{A}$ is **faithful** to the **beautiful** 10-tuple A. If $(\psi, W) \in W^{\tilde{s},p}_{\tilde{\delta}} \times \mathbf{W}^{\tilde{e},\tilde{q}}_{\tilde{\beta}}$ is a solution of the constraint equations with data (τ, σ, ρ, J) (which is the same as $(\tilde{\tau}, \tilde{\sigma}, \tilde{\rho}, \tilde{J})$), then $(\psi, W) \in W^{s,p}_{\delta} \times \mathbf{W}^{e,q}_{\beta}$.*

Claim 2: *If A is a **beautiful** 10-tuple with $s > 2$, then there exists a 10-tuple $\tilde{A}$ that is **extremely faithful** to A.*

Proof of the Theorem under Claims 1 and 2. The argument to complete the proof in the case $s > 2$ based on these two claims holding is as follows. Let A denote the 10-tuple associated to the given data in the statement of the theorem. By **Claim 2**, there exists a finite chain

$$A = A_0 \to A_1 = (s_1, p_1, \ldots) \to A_2 = (s_2, p_2, \ldots) \to \ldots \to A_m = (s_m.p_m, \ldots)$$

of 10-tuples such that $s_m = 2$ and each A_i is **extremely faithful** to A_{i-1}. Now since A_m is **beautiful** and $s_m = 2$, by what was proved in the previous case we can choose μ small enough so that (3.3) and (3.4) have a solution $(\psi, W) \in W^{s_m=2,p_m}_{\delta_m} \times \mathbf{W}^{e_m,q_m}_{\beta_m}$ (note that according to Remark 7.6 by assuming $\|\sigma\|_{L^{\infty}_{\beta-1}}$, $\|\rho\|_{L^{\infty}_{2\beta-2}}$, $\|J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$ are sufficiently small, we can ensure that $\|\sigma\|_{L^{\infty}_{\beta_m-1}}$, $\|\rho\|_{L^{\infty}_{2\beta_m-2}}$, $\|J\|_{\mathbf{W}^{e_m-2,q_m}_{\beta_m-2}}$ are as small as needed). By **Claim 1**, since each A_i is **faithful** to A_{i-1}, we can conclude that $(\psi, W) \in W^{s,p}_{\delta} \times \mathbf{W}^{e,q}_{\beta}$. The main claim of the theorem in the case of $s > 2$ now follows.

Therefore, in the case $s > 2$ it is enough to prove **Claim 1** and **Claim 2**, which we now proceed to do. Before we begin, note that since in both claims A is assumed to be

beautiful, we have $-1<\beta\leq\delta<0$ and so clearly $-1<\tilde{\beta}\leq\tilde{\delta}<0$; moreover $\beta<\tilde{\beta}$ and $\delta<\tilde{\delta}$.

Proof of Claim 1.

Step 1: $b_\tau(\psi+\mu)^6+b_J\in\mathbf{W}^{e-2,q}_{\beta-2}$.

Note that $b_\tau, b_J\in\mathbf{W}^{e-2,q}_{\beta-2}$ and $\psi\in W^{\tilde{s},\tilde{p}}_{\tilde{\delta}}$. By Lemma A.28 in order to show that $b_\tau(\psi+\mu)^6\in\mathbf{W}^{e-2,q}_{\beta-2}$ it is enough to prove the following:

(i) $e-2\in[-\tilde{s},\tilde{s}]$ ($e-2\in(-\tilde{s},\tilde{s})$ if $\tilde{s}\notin\mathbb{N}_0$), (ii) $e-2-\frac{3}{q}\in[-3-\tilde{s}+\frac{3}{\tilde{p}},\tilde{s}-\frac{3}{\tilde{p}}]$.

For (ii) we have

$$e-\frac{3}{q}>1\Rightarrow e-\frac{3}{q}-2>-1>-3-(\tilde{s}-\frac{3}{\tilde{p}})\quad(\text{note that } \tilde{s}>\frac{3}{\tilde{p}}),$$

$$e\leq s+\frac{3}{q}-\frac{3}{p}\leq\tilde{s}+2+\frac{3}{q}-\frac{3}{\tilde{p}}\Rightarrow e-2-\frac{3}{q}\leq\tilde{s}-\frac{3}{\tilde{p}}\quad(\text{note } s\leq\tilde{s}+2 \text{ and } \frac{1}{\tilde{p}}\leq\frac{1}{p}).$$

In order to prove (i) we consider two cases:

Case 1: $0<s-2\leq 2$. In this case $\tilde{s}=2$ and therefore

$$e-2\in[-\tilde{s},\tilde{s}]\Leftrightarrow e-2\in[-2,2]\Leftrightarrow e\in[0,4] \text{ (clearly true since } e\in[s-1,s]).$$

Case 2: $s-2>2$. In this case $\tilde{s}=s-2$. Therefore

$$e-2\in[-\tilde{s},\tilde{s}]\Leftrightarrow e-2\in[-s+2,s-2]\Leftrightarrow e\in[4-s,s].$$

$e\leq s$ is true by assumption. Also by assumption $e\geq s-1$ and since $s>4$ we have $s-1>4-s$. It follows that $e>4-s$. Note that if $\tilde{s}=s-2>0$ is not in $\mathbb{N}_0$, then $s\notin\mathbb{N}_0$ and so since A is **beautiful** and $s>2$ we can conclude that $e<s$. That is, in this case we have $e-2\in(-\tilde{s},\tilde{s})$ exactly as desired.

Step 2: $W\in\mathbf{W}^{e,q}_{\beta}$.

By what was shown in the previous step we know that $\mathcal{A}_L W=b_\tau(\psi+\mu)^6+b_J\in\mathbf{W}^{e-2,q}_{\beta-2}$. It follows from Remark B.8 that $W\in\mathbf{W}^{e,q}_{\beta}$.

Step 3: $\psi\in W^{s,p}_{\delta}$.

Since $W\in\mathbf{W}^{e,q}_{\beta}$ according to the argument that we had in deriving **Weak Formulation 1** we have $a_W\in W^{s-2,p}_{\delta-2}$. It follows that $A_L\psi\in W^{s-2,p}_{\delta-2}$. So again by Remark B.8, we can conclude that $\psi\in W^{s,p}_{\delta}$.

Therefore, we have shown that **Claim 1** holds. We now proceed to **Claim 2**.

Proof of Claim 2. We want to find a 10-tuple $\tilde{A}$ that is **extremely faithful** to A. Note that all the components of $\tilde{A}$, except $\tilde{p}$ and $\tilde{q}$, are automatically determined by A. So we need to find $\tilde{p}$ and $\tilde{q}$ so that $\tilde{A}$ becomes **extremely faithful** to A. We must consider three cases:

Case 1: $0<s-2\leq 2$, $e-2\leq 2$ (so $\tilde{s}=\tilde{e}=2$)

Select $\tilde{p}$ and $\tilde{q}$ to satisfy

$$\frac{1}{\tilde{p}}\in[\frac{1}{p}-\frac{s-2}{3},\frac{1}{3})\cap(0,\frac{1}{p}),\qquad\qquad \frac{1}{\tilde{q}}\in[\frac{1}{q}-\frac{e-2}{3},\frac{1}{3})\cap(\frac{1}{\tilde{p}},\infty).$$

Our claim is that the 10-tuple $\tilde{A}=(\tilde{s}=2,\tilde{p},\tilde{e}=2,\tilde{q},\tilde{\delta}=\delta+\frac{|\delta|}{2},\tilde{\beta}=\beta+\frac{|\delta|}{2},\tilde{\tau}=\tau,\tilde{\sigma}=\sigma,\tilde{\rho}=\rho,\tilde{J}=J)$ is **extremely faithful** to A.

First note that it is possible to pick such $\tilde{p}$ and $\tilde{q}$. Indeed,

$$[\frac{1}{p}-\frac{s-2}{3},\frac{1}{3})\neq\varnothing,\qquad \text{since } s>1+\frac{3}{p},$$
$$[\frac{1}{p}-\frac{s-2}{3},\frac{1}{3})\cap(0,\frac{1}{p})\neq\varnothing,\qquad \text{since for } s>2, \text{ we have } \frac{1}{p}-\frac{s-2}{3}<\frac{1}{p},$$
$$[\frac{1}{q}-\frac{e-2}{3},\frac{1}{3})\neq\varnothing,\qquad \text{since } e>1+\frac{3}{q},$$
$$[\frac{1}{q}-\frac{e-2}{3},\frac{1}{3})\cap(\frac{1}{\tilde{p}},\infty)\neq\varnothing,\qquad \text{since } \frac{1}{\tilde{p}}<\frac{1}{3}.$$

In order to show that $\tilde{A}$ is **extremely faithful** to A we need to show that 1) $\tilde{A}$ is **faithful** to A and 2) $\tilde{A}$ is **beautiful**.

1) $\tilde{A}$ is **faithful** to A:

(i) By definition of $\tilde{p}$ we have $\frac{1}{\tilde{p}}\leq\frac{1}{p}$.

(ii) Clearly $\tilde{e}=2=\max\{2,e-2\},\quad \tilde{s}=2=\max\{2,s-2\}$.

(iii) $\frac{1}{p}-\frac{s-2}{3}\leq\frac{1}{\tilde{p}}<\frac{1}{3}\Rightarrow -1<\frac{-3}{\tilde{p}}\leq s-\frac{3}{p}-2$

$$\Rightarrow 1<2-\frac{3}{\tilde{p}}\leq s-\frac{3}{p}\Rightarrow 1<\tilde{s}-\frac{3}{\tilde{p}}\leq s-\frac{3}{p}\quad(\tilde{s}=2).$$

(iv) Similarly $\frac{1}{q}-\frac{e-2}{3}\leq\frac{1}{\tilde{q}}<\frac{1}{3}\Rightarrow 1<\tilde{e}-\frac{3}{\tilde{q}}\leq e-\frac{3}{q}$.

2) $\tilde{A}$ is **beautiful**:

(i) Clearly $\tilde{p},\tilde{q}\in(1,\infty)$.

In addition, by what was proved above, $\tilde{s}>1+\frac{3}{\tilde{p}}$ and $\tilde{e}>1+\frac{3}{\tilde{q}}$.

(ii) $\tilde{e}\in[\tilde{s}-1,\tilde{s}]\Leftrightarrow 2\in[2-1,2]$ (which is clearly true).

(iii) $\tilde{e}\in[\frac{3}{\tilde{q}}+\tilde{s}-\frac{3}{\tilde{p}}-1,\frac{3}{\tilde{q}}+\tilde{s}-\frac{3}{\tilde{p}}]\Leftrightarrow 2\in[\frac{3}{\tilde{q}}-\frac{3}{\tilde{p}}+1,\frac{3}{\tilde{q}}-\frac{3}{\tilde{p}}+2]$

$$\Leftrightarrow 0\leq\frac{3}{\tilde{q}}-\frac{3}{\tilde{p}}\leq 1\quad(\tilde{s}=\tilde{e}=2)$$
$$\Leftrightarrow \frac{1}{\tilde{p}}\leq\frac{1}{\tilde{q}} \text{ and } \frac{1}{\tilde{q}}\leq\frac{1}{3}+\frac{1}{\tilde{p}}\quad(\text{since we know } \frac{1}{3}>\frac{1}{\tilde{q}}>\frac{1}{\tilde{p}}).$$

Also since $\tilde{s}-\frac{3}{\tilde{p}}\leq s-\frac{3}{p}$, $\tilde{e}-\frac{3}{\tilde{q}}\leq e-\frac{3}{q}$, $\beta<\tilde{\beta}$ and $\delta<\tilde{\delta}$, it follows from the embedding theorem that

$$W^{s,p}_{\delta}\hookrightarrow W^{\tilde{s},\tilde{p}}_{\tilde{\delta}},\quad W^{s-2,p}_{\beta-2}\hookrightarrow W^{\tilde{s},\tilde{p}}_{\tilde{\beta}-2},$$
$$W^{e,q}_{\beta}\hookrightarrow W^{\tilde{e},q}_{\tilde{\beta}},\quad W^{e-1,q}_{\beta-1}\hookrightarrow W^{\tilde{e}-1,q}_{\tilde{\beta}-1},\quad W^{e-2,q}_{\beta-2}\hookrightarrow W^{\tilde{e}-2,q}_{\tilde{\beta}-2}.$$

Therefore τ, σ, ρ and J are in the correct spaces.

Case 2: $s-2>2$, $e-2\leq 2$ (so $\tilde{s}=s-2$, $\tilde{e}=2$)

Select $\tilde{q}$ such that $\frac{1}{\tilde{q}}\in[\frac{1}{q}-\frac{e-2}{3},\frac{1}{3})\cap[\frac{1}{p}-\frac{2}{3},\frac{1}{p})$. Let $\tilde{p}:=\tilde{q}$. Our claim is that the 10-tuple $\tilde{A}=(\tilde{s}=s-2,\tilde{p},\tilde{e}=2,\tilde{q},\tilde{\delta}=\delta+\frac{|\delta|}{2},\tilde{\beta}=\beta+\frac{|\delta|}{2},\tilde{\tau}=\tau,\tilde{\sigma}=\sigma,\tilde{\rho}=\rho,\tilde{J}=J)$ is **extremely faithful** to A.

First note that it is possible to pick such $\tilde{q}$. Indeed, $[\frac{1}{q}-\frac{e-2}{3},\frac{1}{3})\neq\varnothing$ because $e>1+\frac{3}{q}$. For the intersection to be nonempty we need to check $\frac{1}{p}-\frac{2}{3}<\frac{1}{3}$ and $\frac{1}{q}-\frac{e-2}{3}<\frac{1}{p}$. The first inequality is clearly true. The second inequality is also true because

$$e>\frac{3}{q}-\frac{3}{p}+s-1>\frac{3}{q}-\frac{3}{p}+2 \quad \text{(note that } s>4)$$
$$\Rightarrow \frac{3}{q}-(e-2)<\frac{3}{p}\Rightarrow \frac{1}{q}-\frac{e-2}{3}<\frac{1}{p}.$$

1) $\tilde{A}$ is **faithful** to A:

(i) $\tilde{p}=\tilde{q}$, and $\frac{1}{\tilde{q}}<\frac{1}{p}\Rightarrow\frac{1}{\tilde{p}}\leq\frac{1}{p}$.

(ii) $\frac{1}{q}-\frac{e-2}{3}\leq\frac{1}{\tilde{q}}<\frac{1}{3}\Rightarrow 1<\tilde{e}-\frac{3}{\tilde{q}}\leq e-\frac{3}{q}.\quad(\tilde{e}=2)$

(iii) $\tilde{s}=s-2>2\Rightarrow\tilde{s}-\frac{3}{\tilde{q}}>2-\frac{3}{\tilde{q}}>1\Rightarrow\tilde{s}-\frac{3}{\tilde{p}}>1.\quad$ (note $\frac{1}{\tilde{q}}<\frac{1}{3}$ and $\tilde{q}=\tilde{p}$)

(iv) $\frac{1}{\tilde{q}}\geq\frac{1}{p}-\frac{2}{3}\Rightarrow\frac{3}{p}\leq 2+\frac{3}{\tilde{q}}$

$$\Rightarrow s-2-\frac{3}{\tilde{q}}\leq s-\frac{3}{p}\Rightarrow\tilde{s}-\frac{3}{\tilde{p}}\leq s-\frac{3}{p}.\quad \text{(note } \tilde{s}=s-2 \text{ and } \tilde{q}=\tilde{p})$$

2) $\tilde{A}$ is **beautiful**:

(i) Clearly $\tilde{p},\tilde{q}\in(1,\infty)$. By what was proved above $\tilde{s}>1+\frac{3}{\tilde{p}}$ and $\tilde{e}>1+\frac{3}{\tilde{q}}$.

(ii) $\tilde{e}\in[\tilde{s}-1,\tilde{s}]\Leftrightarrow 2\in[s-3,s-2]\Leftrightarrow 4\leq s\leq 5$.

(by assumption $s>4$; also $s-1\leq e\leq 4$ and so $s\leq 5$).

(iii) $\tilde{e}\in[\frac{3}{\tilde{q}}+\tilde{s}-\frac{3}{\tilde{p}}-1,\frac{3}{\tilde{q}}+\tilde{s}-\frac{3}{\tilde{p}}]\Leftrightarrow 2\in[\frac{3}{\tilde{q}}+s-3-\frac{3}{\tilde{p}},\frac{3}{\tilde{q}}+s-2-\frac{3}{\tilde{p}}]$

$$\Leftrightarrow 4\leq s+\frac{3}{\tilde{q}}-\frac{3}{\tilde{p}}\leq 5\Leftrightarrow 4\leq s\leq 5.$$

(which is true; note that $\tilde{s}=s-2$, $\tilde{e}=2$, $\tilde{q}=\tilde{p}$)

The proof that τ,σ,ρ and J belong to the correct spaces is exactly the same as Case 1.

Case 3: $s-2>2$, $e-2>2$ (so $\tilde{s}=s-2$, $\tilde{e}=e-2$).

Select $\tilde{q}$ to satisfy

$$\frac{1}{\tilde{q}}\in[\frac{1}{q}-\frac{2}{3},\frac{e}{3}-1)\cap(0,\frac{1}{q})\cap(\frac{1}{q}-\frac{1}{p},\infty).$$

Define $\tilde{p}$ by $\frac{1}{\tilde{p}}:=\frac{1}{\tilde{q}}-\frac{1}{q}+\frac{1}{p}$. Our claim is that the 10-tuple $\tilde{A}=(\tilde{s}=s-2,\tilde{p},\tilde{e}=e-2,\tilde{q},\tilde{\delta}=\delta+\frac{|\delta|}{2},\tilde{\beta}=\beta+\frac{|\delta|}{2},\tilde{\tau}=\tau,\tilde{\sigma}=\sigma,\tilde{\rho}=\rho,\tilde{J}=J)$ is **extremely faithful** to A.

First note that it is possible to pick such $\tilde{q}$. Indeed, $[\frac{1}{q}-\frac{2}{3},\frac{e}{3}-1)\neq\varnothing$ because $e>1+\frac{3}{q}$. In order to show that the intersection of the three intervals is nonempty we consider two possibilities:

- Possibility 1: $\frac{1}{q}-\frac{1}{p}>0$. In this case

$$(0,\frac{1}{q})\cap(\frac{1}{q}-\frac{1}{p},\infty)=(\frac{1}{q}-\frac{1}{p},\frac{1}{q}),$$

and so it is enough to show that

$$[\frac{1}{q}-\frac{2}{3},\frac{e}{3}-1)\cap(\frac{1}{q}-\frac{1}{p},\frac{1}{q})\neq\varnothing.$$

This is true because

(i) Clearly $\frac{1}{q}-\frac{2}{3}<\frac{1}{q}$,

(ii) $e\geq\frac{3}{q}-\frac{3}{p}+s-1>\frac{3}{q}-\frac{3}{p}+3\Rightarrow\frac{e}{3}-1>\frac{1}{q}-\frac{1}{p}$. (note that $s>4$)

- Possibility 2: $\frac{1}{q}-\frac{1}{p}\leq 0$. In this case

$$(0,\frac{1}{q})\cap(\frac{1}{q}-\frac{1}{p},\infty)=(0,\frac{1}{q}),$$

and so it is enough to show that

$$[\frac{1}{q}-\frac{2}{3},\frac{e}{3}-1)\cap(0,\frac{1}{q})\neq\varnothing.$$

This is true because

(i) Clearly $\frac{1}{q}-\frac{2}{3}<\frac{1}{q}$, and (ii) $e>3\Rightarrow\frac{e}{3}-1>0$.

1) $\tilde{A}$ is **faithful** to A:

(i) $\frac{1}{\tilde{q}}<\frac{1}{q}\Rightarrow\frac{1}{\tilde{q}}-\frac{1}{q}+\frac{1}{p}<\frac{1}{p}\Rightarrow\frac{1}{\tilde{p}}<\frac{1}{p}$.

(ii) $\frac{1}{\tilde{q}}<\frac{e}{3}-1\Rightarrow e-2-\frac{3}{\tilde{q}}>1\Rightarrow\tilde{e}-\frac{3}{\tilde{q}}>1$. $(\tilde{e}=e-2)$

(iii) $\frac{1}{q}-\frac{2}{3}\leq\frac{1}{\tilde{q}}\Rightarrow\frac{3}{q}-2\leq\frac{3}{\tilde{q}}\Rightarrow e-2-\frac{3}{\tilde{q}}\leq e-\frac{3}{q}\Rightarrow\tilde{e}-\frac{3}{\tilde{q}}\leq e-\frac{3}{q}$. $(\tilde{e}=e-2)$

(iv) $3+\frac{3}{\tilde{q}}<e<s+\frac{3}{q}-\frac{3}{p}\Rightarrow 3+\frac{3}{\tilde{q}}<s+\frac{3}{q}-\frac{3}{p}\Rightarrow 1<s-2-\frac{3}{\tilde{q}}+\frac{3}{q}-\frac{3}{p}$

$\Rightarrow 1<s-2-\frac{3}{\tilde{p}}\Rightarrow 1<\tilde{s}-\frac{3}{\tilde{p}}$. (note that $\frac{1}{\tilde{p}}:=\frac{1}{\tilde{q}}-\frac{1}{q}+\frac{1}{p}$ and $\tilde{s}=s-2$)

(v) $\frac{1}{q}-\frac{2}{3}\leq\frac{1}{\tilde{q}}\Rightarrow 0\leq\frac{3}{\tilde{q}}-\frac{3}{q}+2\Rightarrow\frac{3}{p}\leq\frac{3}{\tilde{q}}-\frac{3}{q}+\frac{3}{p}+2\Rightarrow\frac{3}{p}\leq\frac{3}{\tilde{p}}+2$

$\Rightarrow s-2-\frac{3}{\tilde{p}}\leq s-\frac{3}{p}\Rightarrow\tilde{s}-\frac{3}{\tilde{p}}\leq s-\frac{3}{p}$.

2) $\tilde{A}$ is **beautiful**:

(i) Clearly $\tilde{p},\tilde{q}\in(1,\infty)$. By what was proved above, $\tilde{s}>1+\frac{3}{\tilde{p}}$ and $\tilde{e}>1+\frac{3}{\tilde{q}}$.

(ii) $\tilde{e}\in[\tilde{s}-1,\tilde{s}]\Leftrightarrow e-2\in[s-3,s-2]\Leftrightarrow e\in[s-1,s]$. (which is clearly true)

(iii) $\tilde{e}\in[\frac{3}{\tilde{q}}+\tilde{s}-\frac{3}{\tilde{p}}-1,\frac{3}{\tilde{q}}+\tilde{s}-\frac{3}{\tilde{p}}]\Leftrightarrow e-2\in[\frac{3}{\tilde{q}}+s-3-\frac{3}{\tilde{p}},\frac{3}{\tilde{q}}+s-2-\frac{3}{\tilde{p}}]$

$\Leftrightarrow e\in[\frac{3}{\tilde{q}}-\frac{3}{\tilde{p}}+s-1,s+\frac{3}{\tilde{q}}-\frac{3}{\tilde{p}}]\Leftrightarrow e\in[\frac{3}{q}-\frac{3}{p}+s-1,s+\frac{3}{q}-\frac{3}{p}]$.

(note that $\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}=\frac{1}{q}-\frac{1}{p}$).

The last inclusion is true because A is **beautiful**. The proof of the fact that τ, σ, ρ and J belong to the correct spaces is exactly the same as Case 1.

Note that $e \leq s$, so if $s-2 \leq 2$ then $e-2 \leq 2$ and therefore the case where $s-2 \leq 2$, $e-2 > 2$ does not happen.

This establishes **Claim 2**, and by earlier arguments the main claim of the Theorem now follows. □

8. Two Auxiliary Results

We now state and prove two auxiliary lemmas that were used in the proof of Theorem 7.3.

Lemma 8.1. *Let $\chi \in W^{s,p}_{\delta}$, $\chi > -1$ and let $f \in W^{s-2,p}_{\delta-2}$. Then $\frac{1}{1+\chi} f \in W^{s-2,p}_{\delta-2}$ and*

$$\|\frac{1}{1+\chi} f\|_{s-2,p,\delta-2} \preceq (1 + \|\frac{\chi}{\chi+1}\|_{s,p,\delta})\|f\|_{s-2,p,\delta-2}.$$

In particular, for a fixed χ, the mapping $f \mapsto \frac{1}{1+\chi} f$ (from $W^{s-2,p}_{\delta-2} to W^{s-2,p}_{\delta-2}$) sends bounded sets to bounded sets.

Proof. **(Lemma 8.1)** By Lemma A.28 $\frac{1}{1+\chi} f \in W^{s-2,p}_{\delta-2}$. Moreover

$$\|\frac{1}{1+\chi} f\|_{s-2,p,\delta-2} = \|(\frac{1}{1+\chi} - 1 + 1) f\|_{s-2,p,\delta-2} = \|\frac{-\chi}{\chi+1} f + f\|_{s-2,p,\delta-2}.$$

It follows from Lemma A.28 that $\frac{-\chi}{\chi+1} \in W^{s,p}_{\delta}$. Also by Lemma A.24 $W^{s,p}_{\delta} \times W^{s-2,p}_{\delta-2} \to W^{s-2,p}_{\delta-2}$. Thus

$$\begin{aligned}
\|\frac{-\chi}{\chi+1} f + f\|_{s-2,p,\delta-2} &\leq \|\frac{-\chi}{\chi+1} f\|_{s-2,p,\delta-2} + \|f\|_{s-2,p,\delta-2} \\
&\preceq \|\frac{-\chi}{\chi+1}\|_{s,p,\delta}\|f\|_{s-2,p,\delta-2} + \|f\|_{s-2,p,\delta-2} \\
&= (1 + \|\frac{\chi}{\chi+1}\|_{s,p,\delta})\|f\|_{s-2,p,\delta-2}.
\end{aligned}$$

□

Lemma 8.2. *There exists a constant C independent of W such that*

$$\forall W \in S([\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}), \quad \|\tilde{a}_W\|_{s-2,p,\delta-2} \leq C.$$

Proof. **(Lemma 8.2)** By Corollary 4.4 if $W \in S([\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}})$, that is, if W is the solution to the momentum constraint with some source $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$, then

$$\begin{aligned}
\|W\|_{e,q,\beta} &\leq C_1\big[(\mu + \|\psi\|_{L^\infty_\delta})^6 \|b_\tau\|_{L^z_{\beta-2}} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}\big] \\
&\leq C_1\big[(\mu + \max\{\|\psi_-\|_{L^\infty_\delta}, \|\psi_+\|_{L^\infty_\delta}\})^6 \|b_\tau\|_{L^z_{\beta-2}} + \|b_J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}\big].
\end{aligned}$$

Here we used the fact that $|\psi| \leq \max\{|\psi_+|, |\psi_-|\}$ and so $\|\psi\|_{L^\infty_\delta} \leq \max\{\|\psi_-\|_{L^\infty_\delta}, \|\psi_+\|_{L^\infty_\delta}\}$. Consequently there is a constant C_2 such that for all $W \in S([\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}})$ we have $\|W\|_{e,q,\beta} \leq C_2$.

Considering the restrictions on the exponents $s, p, \delta, e, q, \beta$ and using our embedding theorem and multiplication lemma, it is easy to check $W^{s-2,p}_{\beta-2} \hookrightarrow W^{s-2,p}_{\delta-2}$, $W^{e-1,q}_{2\beta-2} \hookrightarrow$

$W^{s-2,p}_{\beta-2}$, and $W^{e-1,q}_{\beta-1} \times W^{e-1,q}_{\beta-1} \hookrightarrow W^{e-1,q}_{2\beta-2}$. Therefore we can write

$$\begin{aligned}\|a_W\|_{s-2,p,\delta-2} &\preceq \|a_W\|_{s-2,p,\beta-2} \preceq \|a_W\|_{e-1,q,2\beta-2}\\ &\preceq \|\sigma+\mathcal{L}W\|^2_{e-1,q,\beta-1} \preceq (\|\sigma\|_{e-1,q,\beta-1}+\|\mathcal{L}W\|_{e-1,q,\beta-1})^2\\ &\preceq \|\sigma\|^2_{e-1,q,\beta-1}+\|\mathcal{L}W\|^2_{e-1,q,\beta-1} \preceq \|\sigma\|^2_{e-1,q,\beta-1}+\|W\|^2_{e,q,\beta}\\ &\leq \|\sigma\|^2_{e-1,q,\beta-1}+C_2.\end{aligned}$$

Hence there is a constant C_3 such that for all $W \in S([\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}})$ we have $\|a_W\|_{s-2,p,\delta-2} \leq C_3$. Now notice that $\tilde{a}_W = (1+\xi)^{-12}a_W$, that is, $\tilde{a}_W$ is obtained from a_W by applying the mapping $f \mapsto \frac{1}{1+\xi}f$ twelve times. But by Lemma 8.1 the mapping $f \mapsto \frac{1}{1+\xi}f$ sends bounded sets in $W^{s-2,p}_{\delta-2}$ to bounded sets in $W^{s-2,p}_{\delta-2}$. Consequently there exists a constant C such that

$$\forall\, W \in S([\psi_-,\psi_+]_{\tilde{s},p,\tilde{\delta}}), \quad \|\tilde{a}_W\|_{s-2,p,\delta-2} \leq C.$$

□

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APPENDIX A. WEIGHTED SOBOLEV SPACES

We first assemble some basic results we need for weighted Sobolev spaces. We limit our selves to simply stating the results we need, unless the proof of the result is either unavailable or difficult to find in the form we need, in which case we include a concise proof.

Consider an open cover of $\mathbb{R}^n$ that consists of the following sets:

$$B_2, \quad B_4 \setminus \bar{B}_1, \quad B_8 \setminus \bar{B}_2,\ ...,\ B_{2^{j+1}} \setminus \bar{B}_{2^{j-1}},$$

where B_r is the open ball of radius r centered at the origin. For all r let $S_r f(x) := f(rx)$. Consider the following partition of unity subordinate to the above cover of $\mathbb{R}^n$[38]:

$$\begin{aligned}&\varphi_0 = 1 \quad \text{on} \quad B_1, \quad \operatorname{supp}\varphi_0 \subseteq B_2,\\ &\varphi(x) = \varphi_0(x) - \varphi_0(2x) \quad (\operatorname{supp}\varphi \subseteq B_2, \quad \varphi = 0 \text{ on } B_{\frac{1}{2}}),\\ &\forall j \geq 1 \quad \varphi_j = S_{2^{-j}}\varphi.\end{aligned}$$

One can easily check that $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.

For $s \in \mathbb{R}$, $p \in (1,\infty)$, the weighted Sobolev space $W^{s,p}_\delta(\mathbb{R}^n)$ is defined as follows:

$$W^{s,p}_\delta(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \|u\|^p_{W^{s,p}_\delta(\mathbb{R}^n)} = \sum_{j=0}^{\infty} 2^{-p\delta j}\|S_{2^j}(\varphi_j u)\|^p_{W^{s,p}(\mathbb{R}^n)} < \infty\}.$$

Here $W^{s,p}(\mathbb{R}^n)$ is the Sobolev-Slobodeckij space which is defined as follows:

- If $s = k \in \mathbb{N}_0$, $p \in [1, \infty]$,

$$W^{k,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \|u\|_{W^{k,p}(\mathbb{R}^n)} := \sum_{|\nu| \leq k} \|\partial^\nu u\|_p < \infty\}$$

- If $s = \theta \in (0, 1)$, $p \in [1, \infty)$,

$$W^{\theta,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : |u|_{W^{\theta,p}(\mathbb{R}^n)} := \Big(\int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\theta p}} dxdy \Big)^{\frac{1}{p}} < \infty\}$$

- If $s = \theta \in (0, 1)$, $p = \infty$,

$$W^{\theta,\infty}(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n) : |u|_{W^{\theta,\infty}(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x,y \in \mathbb{R}^n, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\theta} < \infty\}$$

- If $s = k + \theta$, $k \in \mathbb{N}_0$, $\theta \in (0, 1)$, $p \in [1, \infty]$,

$$W^{s,p}(\mathbb{R}^n) = \{u \in W^{k,p}(\mathbb{R}^n) : \|u\|_{W^{s,p}(\mathbb{R}^n)} := \|u\|_{W^{k,p}(\mathbb{R}^n)} + \sum_{|\nu| = k} |\partial^\nu u|_{W^{\theta,p}(\mathbb{R}^n)} < \infty\}$$

- If $s < 0$ and $p \in (1, \infty)$,

$$W^{s,p}(\mathbb{R}^n) = (W^{-s,p'}(\mathbb{R}^n))^* \quad (\frac{1}{p} + \frac{1}{p'} = 1).$$

Alternatively, we could have defined $W^{s,p}(\mathbb{R}^n)$ as a Bessel potential space, that is,

$$W^{s,p}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \|u\|_{W^{s,p}(\mathbb{R}^n)} := \|\mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F} u)\|_{L^p} < \infty\},$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. It is a well known fact that for $k \in \mathbb{N}_0$ the above definition of $W^{k,p}(\mathbb{R}^n)$ agrees with the first definition [26]. Also for $s \in \mathbb{R}$ and $p = 2$ the two definitions agree[26]. It is customary to use $H^{s,p}$ instead of $W^{s,p}$ for unweighted Bessel potential spaces. We denote the corresponding weighted spaces by $H^{s,p}_\delta$. In this paper (except in Appendix E) we use the first definition. The norm on $W^{k,p}_\delta(\mathbb{R}^n)$ is equivalent to the following norm [38, 54]: (since the norms are equivalent we use the same notation for both norms)

$$\|u\|_{W^{k,p}_\delta(\mathbb{R}^n)} = \sum_{|\beta| \leq k} \|\langle x \rangle^{-\delta - \frac{n}{p} + |\beta|} \partial^\beta u\|_{L^p(\mathbb{R}^n)}.$$

When $s = 0$, we denote $W^{s,p}_\delta(\mathbb{R}^n)$ by $L^p_\delta(\mathbb{R}^n)$. In particular we have

$$\|u\|_{L^p_\delta(\mathbb{R}^n)} = \|\langle x \rangle^{-\delta - \frac{n}{p}} u\|_{L^p(\mathbb{R}^n)}.$$

Remark A.1. *We take a moment to make the following three observations.*

- ◦ *Considering the above formula for the norm, it is obvious that if* $\delta \leq -\frac{n}{p}$ *then* $\langle x \rangle^{-\delta - \frac{n}{p} + |\beta|} \geq 1$ *and therefore* $\|u\|_{W^{k,p}(\mathbb{R}^n)} \leq \|u\|_{W^{k,p}_\delta(\mathbb{R}^n)}$ *and* $W^{k,p}_\delta(\mathbb{R}^n) \hookrightarrow W^{k,p}(\mathbb{R}^n)$.
- ◦ *Note that if* $u \in L^p_\delta(\mathbb{R}^n)$ *and* $v \in L^\infty(\mathbb{R}^n)$*, then*

$$\begin{aligned} \|vu\|_{L^p_\delta(\mathbb{R}^n)} &= \|\langle x \rangle^{-\delta - \frac{n}{p}} vu\|_{L^p(\mathbb{R}^n)} \\ &\leq \|v\|_\infty \|\langle x \rangle^{-\delta - \frac{n}{p}} u\|_{L^p(\mathbb{R}^n)} \\ &= \|v\|_\infty \|u\|_{L^p_\delta(\mathbb{R}^n)}. \end{aligned}$$

- ◦ *It is easy to show that for* $p \in (1, \infty)$, $\langle x \rangle^{\delta'} \in L^p_\delta(\mathbb{R}^n)$ *for every* $\delta' < \delta$*, but* $\langle x \rangle^\delta \notin L^p_\delta(\mathbb{R}^n)$ [38].

Remark A.2. *Suppose $1 < p < \infty$. Note that in the case of unweighted Sobolev spaces, for $s < 0$, $W^{s,p}(\mathbb{R}^n)$ is **defined** as the dual of $W^{-s,p'}(\mathbb{R}^n)$. In fact, since $W^{s,p}(\mathbb{R}^n)$ is reflexive, we have $(W^{s,p}(\mathbb{R}^n))^* = W^{-s,p'}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. Contrary to the unweighted case, in case of weighted Sobolev spaces our definition of $W^{s,p}_\delta(\mathbb{R}^n)$ for $s < 0$ is not based on duality. Nevertheless, as it is stated in the next theorem, $(W^{s,p}_\delta(\mathbb{R}^n))^*$ can be **identified** with $W^{-s,p'}_{-n-\delta}(\mathbb{R}^n)$. This identification can be done by defining a suitable bilinear form $W^{-s,p'}_{-n-\delta}(\mathbb{R}^n) \times W^{s,p}_\delta(\mathbb{R}^n) \to \mathbb{R}$* [55].

Remark A.3. *In the literature, the **growth parameter** δ has been incorporated in the definition of weighted spaces in more than one way. Our convention for the growth parameter agrees with Bartnik's convention* [4] *and Maxwell's convention* [38, 41, 40]. *The following items describe how our definition corresponds with the other related definitions of weighted spaces in the literature:*

- *For $s \in \mathbb{Z}$ our spaces $W^{s,p}_\delta(\mathbb{R}^n)$ correspond with the spaces $h^s_{p,ps-p\delta-n}(\mathbb{R}^n)$ in* [54, 55] *and $H^{s,p}_\delta(\mathbb{R}^n)$ in* [38].
- *For $s \notin \mathbb{Z}$ our spaces $W^{s,p}_\delta(\mathbb{R}^n)$ correspond with the spaces $b^s_{p,p,ps-p\delta-n}(\mathbb{R}^n)$ in* [54, 55] *and $W^p_{s,-\delta-\frac{n}{p}}(\mathbb{R}^n)$ in* [8].
- *For $s \in \mathbb{R}$ and $p = 2$ our spaces $W^{s,p}_\delta(\mathbb{R}^n)$ correspond with the spaces $H^s_\delta(\mathbb{R}^n)$ in* [38, 41].

The space $W^{s,p}_{loc}(\mathbb{R}^n)$ is defined as the set of distributions $u \in D'(\mathbb{R}^n)$ for which $\chi u \in W^{s,p}(\mathbb{R}^n)$ for all $\chi \in C^\infty_c(\mathbb{R}^n)$. $W^{s,p}_{loc}(\mathbb{R}^n)$ is a Frechet space with the topology defined by the seminorms $p_\chi(u) = \|\chi u\|_{W^{s,p}(\mathbb{R}^n)}$ for $\chi \in C^\infty_c(\mathbb{R}^n)$ [27]. Also $C^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}_{loc}(\mathbb{R}^n)$.

Theorem A.4. [38, 41, 40, 4, 8, 54, 55] *Let $p_1, p_2, p, q \in (1,\infty)$, $\delta, \delta_1, \delta_2, \delta' \in \mathbb{R}$.*

(1) If $p \geq q$ and $\delta' < \delta$ then $L^P_{\delta'}(\mathbb{R}^n) \subseteq L^q_\delta(\mathbb{R}^n)$ is continuous.
(2) For $s \geq s'$ and $\delta \leq \delta'$ the inclusion $W^{s,p}_\delta(\mathbb{R}^n) \subseteq W^{s',p}_{\delta'}(\mathbb{R}^n)$ is continuous.
(3) For $s > s'$ and $\delta < \delta'$ the inclusion $W^{s,p}_\delta(\mathbb{R}^n) \subseteq W^{s',p}_{\delta'}(\mathbb{R}^n)$ is compact.
(4) If $0 \leq sp < n$ then $W^{s,p}_\delta(\mathbb{R}^n) \subseteq L^r_\delta(\mathbb{R}^n)$ is continuous for every r with $\frac{1}{p} - \frac{s}{n} \leq \frac{1}{r} \leq \frac{1}{p}$.
(5) If $sp = n$ then $W^{s,p}_\delta(\mathbb{R}^n) \subseteq L^r_\delta(\mathbb{R}^n)$ is continuous for every $r \geq p$.
(6) If $sp > n$ then $W^{s,p}_\delta(\mathbb{R}^n) \subseteq L^r_\delta(\mathbb{R}^n)$ is continuous for every $r \geq p$. Moreover $W^{s,p}_\delta(\mathbb{R}^n) \subseteq C^0_\delta(\mathbb{R}^n)$ is continuous where $C^0_\delta(\mathbb{R}^n)$ is the set of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ for which $\|f\|_{C^0_\delta} := \sup_{x\in\mathbb{R}^n}(\langle x\rangle^{-\delta}|f|) < \infty$.
(7) If $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} < 1$, then pointwise multiplication is a continuous bilinear map $L^{p_1}_{\delta_1}(\mathbb{R}^n) \times L^{p_2}_{\delta_2}(\mathbb{R}^n) \to L^r_{\delta_1+\delta_2}(\mathbb{R}^n)$.
(8) Pointwise multiplication is a continuous bilinear map $C^0_{\delta_1}(\mathbb{R}^n) \times L^p_{\delta_2}(\mathbb{R}^n) \to L^p_{\delta_1+\delta_2}(\mathbb{R}^n)$.
(9) For $s \in \mathbb{R}$ (and $p \in (1,\infty)$), $W^{s,p}_\delta(\mathbb{R}^n)$is a reflexive space and $(W^{s,p}_\delta(\mathbb{R}^n))^* = W^{-s,p'}_{-n-\delta}(\mathbb{R}^n)$.
*(10) **Real Interpolation**: Suppose $\theta \in (0,1)$. If*

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \delta = (1-\theta)\delta_0 + \theta\delta_1$$

then $W^{s,p}_\delta(\mathbb{R}^n) = (W^{s_0,p_0}_{\delta_0}(\mathbb{R}^n), W^{s_1,p_1}_{\delta_1}(\mathbb{R}^n))_{\theta,p}$ unless $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$ and $s \in \mathbb{Z}$. In the case where s_0 and s_1 are not both positive and exactly one of s_0 and s_1 is an integer, we additionally assume that $p_0 = p_1 = p$.

(11) ***Complex Interpolation***: *Suppose* $\theta \in (0,1)$. *If*

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \delta = (1-\theta)\delta_0 + \theta\delta_1$$

then $W^{s,p}_{\delta}(\mathbb{R}^n) = [W^{s_0,p_0}_{\delta_0}(\mathbb{R}^n), W^{s_1,p_1}_{\delta_1}(\mathbb{R}^n)]_\theta$ *provided* $s_0, s_1, s \in \mathbb{Z}$ *or* $s_0, s_1, s \notin \mathbb{Z}$.

Note: The above interpolation facts do not say anything about the case where s_0 *or* $s_1 \in \mathbb{R} \setminus \mathbb{Z}$ ***and*** $s \in \mathbb{Z}$.

(12) $C^\infty_c(\mathbb{R}^n)$ *is dense in* $W^{s,p}_{\delta}(\mathbb{R}^n)$ *for all* $s \in \mathbb{R}$.

Remark A.5. *We define* $L^\infty_\delta(\mathbb{R}^n)$ *as follows:* $f \in L^\infty_\delta(\mathbb{R}^n) \Leftrightarrow \langle x\rangle^{-\delta} f \in L^\infty(\mathbb{R}^n)$. *We equip this space with the norm* $\|f\|_{L^\infty_\delta(\mathbb{R}^n)} := \|\langle x\rangle^{-\delta} f\|_{L^\infty(\mathbb{R}^n)}$. *More generally, for all* $k \in \mathbb{N}_0$

$$W^{k,\infty}_\delta(\mathbb{R}^n) := \{u \in L^\infty_\delta(\mathbb{R}^n) : \partial^\alpha u \in L^\infty_{\delta-|\alpha|}(\mathbb{R}^n) \quad \forall\, |\alpha| \leq k\},$$

$$\|u\|_{W^{k,\infty}_\delta(\mathbb{R}^n)} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^\infty_{\delta-|\alpha|}(\mathbb{R}^n)}.$$

It is easy to show that $C^0_\delta(\mathbb{R}^n)$ *is a subspace of* $L^\infty_\delta(\mathbb{R}^n)$*, pointwise multiplication is a continuous bilinear map* $L^\infty_{\delta_1}(\mathbb{R}^n) \times L^p_{\delta_2}(\mathbb{R}^n) \to L^p_{\delta_1+\delta_2}(\mathbb{R}^n)$ *and the inclusion* $L^\infty_{\tilde{\delta}}(\mathbb{R}^n) \subseteq L^p_\delta(\mathbb{R}^n)$ *is continuous for* $\tilde{\delta} < \delta$ *and* $p \in (1,\infty)$ [4]. *Also if* $sp > n$*, then the inclusions* $W^{s,p}_\delta(\mathbb{R}^n) \subseteq C^0_\delta(\mathbb{R}^n) \subseteq L^\infty_\delta(\mathbb{R}^n)$ *are continuous.*

Note that if we let $r := \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$*, then for* $u \in L^\infty_\delta(\mathbb{R}^n)$ *we have*

$$\|u\|_{L^\infty_\delta(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}(r^{-\delta}|u|) \Longrightarrow |u| \leq r^\delta \|u\|_{L^\infty_\delta(\mathbb{R}^n)} \text{ a.e.}$$

Definition A.6. *Let* Ω *be an open subset of* $\mathbb{R}^n$. $W^{s,p}_\delta(\Omega)$ *is defined as the restriction of* $W^{s,p}_\delta(\mathbb{R}^n)$ *to* Ω *and is equipped with the following norm:*

$$\|u\|_{W^{s,p}_\delta(\Omega)} = \inf_{v \in W^{s,p}_\delta(\mathbb{R}^n), v|_\Omega = u} \|v\|_{W^{s,p}_\delta(\mathbb{R}^n)}.$$

When there is no ambiguity about the domain we may write

- $W^{s,p}$ instead of $W^{s,p}(\Omega)$,
- $W^{s,p}_\delta$ instead of $W^{s,p}_\delta(\Omega)$,
- $\|.\|_{W^{s,p}}$ or $\|\cdot\|_{s,p}$ instead of $\|\cdot\|_{W^{s,p}(\Omega)}$,
- $\|.\|_{W^{s,p}_\delta}$ or $\|\cdot\|_{s,p,\delta}$ instead of $\|\cdot\|_{W^{s,p}_\delta(\Omega)}$.

Definition A.7. *Let* (M,h) *be an n-dimensional AF manifold of class* $W^{\alpha,\gamma}_\rho$. *In addition, let* $\{(U_i,\phi_i)\}_{i=1}^m$ *be the collection of end charts. We can extend this set to an atlas* $\{(U_i,\phi_i)\}_{i=1}^k$ *such that for* $i > m$ *the set* $\bar{U}_i$ *is compact and* $\phi_i(U_i) = B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$. *Let* $\{\chi_i\}_{i=1}^k$ *be a partition of unity subordinate to the cover* $\{U_i\}_{i=1}^k$. *The weighted Sobolev space* $W^{s,p}_\delta(M)$ *is the subset of* $W^{s,p}_{loc}(M)$ *consisting of functions* u *that satisfy*

$$\|u\|_{W^{s,p}_\delta(M)} := \sum_{i=1}^m \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s,p}_\delta(\mathbb{R}^n)} + \sum_{i=m+1}^k \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s,p}(B_1)} < \infty$$

The collection $\{(U_i,\phi_i)\}_{i=1}^k$ *is called an* ***AF atlas*** *for* M.

Remark A.8. *The above definition of* $W^{s,p}_\delta(M)$ *does not depend on the metric* h *and its class and it is also independent of the chosen partition of unity, but it is based on the specific charts that were introduced in the definition of AF manifolds. This definition is not necessarily coordinate independent (of course see Remark A.10). Indeed, as for the* *case of compact manifolds, one can easily show that different choices for* $\{U_i, \phi_i\}_{i=m+1}^{k}$ *result in equivalent norms; but the dependence of the norm on the end charts is more critical. In what follows we always assume that one fixed AF atlas is given and we just work with that fixed atlas.*

Remark A.9. *By using partition of unity arguments one can prove all the items in Theorem A.4 for AF manifolds (see below; also for item 9. there are several ways to construct an isomorphism between* $(W^{s,p}_{\delta}(M))^*$ *and* $W^{-s,p'}_{-n-\delta}(M)$*, see our discussion about duality pairing in Appendix B). Of course note that for instance we have* $\|f\|_{C^0_\delta(M)} := \sup_{x\in M}([(1+|x|^2)^{\frac{1}{2}}]^{-\delta}|f|)$*, where* $|x|$ *is the geodesic distance from* x *to a fixed point* O *in the compact core. As opposed to* $\mathbb{R}^n$*, in a general Riemannian manifold* $|x|^2$ *is not smooth, so there is no advantage in using* $(1+|x|^2)^{\frac{1}{2}}$ *instead of for example* $1+|x|$*. In the literature the norms* $\|f\|_{C^0_\delta(M)} := \sup_{x\in M}((1+|x|)^{-\delta}|f|)$ *and* $\|f\|_{L^\infty_\delta(M)} = \|(1+|x|)^{-\delta} f\|_\infty$ *have also been used for* $C^0_\delta(M)$ *and* $L^\infty_\delta(M)$*, respectively. Clearly these norms are equivalent to the original ones.*

Remark A.10. *Item (3) in the definition of AF manifolds (Definition 3.1) guarantees that* $L^p_\delta(M)$ *is independent of the chosen AF atlas and in fact* $\|u\|_{L^p_\delta(M)}$ *agrees with the following norm that uses the natural volume form of* M [4] *:*

$$\|u\|_{L^p_\delta(M)} = \|\langle x\rangle^{-\delta-\frac{n}{p}} u\|_{L^p(M)}. \quad \Big(\|u\|_{L^p(M)} = (\int_M |u|^p dV_h)^{\frac{1}{p}}\Big).$$

Of course it is not necessary to single out weighted Lebesgue spaces and require their definition to be coordinate independent. One may choose to treat the spaces $L^p_\delta(M)$ *as general* $W^{s,p}_\delta(M)$ *spaces are treated. This is the reason why in some of the literature item (3) in Definition 3.1 is not considered as part of the definition.*

Here we just show two of the previously stated facts for weighted spaces on $\mathbb{R}^n$ are also true for weighted spaces on AF manifolds. The other items in Theorem A.4 and Remark A.5 can be proved for AF manifolds in a similar way.

- **Continuous Embedding:** For $s \geq s'$ and $\delta \leq \delta'$the inclusion $W^{s,p}_\delta(M) \subseteq W^{s',p}_{\delta'}(M)$ is continuous:

$$\begin{aligned}\|u\|_{W^{s',p}_{\delta'}(M)} &= \sum_{i=1}^{m} \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s',p}_{\delta'}(\mathbb{R}^n)} + \sum_{i=m+1}^{k} \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s',p}(B_1)} \\ &\preceq \sum_{i=1}^{m} \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s,p}_{\delta}(\mathbb{R}^n)} + \sum_{i=m+1}^{k} \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s,p}(B_1)} \\ &= \|u\|_{W^{s,p}_\delta(M)}.\end{aligned}$$

- **Compact Embedding:** For $s > s'$ and $\delta < \delta'$ the inclusion $W^{s,p}_\delta(M) \subseteq W^{s',p}_{\delta'}(M)$ is compact:
  Let $\{u_j\}$ be a bounded sequence in $W^{s,p}_\delta$: $\|u_j\|_{W^{s,p}_\delta} \leq \tilde{M}$. We must prove that there exists a subsequence of $\{u_j\}$ that is Cauchy in $W^{s',p}_{\delta'}$ (recall that $W^{s',p}_{\delta'}$ is complete).

$$\tilde{M} \geq \|u_j\|_{W^{s,p}_\delta} = \sum_{i=1}^{m} \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}_\delta(\mathbb{R}^n)} + \sum_{i=m+1}^{k} \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}(B_1)}.$$

Therefore

$$\begin{cases}\forall\ 1\leq i\leq m \quad \forall j \quad \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}_{\delta}(\mathbb{R}^n)}\leq \tilde{M},\\ \forall\ m+1\leq i\leq k \quad \forall j \quad \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}(B_1)}\leq \tilde{M}.\end{cases}$$

Since $W^{s,p}_{\delta}(\mathbb{R}^n)\hookrightarrow W^{s',p}_{\delta'}(\mathbb{R}^n)$ and $W^{s,p}(B_1)\hookrightarrow W^{s',p}(B_1)$ are compact (by Theorem A.4 and Rellich-Kondrachov theorem, respectively), we can conclude that

$$\begin{cases}\forall\ 1\leq i\leq m, \exists \text{ a subsequence of } \{(\phi_i^{-1})^*(\chi_i u_j)\}_{j=1}^{\infty} \text{ that converges in } W^{s',p}_{\delta'}(\mathbb{R}^n),\\ \forall\ m+1\leq i\leq k, \exists \text{ a subsequence of } \{(\phi_i^{-1})^*(\chi_i u_j)\}_{j=1}^{\infty} \text{ that converges in } W^{s',p}(B_1).\end{cases}$$

In fact, by a diagonalization argument we can construct a subsequence $\{v_j\}$ that converges in the corresponding space for all $1\leq i\leq k$ (Start with $i=1$ and find a subsequence that converges. Then for $i=2$ find a subsequence from the preceding subsequence that converges and so on. At each step we find a subsequence of the preceding subsequence). So

$$\begin{cases}\forall\ 1\leq i\leq m \quad \{(\phi_i^{-1})^*(\chi_i v_j)\}_{j=1}^{\infty} \quad \text{converges in } W^{s',p}_{\delta'}(\mathbb{R}^n),\\ \forall m+1\leq i\leq k \quad \{(\phi_i^{-1})^*(\chi_i v_j)\}_{j=1}^{\infty} \quad \text{converges in } W^{s',p}(B_1).\end{cases}$$

We claim that $\{v_j\}$ is Cauchy in $W^{s',p}_{\delta'}(M)$. Let $\epsilon>0$ be given. For each $1\leq i\leq m$, let N_i be such that if $l,\tilde{l}>N_i$ then

$$\|(\phi_i^{-1})^*(\chi_i v_{\tilde{l}})-(\phi_i^{-1})^*(\chi_i v_l)\|_{W^{s',p}_{\delta'}(\mathbb{R}^n)}<\frac{\epsilon}{k}.$$

Also for each $m+1\leq i\leq k$, let N_i be such that if $l,\tilde{l}>N_i$ then

$$\|(\phi_i^{-1})^*(\chi_i v_{\tilde{l}})-(\phi_i^{-1})^*(\chi_i v_l)\|_{W^{s',p}(B_1)}<\frac{\epsilon}{k}.$$

Now let $N=\max\{N_1,...,N_k\}$. Clearly for all $l,\tilde{l}>N$ we have

$$\begin{aligned}\|v_l-v_{\tilde{l}}\|_{W^{s',p}_{\delta'}(M)}&=\sum_{i=1}^{m}\|(\phi_i^{-1})^*(\chi_i(v_l-v_{\tilde{l}}))\|_{W^{s',p}_{\delta'}(\mathbb{R}^n)}\\&\quad+\sum_{i=m+1}^{k}\|(\phi_i^{-1})^*(\chi_i(v_l-v_{\tilde{l}}))\|_{W^{s',p}(B_1)}\\&<k\frac{\epsilon}{k}=\epsilon.\end{aligned}$$

This proves that $\{v_j\}$ is Cauchy in $W^{s',p}_{\delta'}(M)$.

Theorem A.11. [7, 61] *If $s_1-\frac{n}{p_1}\geq s_0-\frac{n}{p_0}$, $1<p_1\leq p_0<\infty$, $s_1\geq s_0\geq 0$, then $W^{s_1,p_1}(\Omega)\hookrightarrow W^{s_0,p_0}(\Omega)$ (that is, $W^{s_1,p_1}(\Omega)\subseteq W^{s_0,p_0}(\Omega)$ and the inclusion map is continuous).*

Remark A.12. *Note that if Ω is a bounded domain, then the restriction $p_1\leq p_0$ can be removed. Indeed, if Ω is bounded and $p_1>p_0$ then $L^{p_1}\subseteq L^{p_0}$ and consequently if $k\geq l\in\mathbb{N}_0$ then $W^{k,p_1}(\Omega)\hookrightarrow W^{k,p_0}(\Omega)\hookrightarrow W^{l,p_0}(\Omega)$. The claim can be proved by interpolation for the cases where s_0 or s_1 are not integers (the details are similar to the proof of Lemma A.14 below).*

Lemma A.13. *Let $k\in\mathbb{N}_0$, $\delta\in\mathbb{R}$ and $p\in(1,\infty)$. Then*

$$u\in W^{k,p}_{\delta}(\mathbb{R}^n)\iff \partial^{\alpha}u\in L^p_{\delta-|\alpha|}(\mathbb{R}^n)\quad \forall\,|\alpha|\leq k.$$

Proof. **(Lemma A.13)** The case $k = 0$ is obvious. In general we have

$$\begin{aligned} u \in W^{k,p}_\delta &\iff \|u\|_{W^{k,p}_\delta} < \infty \iff \forall\,|\alpha| \le k \quad \|\langle x\rangle^{-\delta-\frac{n}{p}+|\alpha|}\partial^\alpha u\|_{L^p} < \infty \\ &\iff \forall\,|\alpha| \le k \quad \|\langle x\rangle^{-(\delta-|\alpha|)-\frac{n}{p}}\partial^\alpha u\|_{L^p} < \infty \\ &\iff \forall\,|\alpha| \le k \quad \partial^\alpha u \in L^p_{\delta-|\alpha|}. \end{aligned}$$

□

Lemma A.14. *Let $s \in \mathbb{R}$, $p, q \in (1,\infty)$. If $p \ge q$ and $\delta' < \delta$, then $W^{s,p}_{\delta'}(\mathbb{R}^n) \hookrightarrow W^{s,q}_\delta(\mathbb{R}^n)$.*

Proof. **(Lemma A.14)** We consider three cases:

- **Case 1:** $s = k \in \mathbb{N}_0$.

$$\begin{aligned} u \in W^{k,p}_{\delta'} &\Rightarrow \forall\,|\alpha| \le k \quad \partial^\alpha u \in L^p_{\delta'-|\alpha|} \\ &\Rightarrow \forall\,|\alpha| \le k \quad \partial^\alpha u \in L^q_{\delta-|\alpha|} \quad \text{(by item1. of Theorem A.4)} \\ &\Rightarrow u \in W^{k,q}_\delta. \end{aligned}$$

  In fact,

$$\begin{aligned} \|u\|_{k,q,\delta} &= \sum_{|\beta|\le k} \|\langle x\rangle^{-\delta-\frac{n}{p}+|\beta|}\partial^\beta u\|_{L^q(\mathbb{R}^n)} = \sum_{|\beta|\le k} \|\langle x\rangle^{-(\delta-|\beta|)-\frac{n}{p}}\partial^\beta u\|_{L^q(\mathbb{R}^n)} \\ &= \sum_{|\beta|\le k} \|\partial^\beta u\|_{L^q_{\delta-|\beta|}(\mathbb{R}^n)} \preceq \sum_{|\beta|\le k} \|\partial^\beta u\|_{L^p_{\delta'-|\beta|}(\mathbb{R}^n)} \quad (L^p_{\delta'-|\beta|} \hookrightarrow L^q_{\delta-|\beta|}) \\ &= \sum_{|\beta|\le k} \|\langle x\rangle^{-\delta'-\frac{n}{p}+|\beta|}\partial^\beta u\|_{L^p(\mathbb{R}^n)} = \|u\|_{k,p,\delta'}. \end{aligned}$$

- **Case 2:** $s \ge 0$, $s \notin \mathbb{N}_0$.
  Let $k = \lfloor s \rfloor$, $\theta = s - k$. By what was proved in the previous case

$$W^{k,p}_{\delta'} \hookrightarrow W^{k,q}_\delta, \quad W^{k+1,p}_{\delta'} \hookrightarrow W^{k+1,q}_\delta.$$

  Since $s = (1-\theta)k + \theta(k+1)$, the claim follows from real interpolation.

- **Case 3:** $s < 0$.
  By assumption $p \ge q$ and $\delta' < \delta$, therefore

$$p' \le q', \quad -n-\delta' > -n-\delta.$$

  Here p' and q' are the conjugates of p and q, respectively. Thus by what was proved in the previous cases we have

$$W^{-s,q'}_{-n-\delta} \hookrightarrow W^{-s,p'}_{-n-\delta'}.$$

  The result follows by taking the dual.

□

Lemma A.15. *Let the following assumptions hold:*

(i) $1 < p \le r < \infty$,
(ii) $t, s \in \mathbb{R}$ *with* $0 \le t \le s$,
(iii) $s - \frac{n}{p} \ge t - \frac{n}{r}$.

Then: For all $\delta \in \mathbb{R}$ $W^{s,p}_\delta \hookrightarrow W^{t,r}_\delta$.

Proof. **(Lemma A.15)** In the proof we use the fact that if $1 \leq \alpha \leq \beta$, then $l^{\alpha} \hookrightarrow l^{\beta}$ (l^{α} denotes the space of α-power summable sequences); in fact for any sequence $a = \{a_j\}$, $\|a\|_{l^{\beta}} \leq \|a\|_{l^{\alpha}}$. From the assumption it follows that $W^{s,p} \hookrightarrow W^{t,r}$ and so

$$\begin{aligned}
\|u\|_{t,r,\delta} &= \Big[\sum_{j=0}^{\infty} 2^{-r\delta j}\|S_{2^j}(\varphi_j u)\|_{t,r}^{r}\Big]^{\frac{1}{r}} \preceq \Big[\sum_{j=0}^{\infty} 2^{-r\delta j}\|S_{2^j}(\varphi_j u)\|_{s,p}^{r}\Big]^{\frac{1}{r}} \\
&= \Big[\sum_{j=0}^{\infty}(2^{-\delta j}\|S_{2^j}(\varphi_j u)\|_{s,p})^{r}\Big]^{\frac{1}{r}} \leq \Big[\sum_{j=0}^{\infty}(2^{-\delta j}\|S_{2^j}(\varphi_j u)\|_{s,p})^{p}\Big]^{\frac{1}{p}} \\
&= \|u\|_{s,p,\delta} \quad (\text{Note that } p \leq r \text{ and so } \|\cdot\|_{l^r} \leq \|\cdot\|_{l^p}).
\end{aligned}$$

□

Theorem A.16 (Embedding Theorem I)**.** *Let the following assumptions hold:*

(i) $1 < p \leq r < \infty$,
(ii) $t, s \in \mathbb{R}$ *with* $t \leq s$,
(iii) $s - \frac{n}{p} \geq t - \frac{n}{r}$.

Then: If $\delta' \leq \delta$ *then* $W^{s,p}_{\delta'} \hookrightarrow W^{t,r}_{\delta}$.

Proof. **(Theorem A.16)** Note that, since $\delta' \leq \delta$, $W^{s,p}_{\delta'} \hookrightarrow W^{s,p}_{\delta}$, so we just need to show that $W^{s,p}_{\delta} \hookrightarrow W^{t,r}_{\delta}$. By Lemma A.15 we know that the claim is true for the case $0 \leq t$. So we just need to consider the case where $t < 0$.

- **Case 1:** $t < 0, s \leq 0$
  It is enough to show that $(W^{t,r}_{\delta})^* \hookrightarrow (W^{s,p}_{\delta})^*$, that is, we need to prove that
  $$W^{-t,r'}_{-n-\delta} \hookrightarrow W^{-s,p'}_{-n-\delta}.$$
  Note that $-t$ and $-s$ are nonnegative so we just need to check that the assumptions of Lemma A.15 hold true:

$$t \leq s \leq 0 \Rightarrow 0 \leq -s \leq -t$$
$$1 < p \leq r \Rightarrow 1 < r' \leq p'$$
$$s - \frac{n}{p} \geq t - \frac{n}{r} \Rightarrow s + \frac{n}{r} - n \geq t + \frac{n}{p} - n \Rightarrow s - \frac{n}{r'} \geq t - \frac{n}{p'} \Rightarrow -t - \frac{n}{r'} \geq -s - \frac{n}{p'}.$$

- **Case 2:** $t < 0, s > 0$
  In this case we will prove that there exists $q \geq 1$ such that
  $$W^{s,p}_{\delta} \hookrightarrow L^{q}_{\delta} \hookrightarrow W^{t,r}_{\delta}$$
  By what was proved previously, in order to make sure that the above inclusions hold true it is enough to find q such that
  $$t - \frac{n}{r} \leq 0 - \frac{n}{q} \leq s - \frac{n}{p} \quad (\Leftrightarrow -\frac{s}{n} + \frac{1}{p} \leq \frac{1}{q} \leq -\frac{t}{n} + \frac{1}{r})$$
  $$p \leq q \leq r \quad (\Leftrightarrow \frac{1}{r} \leq \frac{1}{q} \leq \frac{1}{p})$$
  Note that by assumption $-\frac{s}{n} + \frac{1}{p} \leq -\frac{t}{n} + \frac{1}{r}$. If $-\frac{s}{n} + \frac{1}{p} = -\frac{t}{n} + \frac{1}{r}$, then q defined by $\frac{1}{q} = -\frac{s}{n} + \frac{1}{p} (= -\frac{t}{n} + \frac{1}{r})$ clearly satisfies the desired conditions. So it remains

to consider the case where $-\frac{s}{n}+\frac{1}{p}<-\frac{t}{n}+\frac{1}{r}$. The inequalities in the first line are satisfied if and only if

$$\frac{1}{q}=-\frac{s}{n}+\frac{1}{p}+\sigma(\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}).$$

for some $\sigma\in[0,1]$. The question is "can we choose σ so that the above expression lies between $\frac{1}{r}$ and $\frac{1}{p}$?" We want to find $\sigma\in[0,1]$ such that

$$\frac{1}{r}\leq-\frac{s}{n}+\frac{1}{p}+\sigma(\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p})\leq\frac{1}{p}$$

That is we want to find $\sigma\in[0,1]$ such that

$$\frac{\frac{1}{r}-\frac{1}{p}+\frac{s}{n}}{\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}}\leq\sigma\leq\frac{\frac{s}{n}}{\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}}.$$

Note that since $\frac{1}{r}\leq\frac{1}{p}$ clearly

$$\frac{\frac{1}{r}-\frac{1}{p}+\frac{s}{n}}{\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}}\leq\frac{\frac{s}{n}}{\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}}.$$

So it would be possible to find σ if and only if

$$\frac{\frac{s}{n}}{\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}}\geq 0\quad\text{and}\quad\frac{\frac{1}{r}-\frac{1}{p}+\frac{s}{n}}{\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}}\leq 1.$$

The first inequality is true because by assumption $s>0$ and $s-\frac{n}{p}\geq t-\frac{n}{r}$. The second inequality is true because by assumption $t<0$ and

$$\frac{\frac{1}{r}-\frac{1}{p}+\frac{s}{n}}{\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}}\leq 1\Leftrightarrow\frac{1}{r}-\frac{1}{p}+\frac{s}{n}\leq\frac{s-t}{n}+\frac{1}{r}-\frac{1}{p}\Leftrightarrow\frac{t}{n}\leq 0.$$

□

Theorem A.17 (Embedding Theorem II). *Let the following assumptions hold:*

(i) $1<p,r<\infty$,
(ii) $t,s\in\mathbb{R}$ *with* $t\leq s$,
(iii) $s-\frac{n}{p}\geq t-\frac{n}{r}$,
(iv) δ' *is* ***strictly*** *less than* δ.

Then: $W^{s,p}_{\delta'}\hookrightarrow W^{t,r}_{\delta}$. *(Note that if* $p>r$*, then the third assumption follows from the second assumption.)*

Proof. **(Theorem A.17)** If $p\leq r$, then the claim follows from Theorem A.16. Let's assume $p>r$. Then by Lemma A.14 we have $W^{s,p}_{\delta'}\hookrightarrow W^{s,r}_{\delta}$ and by Theorem A.4 we have $W^{s,r}_{\delta}\hookrightarrow W^{t,r}_{\delta}$. Consequently $W^{s,p}_{\delta'}\hookrightarrow W^{t,r}_{\delta}$. □

Lemma A.18 (Multiplication by bounded smooth functions). *Let* $\sigma\in\mathbb{R}$, $q\in[1,\infty)$ *(if* $\sigma<0, q\neq 1$*). Let* $N=\lceil|\sigma|\rceil$. *If* $f\in BC^N(\mathbb{R}^n)$ *and* $u\in W^{\sigma,q}(\mathbb{R}^n)$, *then* $fu\in W^{\sigma,q}(\mathbb{R}^n)$ *and moreover* $\|fu\|_{\sigma,q}\preceq\|u\|_{\sigma,q}$ *(the implicit constant depends on* f *but it does not depend on* u*).*

Proof. **(Lemma A.18)** The proof consists of four steps:

- **Step 1:** $\sigma=k\in\mathbb{N}_0$. The claim is proved in [20].

- **Step 2:** $0 < \sigma < 1$. The claim has been proved in [48] for the case where $\sigma \in (0,1)$, f is Lipschitz continuous and $0 \leq f \leq 1$. With an obvious modification that proof also works for the case where $f \in BC^1(\mathbb{R}^n)$.
- **Step 3:** $1 < \sigma \notin \mathbb{N}$. In this case we can proceed as follows: Let $k = \lfloor \sigma \rfloor$, $\theta = \sigma - k$.

$$\begin{aligned}
\|fu\|_{\sigma,q} &= \|fu\|_{k,q} + \sum_{|\nu|=k} \|\partial^\nu (fu)\|_{\theta,q} \\
&\leq \|fu\|_{k,q} + \sum_{|\nu|=k} \sum_{\beta \leq \nu} \|\partial^{\nu-\beta} f \partial^\beta u\|_{\theta,q} \\
&\preceq \|u\|_{k,q} + \sum_{|\nu|=k} \sum_{\beta \leq \nu} \|\partial^\beta u\|_{\theta,q} \quad \text{(by Step1 and Step2)} \\
&= \|u\|_{\sigma,q} + \sum_{|\nu|=k} \sum_{\beta < \nu} \|\partial^\beta u\|_{\theta,q} \\
&\preceq \|u\|_{\sigma,q} + \sum_{|\nu|=k} \sum_{\beta < \nu} \|u\|_{\theta+|\beta|,q} \quad (\partial^\beta : W^{\theta+|\beta|,q} \to W^{\theta,q} \text{is continuous}) \\
&\preceq \|u\|_{\sigma,q} + \sum_{|\nu|=k} \sum_{\beta < \nu} \|u\|_{\sigma,q} \quad (\theta + |\beta| < \sigma \Rightarrow W^{\sigma,q} \hookrightarrow W^{\theta+|\beta|,q}) \\
&\preceq \|u\|_{\sigma,q}.
\end{aligned}$$

- **Step 4:** $\sigma < 0$. For this case we use a duality argument:

$$\begin{aligned}
\|fu\|_{\sigma,q} &= \sup_{v \in W^{-\sigma,q} \setminus \{0\}} \frac{|\langle fu, v \rangle|}{\|v\|_{-\sigma,q}} = \sup_{v \in W^{-\sigma,q} \setminus \{0\}} \frac{|\langle u, fv \rangle|}{\|v\|_{-\sigma,q}} \\
&\leq \sup_{v \in W^{-\sigma,q} \setminus \{0\}} \frac{\|u\|_{\sigma,q} \|fv\|_{-\sigma,q}}{\|v\|_{-\sigma,q}} \preceq \sup_{v \in W^{-\sigma,q} \setminus \{0\}} \frac{\|u\|_{\sigma,q} \|v\|_{-\sigma,q}}{\|v\|_{-\sigma,q}} = \|u\|_{\sigma,q}.
\end{aligned}$$

□

Lemma A.19. *Let $\sigma, \delta \in \mathbb{R}$, $q \in (1,\infty)$. Let $N = \lfloor |\sigma| \rfloor + 1$. Suppose $f \in C^N(\mathbb{R}^n)$ is such that for all multi-indices ν with $|\nu| \leq N$*

$$|\partial^\nu f(x)| \leq b(\nu) |x|^{-|\nu|},$$

where $b(\nu)$ are appropriate numbers independent of x. If $u \in W^{\sigma,q}_\delta(\mathbb{R}^n)$, then $fu \in W^{\sigma,q}_\delta(\mathbb{R}^n)$ and moreover $\|fu\|_{\sigma,q,\delta} \preceq \|u\|_{\sigma,q,\delta}$ where the implicit constant depends on $b(\nu)$.

Proof. **(Lemma A.19)** The case $\sigma \geq 0$ is a special case of Lemma 3 in [54]. For the case $\sigma < 0$ we may use a duality argument exactly similar to the proof of Lemma A.18. □

Most of the claims of the following lemma are discussed in [61] for $s \geq 0$. The argument in [61] in part is based on a similar multiplication lemma for Besov spaces. An entirely different approach to the proof which includes some cases that are not considered in [61] can be found in [6]. By using a duality argument one can extend the proof to negative values of s [32, 2, 6].

Lemma A.20 (Multiplication Lemma, Unweighted spaces)**.** *Let $s_i \geq s$ with $s_1+s_2 \geq 0$, and $1 < p, p_i < \infty$ ($i = 1,2$) be real numbers satisfying*

$$s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p}), \quad (\textit{if } s_i = s \notin \mathbb{Z}, \textit{ then let } p_i \leq p)$$
$$s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0.$$

In case $s < 0$, in addition let

$$s_1 + s_2 > n(\frac{1}{p_1} + \frac{1}{p_2} - 1) \quad (\textit{equality is allowed if } min(s_1, s_2) < 0).$$

Also in case where $s_1 + s_2 = 0$ and $min(s_1, s_2) \notin \mathbb{Z}$, in addition let $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$. Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n).$$

Remark A.21. *Note that in case $s_i = s \notin \mathbb{Z}$, the condition $p_i \leq p$ together with $s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p})$ in fact implies that we must have $p_i = p$.*

Corollary A.22. *Let $s_i \geq s$ with $s_1 + s_2 \geq 0$, and $2 \leq p < \infty$ ($i = 1,2$) be real numbers satisfying*

$$s_1 + s_2 - s > \frac{n}{p}.$$

Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$W^{s_1,p}(\mathbb{R}^n) \times W^{s_2,p}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n).$$

Corollary A.23. *As a direct consequence of the multiplication lemma we have:*

- *If $p \in (1,\infty)$ and $s \in (\frac{n}{p}, \infty)$, then $W^{s,p}(\mathbb{R}^n)$ is a Banach algebra.*
- *Let $p \in (1,\infty)$ and $s \in (\frac{n}{p}, \infty)$. Suppose $q \in (1,\infty)$ and $\sigma \in [-s, s]$ satisfy $\sigma - \frac{n}{q} \in [-n - s + \frac{n}{p}, s - \frac{n}{p}]$; in case $s \notin \mathbb{N}_0$, $q < p$, in addition assume $\sigma \neq \pm s$. Then the pointwise multiplication is bounded as a map $W^{s,p}(\mathbb{R}^n) \times W^{\sigma,q}(\mathbb{R}^n) \to W^{\sigma,q}(\mathbb{R}^n)$.*

Lemma A.24 (Multiplication Lemma, Weighted spaces)**.** *Assume that s, s_1, s_2 and that $1 < p, p_1, p_2 < \infty$ are real numbers satisfying*

(i) $s_i \geq s$ *($i = 1,2$) (if $s_i = s \notin \mathbb{Z}$, then let $p_i \leq p$),*
(ii) $s_1 + s_2 \geq 0$ *(if $s_1 + s_2 = 0$ and $min(s_1, s_2) \notin \mathbb{Z}$, then let $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$),*
(iii) $s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p})$ *($i = 1,2$),*
(iv) $s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0$.

In case $min(s_1, s_2) < 0$, in addition let

(v) $s_1 + s_2 \geq n(\frac{1}{p_1} + \frac{1}{p_2} - 1)$.

In case $s < 0$ and $min(s_1, s_2) \geq 0$, we assume the above inequality is strict ($s_1 + s_2 > n(\frac{1}{p_1} + \frac{1}{p_2} - 1)$). Then for all $\delta_1, \delta_2 \in \mathbb{R}$, the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$W^{s_1,p_1}_{\delta_1}(\mathbb{R}^n) \times W^{s_2,p_2}_{\delta_2}(\mathbb{R}^n) \to W^{s,p}_{\delta_1+\delta_2}(\mathbb{R}^n).$$

Proof. **(Lemma A.24)** A proof for the case $p_1 = p_2 = p = 2$ is given in [38] [Lemma 3.5]. In what follows we use Lemma A.20 to extend that proof to our general setting. In our proof we will make use of the following facts:

- **Fact 1**: If f is a smooth function with compact support and $u \in W^{t,q}$ then $fu \in W^{t,q}$ and $\|fu\|_{W^{t,q}} \preceq \|u\|_{W^{t,q}}$ (this is in fact a special case of Lemma A.18).
- **Fact 2**: For all j, $S_{2^j}\varphi_j = S_{2^j}S_{2^{-j}}\varphi = \varphi$. So $S_{2^j}\varphi_j$ is zero if $x \notin B_2 \setminus B_{\frac{1}{2}}$.
- **Fact 3**: If $1 \leq \alpha \leq \beta$, then $l^\alpha \hookrightarrow l^\beta$; in fact for any sequence $a = \{a_j\}$, $\|a\|_{l^\beta} \leq \|a\|_{l^\alpha}$.
- **Fact 4**: For $a_k > 0$ we have :$\sum_{k=1}^m a_k^p \sim (\sum_{k=1}^m a_k)^p$ (that is $\sum_{k=1}^m a_k^p \preceq (\sum_{k=1}^m a_k)^p \preceq \sum_{k=1}^m a_k^p$).
- **Fact 5**: $\|S_r u\|_{W^{s,p}} \leq C(r,s,p,n)\|u\|_{W^{s,p}}$.

Now let's start proving the lemma. Suppose $u_i \in W^{s_i,p_i}_{\delta_i}$. Let $\varphi_j = 0$ for $j < 0$. We have

$$S_{2^j}(\varphi_j u_1 u_2) = S_{2^j}(\varphi_j) S_{2^j} u_1 S_{2^j} u_2.$$

By **Fact 2**, $S_{2^j}\varphi_j$ is zero if $x \notin B_2 \setminus B_{\frac{1}{2}}$. Also it is easy to see that for $x \in B_2 \setminus B_{\frac{1}{2}}$, $\varphi_k(2^j x) = 0$ if $k \notin \{j-1, j, j+1\}$. Since for all x, $\sum_{k=0}^\infty \varphi_k(2^j x) = 1$, we can conclude that for $x \in B_2 \setminus B_{\frac{1}{2}}$

$$\sum_{k=j-1}^{j+1} \varphi_k(2^j x) = 1.$$

Therefore for all x

$$S_{2^j}(\varphi_j u_1 u_2) = S_{2^j}(\varphi_j) \sum_{k=j-1}^{j+1} S_{2^j}(\varphi_k u_1) \sum_{l=j-1}^{j+1} S_{2^j}(\varphi_l u_2).$$

Now by **Fact 1** we have

$$\|S_{2^j}(\varphi_j u_1 u_2)\|^p_{W^{s,p}} \preceq \sum_{k,l=j-1}^{j+1} \|S_{2^j}(\varphi_k u_1) S_{2^j}(\varphi_l u_2)\|^p_{W^{s,p}},$$

and by the multiplication lemma for the corresponding unweighted Sobolev spaces we get

$$\begin{aligned}\|S_{2^j}(\varphi_j u_1 u_2)\|^p_{W^{s,p}} &\preceq \sum_{k,l=j-1}^{j+1} \|S_{2^j}(\varphi_k u_1)\|^p_{W^{s_1,p_1}} \|S_{2^j}(\varphi_l u_2)\|^p_{W^{s_2,p_2}} \\ &\preceq \sum_{k,l=j-1}^{j+1} \|S_{2^{j-k}}S_{2^k}(\varphi_k u_1)\|^p_{W^{s_1,p_1}} \|S_{2^{j-l}}S_{2^l}(\varphi_l u_2)\|^p_{W^{s_2,p_2}}.\end{aligned}$$

$S_{2^{j-k}}$ is one of $S_{2^{-1}}$, S_{2^0}, or S_{2^1}. So, by **Fact 5**

$$\begin{aligned}\sum_{k=j-1}^{j+1} \|S_{2^{j-k}}S_{2^k}(\varphi_k u_1)\|^p_{W^{s_1,p_1}} &\leq \sum_{k=j-1}^{j+1} (\|S_{2^{-1}}S_{2^k}(\varphi_k u_1)\|^p_{W^{s_1,p_1}} + \|S_{2^0}S_{2^k}(\varphi_k u_1)\|^p_{W^{s_1,p_1}} \\ &\quad + \|S_{2^1}S_{2^k}(\varphi_k u_1)\|^p_{W^{s_1,p_1}}) \\ &\preceq \sum_{k=j-1}^{j+1} \|S_{2^k}(\varphi_k u_1)\|^p_{W^{s_1,p_1}}\end{aligned}$$

and the similar result is true for $\sum_{l=j-1}^{j+1}\|S_{2^{j-l}}S_{2^l}(\varphi_l u_2)\|^p_{W^{s_2,p_2}}$. Consequently

$$\begin{aligned}\|S_{2^j}(\varphi_j u_1 u_2)\|^p_{W^{s,p}} &\preceq \sum_{k,l=j-1}^{j+1}\|S_{2^k}(\varphi_k u_1)\|^p_{W^{s_1,p_1}}\|S_{2^l}(\varphi_l u_2)\|^p_{W^{s_2,p_2}}\\ &\preceq (\sum_{k=j-1}^{j+1}\|S_{2^k}(\varphi_k u_1)\|_{W^{s_1,p_1}})^p(\sum_{l=j-1}^{j+1}\|S_{2^l}(\varphi_l u_2)\|_{W^{s_2,p_2}})^p\end{aligned}$$

There fore

$$\begin{aligned}&\sum_{j=0}^{\infty}2^{-p(\delta_1+\delta_2)j}\|S_{2^j}(\varphi_j u_1 u_2)\|^p_{W^{s,p}}\\ &\quad\preceq \sum_{j=0}^{\infty}\big[2^{-p\delta_1 j}(\sum_{k=j-1}^{j+1}\|S_{2^k}(\varphi_k u_1)\|_{W^{s_1,p_1}})^p\,2^{-p\delta_2 j}(\sum_{l=j-1}^{j+1}\|S_{2^l}(\varphi_l u_2)\|_{W^{s_2,p_2}})^p\big].\end{aligned}$$

Let

$$\begin{aligned}a_j &= 2^{-\delta_1 j}\sum_{k=j-1}^{j+1}\|S_{2^k}(\varphi_k u_1)\|_{W^{s_1,p_1}}\\ b_j &= 2^{-\delta_2 j}\sum_{l=j-1}^{j+1}\|S_{2^l}(\varphi_l u_2)\|_{W^{s_2,p_2}}.\end{aligned}$$

So we have

$$\|u_1u_2\|_{W^{s,p}_{\delta_1+\delta_2}} = \big[\sum_{j=0}^{\infty}2^{-p(\delta_1+\delta_2)j}\|S_{2^j}(\varphi_j u_1 u_2)\|^p_{W^{s,p}}\big]^{\frac{1}{p}} \preceq \big[\sum_{j=0}^{\infty}(a_jb_j)^p\big]^{\frac{1}{p}}.$$

Now let r be such that $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}$. By assumption $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\geq 0$ and so $r\leq p$. Thus by **Fact 3** and Holder's inequality we get

$$\begin{aligned}\big[\sum_{j=0}^{\infty}(a_jb_j)^p\big]^{\frac{1}{p}} &\leq \big[\sum_{j=0}^{\infty}(a_jb_j)^r\big]^{\frac{1}{r}}\\ &\leq \big[\sum_{j=0}^{\infty}(a_j)^{p_1}\big]^{\frac{1}{p_1}}\big[\sum_{j=0}^{\infty}(b_j)^{p_2}\big]^{\frac{1}{p_2}}\\ &\preceq \big[\sum_{j=0}^{\infty}2^{-p_1\delta_1 j}\sum_{k=j-1}^{j+1}\|S_{2^k}(\varphi_k u_1)\|^{p_1}_{W^{s_1,p_1}}\big]^{\frac{1}{p_1}}\\ &\qquad\big[\sum_{j=0}^{\infty}2^{-p_2\delta_2 j}\sum_{l=j-1}^{j+1}\|S_{2^l}(\varphi_l u_2)\|^{p_2}_{W^{s_2,p_2}}\big]^{\frac{1}{p_2}}\\ &\preceq \big[\sum_{j=0}^{\infty}2^{-p_1\delta_1 j}\|S_{2^j}(\varphi_j u_1)\|^{p_1}_{W^{s_1,p_1}}\big]^{\frac{1}{p_1}}\big[\sum_{j=0}^{\infty}2^{-p_2\delta_2 j}\|S_{2^j}(\varphi_j u_2)\|^{p_2}_{W^{s_2,p_2}}\big]^{\frac{1}{p_2}}\\ &= \|u_1\|_{W^{s_1,p_1}_{\delta_1}}\|u_2\|_{W^{s_2,p_2}_{\delta_2}}.\end{aligned}$$

This proves $\|u_1u_2\|_{W^{s,p}_{\delta_1+\delta_2}}\preceq\|u_1\|_{W^{s_1,p_1}_{\delta_1}}\|u_2\|_{W^{s_2,p_2}_{\delta_2}}$. □

Remark A.25. *By using partition of unity and charts one can show that the above lemma also holds for AF manifolds.*

Corollary A.26 (The case where $p_1 = p_2 = p$). *Assume $s \leq min\{s_1, s_2\}$, $s_1 + s_2 > s + \frac{n}{p}$, $s_1 + s_2 \geq 0$, $s_1 + s_2 > n(\frac{2}{p} - 1)$ and $\delta_1 + \delta_2 \leq \delta$, then the multiplication*

$$W^{s_1,p}_{\delta_1} \times W^{s_2,p}_{\delta_2} \to W^{s,p}_{\delta},$$

is continuous.

Corollary A.27. *Let $p \in (1, \infty)$, $s \in (\frac{n}{p}, \infty)$, and $\delta < 0$, then the space $W^{s,p}_{\delta}$ is an algebra.*

Lemma A.28. *Let the following assumptions hold:*

- $f : \mathbb{R} \to \mathbb{R}$ *is smooth,*
- $u \in W^{s,p}_{\rho}(\mathbb{R}^n)$, *where* $s > \frac{n}{p}$, $\rho < 0$, *and* $p \in (1, \infty)$,
- $v \in W^{\sigma,q}_{\delta}(\mathbb{R}^n)$, *where* $\delta \in \mathbb{R}$, $q \in (1, \infty)$ *and* (i) $\sigma \in [-s, s]$ *(*$\sigma \neq \pm s$ *if* $s \notin \mathbb{N}_0$ *and* $q < p$*),* (ii) $\sigma - \frac{n}{q} \in [-n - s + \frac{n}{p}, s - \frac{n}{p}]$.

Then: $f(u)v \in W^{\sigma,q}_{\delta}(\mathbb{R}^n)$ and moreover the map taking (u, v) to $f(u)v$ is continuous.

Proof. **(Lemma A.28)** A proof for the case $p = q = 2$ is given in [38] [Lemma 3.6]. Here we use the multiplication lemma to extend that proof to our general setting. In the proof we make use of the following facts:

- **Fact 1**: If η is a smooth function with compact support, f is as in the statement of lemma, and $u \in W^{t,q}$ with $tq > n$, then $\eta f(u) \in W^{t,q}$ and the map taking u to $\eta f(u)$ is continuous from $W^{t,q}$ to $W^{t,q}$.
- **Fact 2**: For all j, $S_{2^j}\varphi_j = S_{2^j}S_{2^{-j}}\varphi = \varphi$. So $S_{2^j}\varphi_j$ is zero if $x \notin B_2 \setminus B_{\frac{1}{2}}$. Also it is easy to see that for $x \in B_2 \setminus B_{\frac{1}{2}}$, $\varphi_k(2^j x) = 0$ if $k \notin \{j-1, j, j+1\}$. Since for all x, $\sum_{k=0}^{\infty} \varphi_k(2^j x) = 1$, we can conclude that for $x \in B_2 \setminus B_{\frac{1}{2}}$

$$\sum_{k=j-1}^{j+1} \varphi_k(2^j x) = 1.$$

- **Fact 3**: $\|S_r u\|_{W^{t,e}} \leq C(r, t, e, n)\|u\|_{W^{t,e}}$.

We prove the lemma in six steps:

Step 1: Suppose u and v satisfy the hypotheses of the lemma. Then considering **Fact 2** and the fact that $W^{s,p} \times W^{\sigma,q} \hookrightarrow W^{\sigma,q}$, we can write

$$\begin{aligned}
\|f(u)v\|^q_{W^{\sigma,q}_{\delta}} &= \sum_{j=0}^{\infty} 2^{-q\delta j}\|S_{2^j}(\varphi_j f(u)v)\|^q_{W^{\sigma,q}} \\
&= \sum_{j=0}^{\infty} 2^{-q\delta j}\| \sum_{k=j-1}^{j+1} (S_{2^j}\varphi_k) f(\sum_{i=j-1}^{j+1} S_{2^{j-i}}S_{2^i}(\varphi_i u))S_{2^j}(\varphi_j v)\|^q_{W^{\sigma,q}} \\
&\preceq \sum_{j=0}^{\infty} 2^{-q\delta j}\| \sum_{k=j-1}^{j+1} (S_{2^j}\varphi_k) f(\sum_{i=j-1}^{j+1} S_{2^{j-i}}S_{2^i}(\varphi_i u))\|^q_{W^{s,p}}\|S_{2^j}(\varphi_j v)\|^q_{W^{\sigma,q}}.
\end{aligned}$$

In the second line above we made use of the following equality over $B_2 \setminus B_{\frac{1}{2}}$

$$S_{2^j} f(u) = f(S_{2^j} u) = f(u(2^j x)) = f(\sum_{i=j-1}^{j+1} \varphi_i(2^j x) u(2^j x)) = f(\sum_{i=j-1}^{j+1} S_{2^j}(\varphi_i u))$$
$$= f(\sum_{i=j-1}^{j+1} S_{2^{j-i}} S_{2^i}(\varphi_i u)).$$

For all j define

$$R_j u = \sum_{i=j-1}^{j+1} S_{2^{j-i}} S_{2^i}(\varphi_i u).$$

So we have

$$\|f(u)v\|^q_{W^{\sigma,q}_\delta} \preceq \sum_{j=0}^{\infty} 2^{-q\delta j} \| \sum_{k=j-1}^{j+1} (S_{2^j}\varphi_k) f(R_j u)\|^q_{W^{s,p}} \|S_{2^j}(\varphi_j v)\|^q_{W^{\sigma,q}}.$$

Step 2: Note that if $g \in W^{s,p}_\rho$ since $\rho < 0$ we have $S_{2^i}(\varphi_i g) \to 0$ in $W^{s,p}$ as $i \to \infty$. Indeed, $2^{-p\rho i} \geq 1$ and therefore we may write

$$\|g\|^p_{W^{s,p}_\rho} < \infty \Rightarrow \sum_{i=0}^{\infty} 2^{-p\rho i} \|S_{2^i}(\varphi_i g)\|^p_{W^{s,p}} < \infty \Rightarrow \sum_{i=0}^{\infty} \|S_{2^i}(\varphi_i g)\|^p_{W^{s,p}} < \infty$$
$$\Rightarrow \lim_{i\to\infty} S_{2^i}(\varphi_i g) = 0 \quad \text{in } W^{s,p}$$

Moreover it follows from $2^{-p\rho i} \geq 1$ that if $g \in W^{s,p}_\rho$ with $\rho < 0$ then it holds that $\|S_{2^i}(\varphi_i g)\|_{W^{s,p}} \leq \|g\|_{W^{s,p}_\rho}$ for all $i \geq 0$. Also we have

$$\|R_j g - 0\|_{W^{s,p}} = \| \sum_{i=j-1}^{j+1} S_{2^{j-i}} S_{2^i}(\varphi_i g)\|_{W^{s,p}} \leq \sum_{i=j-1}^{j+1} \|S_{2^{j-i}} S_{2^i}(\varphi_i g)\|_{W^{s,p}}$$
$$\leq \sum_{i=j-1}^{j+1} (\|S_{2^{-1}} S_{2^i}(\varphi_i g)\|_{W^{s,p}} + \|S_{2^0} S_{2^i}(\varphi_i g)\|_{W^{s,p}}$$
$$+ \|S_{2^1} S_{2^i}(\varphi_i g)\|_{W^{s,p}})$$
$$\preceq \sum_{i=j-1}^{j+1} \|S_{2^i}(\varphi_i g)\|_{W^{s,p}} \to 0.$$

Step 3: Let $\eta_j := \sum_{k=j-1}^{j+1} (S_{2^j}\varphi_k)$. For $j > 1$ we may write

$$\sum_{k=j-1}^{j+1} (S_{2^j}\varphi_k) = \sum_{k=j-1}^{j+1} S_{2^j} S_{2^{-k}} \varphi = \sum_{k=j-1}^{j+1} S_{2^j} S_{2^{-k}} S_2 \varphi_1 = \sum_{k-j=-1}^{k-j=1} S_{2^{j-k+1}} \varphi_1$$
$$= \sum_{i=0}^{2} S_{2^i} \varphi_1 =: \eta.$$

That is for $j > 1$, η_j does not depend on j. Now, by **Step 2**, we know that $R_j u \to 0$ in $W^{s,p}$. So it follows from **Fact 1** that $\eta f(R_j u) \to \eta f(0)$ in $W^{s,p}$. Consequently $\{\|\eta f(R_j u)\|_{W^{s,p}}\}_{j=2}^{\infty}$ is a bounded sequence:

$$\exists\, M_1 \quad \text{such that} \quad \forall\, j \geq 2 \quad \|\eta f(R_j u)\|_{W^{s,p}} < M_1.$$

Let

$$M=\max\{M_1, \|\eta_1 f(R_1 u)\|_{W^{s,p}}, \|\eta_0 f(R_0 u)\|_{W^{s,p}}\}$$

(M is independent of j but it may depend on u).

So by what was proved in **Step 1** we have

$$\|f(u)v\|^q_{W^{\sigma,q}_\delta} \preceq \sum_{j=0}^{\infty} 2^{-q\delta j} M^q \|S_{2^j}(\varphi_j v)\|^q_{W^{\sigma,q}} = M^q \|v\|^q_{W^{\sigma,q}_\delta}.$$

This shows that $f(u)v$ is in $W^{\sigma,q}_\delta$. Now it remains to prove the continuity.

Step 4: Let (u_k, v_k) be a sequence in $W^{s,p}_\rho \times W^{\sigma,q}_\delta$ that converges to $(u,v) \in W^{s,p}_\rho \times W^{\sigma,q}_\delta$. We must show that $f(u_k)v_k \to f(u)v$ in $W^{\sigma,q}_\delta$. Note that

$$f(u)v - f(u_k)v_k = f(u)(v-v_k) + (f(u)-f(u_k))v_k.$$

By what was proved in **Step 3**, we have

$$\|f(u)(v-v_k)\|_{W^{\sigma,q}_\delta} \preceq \|v-v_k\|_{W^{\sigma,q}_\delta} \to 0.$$

So it remains to show that $\|(f(u)-f(u_k))v_k\|_{W^{\sigma,q}_\delta} \to 0$.

Step 5: By calculations similar to what was done in **Step 1** we have

$$\begin{aligned}\|(f(u)-f(u_k))v_k\|^q_{W^{\sigma,q}_\delta} &\preceq \sum_{j=0}^{\infty} 2^{-q\delta j} \|\eta_j(f(R_j u)-f(R_j u_k))\|^q_{W^{s,p}} \|S_{2^j}(\varphi_j v_k)\|^q_{W^{\sigma,q}} \\ &\preceq \|v_k\|^q_{W^{\sigma,q}_\delta} \sup_{j\geq 0} \|\eta_j(f(R_j u)-f(R_j u_k))\|^q_{W^{s,p}}.\end{aligned}$$

Note that $\{v_k\}$ is convergent and so $\{v_k\}$ is bounded in $W^{\sigma,q}_\delta$. Thus it is enough to show that $\sup_{j\geq 0} \|\eta_j(f(R_j u)-f(R_j u_k))\|_{W^{s,p}} \to 0$ as $k\to\infty$.

Step 6: We need to show

$$\forall\, \epsilon>0\ \exists\, N \quad s.t. \quad \forall\, k\geq N \quad \sup_{j\geq 0} \|\eta_j(f(R_j u)-f(R_j u_k))\|_{W^{s,p}} < \epsilon.$$

Let $\epsilon > 0$ be given. Note that

$$\|\eta_j(f(R_j u)-f(R_j u_k))\|_{W^{s,p}} \leq \|\eta_j(f(R_j u)-f(0))\|_{W^{s,p}} + \|\eta_j(f(0)-f(R_j u_k))\|_{W^{s,p}}. \tag{A.1}$$

Let's start by considering the first term on RHS. By **Fact 1**, there exists $\alpha > 0$ such that if $\|g\|_{W^{s,p}} < \alpha$ then $\|\eta_j(f(g)-f(0))\|_{W^{s,p}} < \frac{\epsilon}{4}$. Note that for $j>1$, η_j does not depend on j and so α can be chosen independent of j. By **Step 2** we know that $R_j u \to 0$ in $W^{s,p}$ and so there exists a number $P\geq 2$ such that for $j \geq P$, $\|R_j u\|_{W^{s,p}} < \frac{\alpha}{2}$. It follows that

$$\forall\, j\geq P \quad \|\eta_j(f(R_j u)-f(0))\|_{W^{s,p}} < \frac{\epsilon}{4}$$

So

$$\sup_{j\geq P} \|\eta_j(f(R_j u)-f(0))\|_{W^{s,p}} \leq \frac{\epsilon}{4}. \tag{A.2}$$

Now we show that there exists N_1 such that if $k\geq N_1$ then it holds that $\sup_{j\geq P}\|\eta(f(0)-f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{4}$. (Note that since $P\geq 2$ we have $\eta_j = \eta$.)

- **Claim:** For all j, $R_j u_k \to R_j u$ in $W^{s,p}$ uniformly with respect to j as $k\to\infty$.

- **Proof of the claim:** By what was stated in **Step 2**, since we have that $\rho < 0$, $\|S_{2^i}(\varphi_i(u_k - u))\|_{W^{s,p}} \leq \|u_k - u\|_{W^{s,p}_\rho}$ for all i, and we have

$$\|R_j(u_k - u)\|_{W^{s,p}} \preceq \sum_{i=j-1}^{j+1} \|S_{2^i}(\varphi_i(u_k - u))\|_{W^{s,p}} \leq \sum_{i=j-1}^{j+1} \|u_k - u\|_{W^{s,p}_\rho}$$
$$= 3\|u_k - u\|_{W^{s,p}_\rho} \to 0 \quad \text{uniform in } j \text{ as} \quad k \to \infty$$

Therefore

$$\exists\, N_1 \ s.t. \ \forall j \quad \forall k \geq N_1 \quad \|R_j(u_k - u)\|_{W^{s,p}} < \frac{\alpha}{2}.$$

In particular, for all $j \geq P$ and $k \geq N_1$ we have

$$\|R_j u_k\|_{W^{s,p}} \leq \|R_j(u_k - u)\|_{W^{s,p}} + \|R_j u\|_{W^{s,p}} < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Consequently for all $j \geq P$ and $k \geq N_1$ we have

$$\|\eta(f(0) - f(R_j u_k))\|_{W^{s,p}} < \frac{\epsilon}{4},$$

which implies

$$\forall\ k \geq N_1 \quad \sup_{j \geq P} \|\eta(f(0) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{4}. \tag{A.3}$$

From (A.1), (A.2), and (A.3) we get

$$\forall\ k \geq N_1 \quad \sup_{j \geq P} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{2}.$$

Now note that by the claim that was proved above, we know that $R_j u_k \to R_j u$ in $W^{s,p}$. So by **Fact 1**, $\|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \to 0$ for any fixed j as $k \to \infty$. In particular for $0 \leq j \leq P - 1$,

$$\exists\, M_j \ s.t. \ \forall\, k \geq M_j \quad \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} < \frac{\epsilon}{2}.$$

So if we let $N = \max\{N_1, M_0, M_1, ..., M_{P-1}\}$, then for all $k \geq N$

$$\sup_{j \geq P} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{2}$$
$$\sup_{0 \leq j \leq P-1} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{2}.$$

That is

$$\forall\ k \geq N \quad \sup_{j \geq 0} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{2} < \epsilon,$$

which is exactly what we wanted to prove. □

Remark A.29. *Obviously the above result also holds true if f is only smooth on an open interval containing the range of u. By using partition of unity and charts one can show that the claim also holds for AF manifolds (of any class).*

Corollary A.30. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth and $f(0) = 0$. If $u \in W^{s,p}_\rho$ where $sp > n$, $\rho < 0$ then $f(u) \in W^{s,p}_\rho$ and the map taking u to $f(u)$ is continuous from $W^{s,p}_\rho$ to $W^{s,p}_\rho$.*

Proof. **(Corollary A.30)** $f(0) = 0$, so by Taylor's theorem we have $f(x) = xF(x)$ where F is smooth. Therefore by the previous lemma, $f(u) = uF(u) \in W^{s,p}_\rho$ and moreover the map taking u to $f(u) = uF(u)$ is continuous from $W^{s,p}_\rho$ to $W^{s,p}_\rho$. □

Lemma A.31. *Let the following assumptions hold:*

- $p \in (1,\infty)$, $s \in (\frac{n}{p},\infty)$, $\delta < 0$ *and* $u \in W^{s,p}_{\delta}$,
- $\nu \in \mathbb{R}$, $\sigma \in [-1,1]$, $\theta = \frac{1}{p} - \frac{s-1}{n}$, $\frac{1}{q} \in (\frac{1+\sigma}{2}\theta, 1 - \frac{1-\sigma}{2}\theta)$ *and* $v \in W^{\sigma,q}_{\nu}$,
- $f : [\inf u, \sup u] \to \mathbb{R}$ *is a smooth function. (Note that* $W^{s,p}_{\delta} \hookrightarrow C^0_{\delta} \hookrightarrow L^\infty$ *and therefore* $\inf u$ *and* $\sup u$ *are finite numbers.)*

Then:

$$\|vf(u)\|_{\sigma,q,\nu} \preceq \|v\|_{\sigma,q,\nu}(\|f(u)\|_\infty + \|f'(u)\|_\infty \|u\|_{s,p,\delta}).$$

Proof. **(Lemma A.31)** First we prove the claim for the case $\sigma = 1$. We have

$$\begin{aligned}
\|vf(u)\|_{1,q,\nu} &\preceq \|\langle x\rangle^{-\nu-\frac{n}{q}} vf(u)\|_{L^q} + \|\langle x\rangle^{-\nu-\frac{n}{q}+1}\nabla(vf(u))\|_{L^q}\\
&\preceq \|\langle x\rangle^{-\nu-\frac{n}{q}} vf(u)\|_{L^q} + \|\langle x\rangle^{-\nu-\frac{n}{q}+1}(\nabla v)f(u)\|_{L^q}\\
&\quad + \|\langle x\rangle^{-\nu-\frac{n}{q}+1} vf'(u)\nabla u\|_{L^q}\\
&\preceq \|\langle x\rangle^{-\nu-\frac{n}{q}} v\|_{L^q}\|f(u)\|_{L^\infty} + \|\langle x\rangle^{-\nu-\frac{n}{q}+1}\nabla v\|_{L^q}\|f(u)\|_{L^\infty}\\
&\quad + \|\langle x\rangle^{-(\nu-1)-\frac{n}{q}} v\nabla u\|_{L^q}\|f'(u)\|_{L^\infty}\\
&\text{(note that } f \text{ is smooth on } [\inf u, \sup u] \text{ so } f(u) \in L^\infty,\ f'(u) \in L^\infty)\\
&\preceq \|v\|_{1,q,\nu}\|f(u)\|_{L^\infty} + \|v\nabla u\|_{L^q_{\nu-1}}\|f'(u)\|_{L^\infty}\\
&\preceq \|v\|_{1,q,\nu}\|f(u)\|_{L^\infty} + \|v\|_{1,q,\nu}\|\nabla u\|_{s-1,p,\delta-1}\|f'(u)\|_{L^\infty}\\
&\qquad (\frac{1}{q} \geq \theta \text{ so } W^{1,q}_{\nu} \times W^{s-1,p}_{\delta-1} \hookrightarrow L^q_{\delta+\nu-1} \hookrightarrow L^q_{\nu-1})\\
&\preceq \|v\|_{1,q,\nu}\|f(u)\|_{L^\infty} + \|v\|_{1,q,\nu}\|u\|_{s,p,\delta}\|f'(u)\|_{L^\infty}\\
&= \|v\|_{1,q,\nu}(\|f(u)\|_{L^\infty} + \|f'(u)\|_{L^\infty}\|u\|_{s,p,\delta}).
\end{aligned}$$

Now we prove the case $\sigma = -1$ by a duality argument. Note that

$$\|vf(u)\|_{-1,q,\nu} = \sup_{\eta \in C^\infty_c} \frac{|\langle vf(u), \eta\rangle_{W^{-1,q}_{\nu} \times W^{1,q'}_{-n-\nu}}|}{\|\eta\|_{1,q',-n-\nu}}.$$

We have

$$\begin{aligned}
\frac{|\langle vf(u), \eta\rangle_{W^{-1,q}_{\nu} \times W^{1,q'}_{-n-\nu}}|}{\|\eta\|_{1,q',-n-\nu}} &= \frac{|\langle v, f(u)\eta\rangle_{W^{-1,q}_{\nu} \times W^{1,q'}_{-n-\nu}}|}{\|\eta\|_{1,q',-n-\nu}}\\
&\leq \frac{\|v\|_{-1,q,\nu}\|f(u)\eta\|_{1,q',-n-\nu}}{\|\eta\|_{1,q',-n-\nu}}
\end{aligned}$$

By assumption $\frac{1}{q} < 1-\theta$, so $\frac{1}{q'} > \theta$ and thus we can apply what was proved for the case $\sigma = 1$ to $\|f(u)\eta\|_{1,q',-n-\nu}$:

$$\begin{aligned}
\frac{\|v\|_{-1,q,\nu}\|f(u)\eta\|_{1,q',-n-\nu}}{\|\eta\|_{1,q',-n-\nu}} &\preceq \frac{\|v\|_{-1,q,\nu}[\|\eta\|_{1,q',-n-\nu}(\|f(u)\|_{L^\infty} + \|f'(u)\|_{L^\infty}\|u\|_{s,p,\delta})]}{\|\eta\|_{1,q',-n-\nu}}\\
&= \|v\|_{-1,q,\nu}(\|f(u)\|_{L^\infty} + \|f'(u)\|_{L^\infty}\|u\|_{s,p,\delta}).
\end{aligned}$$

Therefore

$$\|vf(u)\|_{-1,q,\nu} \preceq \|v\|_{-1,q,\nu}(\|f(u)\|_{L^\infty} + \|f'(u)\|_{L^\infty}\|u\|_{s,p,\delta}).$$

Now we prove the case where $\sigma \in (-1,1)$ by interpolation. According to what was proved we have

$$\|vf(u)\|_{1,q_1,\nu} \preceq \|v\|_{1,q_1,\nu}(\|f(u)\|_{L^\infty} + \|f'(u)\|_{L^\infty}\|u\|_{s,p,\delta}), \tag{A.4}$$

$$\|vf(u)\|_{-1,q_2,\nu} \preceq \|v\|_{-1,q_2,\nu}(\|f(u)\|_{L^\infty} + \|f'(u)\|_{L^\infty}\|u\|_{s,p,\delta}), \tag{A.5}$$

where q_1 and q_2 are any two numbers that satisfy $\theta < \frac{1}{q_1} < 1$ and $0 < \frac{1}{q_2} < 1-\theta$. Let $t = \frac{1-\sigma}{2}$. Clearly $t \in (0,1)$. Also note that if we set $\frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}$ then

$$\frac{1}{q_1} > \theta,\ \frac{1}{q_2} > 0 \Rightarrow \frac{1}{q} > (1-t)\theta = \frac{1+\sigma}{2}\theta.$$
$$\frac{1}{q_2} < 1-\delta,\ \frac{1}{q_1} < 1 \Rightarrow \frac{1}{q} < 1 - t\theta = 1 - \frac{1-\sigma}{2}\theta.$$

So by choosing appropriate q_1 and q_2 we can get any q with the property that $\frac{1}{q} \in (\frac{1+\sigma}{2}\theta, 1-\frac{1-\sigma}{2}\theta)$. This implies if $\frac{1}{q} \in (\frac{1+\sigma}{2}\theta, 1-\frac{1-\sigma}{2}\theta)$ then we may find q_1 and q_2 for which inequalities A.4, A.5 hold true and moreover

$$(W^{1,q_1}_{\nu}, W^{-1,q_2}_{\nu})_{t,q} = W^{\sigma,q}_{\nu} \quad \text{if } \sigma \neq 0 \quad \text{(real interpolation)}$$
$$[W^{1,q_1}_{\nu}, W^{-1,q_2}_{\nu}]_{t} = W^{\sigma,q}_{\nu} \quad \text{if } \sigma = 0 \quad \text{(complex interpolation)}.$$

So by interpolation we get

$$\|vf(u)\|_{\sigma,q,\nu} \preceq \|v\|_{\sigma,q,\nu}(\|f(u)\|_{\infty} + \|f'(u)\|_{\infty}\|u\|_{s,p,\delta}).$$

□

APPENDIX B. DIFFERENTIAL OPERATORS IN WEIGHTED SPACES

We now assemble some results we need for differential operators in Weighted spaces. Again, we limit our selves to simply stating the results we need, unless the proof of the result is either unavailable or difficult to find in the form we need, in which case we include a concise proof.

Let M be an n dimensional smooth AF manifold and let E be a smooth vector bundle over M with fiber dimension k. Consider the differential operator $A : \Gamma(E) \to \Gamma(E)$ of order m where $\Gamma(E)$ denotes the space of smooth sections of E. By definition, we know that in any local coordinates (trivializing E) A can be written as: $A = \sum_{|\nu|\leq m} a_\nu \partial^\nu$.

Definition B.1. *Let $\alpha \in \mathbb{R}$, $\gamma \in (1,\infty)$, and $\rho < 0$.*

- *We say A belongs to the class $D^{\alpha,\gamma}_m(E)$ if and only if $a_\nu \in W^{\alpha-m+|\nu|,\gamma}(\mathbb{R}^n, \mathbb{R}^{k\times k})$ for $|\nu| \leq m$.*
- *We say A belongs to the class $D^{\alpha,\gamma}_{m,\rho}(E)$ if and only if $a_\nu \in W^{\alpha-m+|\nu|,\gamma}_{\rho-m+|\nu|}(\mathbb{R}^n, \mathbb{R}^{k\times k})$ for $|\nu| < m$ and there are constants a^∞_ν such that $a^\infty_\nu - a_\nu \in W^{\alpha,\gamma}_{\rho}(\mathbb{R}^n, \mathbb{R}^{k\times k})$ for all $|\nu| = m$. We call $A_\infty = \sum_{|\nu|=m} a^\infty_\nu \partial_\nu$ the principal part of A at infinity.*
- *If $\alpha\gamma > n$, then the highest order coefficients of $A \in D^{\alpha,\gamma}_m(E)$ are continuous and so it makes sense to talk about their pointwise values. We say A is elliptic if for each x, the constant coefficient operator $\sum_{|\nu|=m} a_\nu(x)\partial^\nu$ is elliptic.*
- *If $\alpha\gamma > n$, then the highest order coefficients of $A \in D^{\alpha,\gamma}_{m,\rho}(E)$ are continuous and so it makes sense to talk about their pointwise values. We say A is elliptic if A_∞ is elliptic and moreover for each x, the constant coefficient operator $\sum_{|\nu|=m} a_\nu(x)\partial^\nu$ is elliptic.*

By using the multiplication lemma, we can prove the following theorem:

Theorem B.2. *If $\delta \in \mathbb{R}$, $\rho < 0$ and if $A \in D^{\alpha,\gamma}_{m,\rho}(E)$ then A can be viewed as a bounded linear map*

$$A : W^{s,q}_{\delta}(E) \to W^{\sigma,q}_{\delta-m}(E),$$

provided

$$(i)\ \gamma, q \in (1,\infty),$$
$$(ii)\ s \geq m-\alpha \quad (\textit{ let } \frac{1}{q}+\frac{1}{\gamma} \geq 1 \textit{ if } s=m-\alpha),$$
$$(iii)\ \sigma \leq \min(s,\alpha)-m \quad (\textit{let } \gamma \leq q \textit{ if } \alpha-m=\sigma \notin \mathbb{Z})$$
$$(iv)\ \sigma < s-m+\alpha-\frac{n}{\gamma},$$
$$(v)\ \sigma-\frac{n}{q} \leq \alpha-\frac{n}{\gamma}-m,$$
$$(vi)\ s-n/q > m-n-\alpha+n/\gamma.$$

If moreover $A_\infty = 0$*, then* A *is a continuous map*

$$A : W^{s,q}_{\delta}(E) \to W^{\sigma,q}_{\delta-m+\rho}(E)$$

Proof. **(Theorem B.2)** First let's consider the case where $A_\infty \neq 0$. The goal is to find sufficient conditions to make sure that $A=\sum_{|\nu|\leq m} a_\nu \partial^\nu$ is a continuous operator from $W^{s,q}_{\delta} \to W^{\sigma,q}_{\beta}$. Clearly this will be true provided

(1) For all $|\nu| < m$

$$W^{\alpha-m+|\nu|,\gamma}_{\rho-m+|\nu|} \times W^{s-|\nu|,q}_{\delta-|\nu|} \hookrightarrow W^{\sigma,q}_{\beta}, \quad (\text{note that } a_\nu \in W^{\alpha-m+|\nu|,\gamma}_{\rho-m+|\nu|},\ \partial^\nu u \in W^{s-|\nu|,q}_{\delta-|\nu|})$$

It follows from the multiplication lemma and previously mentioned embedding theorems that the above embedding holds true provided (the numbering of the items corresponds to the numbering of the assumptions in multiplication lemma)

$$(ii)\ s \geq m-\alpha, \quad (\frac{1}{q}+\frac{1}{\gamma} \geq 1 \text{ if } s=m-\alpha)$$
$$(i)\ \sigma \leq \alpha-m \quad (\gamma \leq q \text{ if } \alpha-m=\sigma \notin \mathbb{Z}),$$
$$(i),(iii)\ \sigma \leq s-(m-1),$$
$$(iv)\ \sigma < s-m+\alpha-\frac{n}{\gamma},$$
$$(iii)\ \sigma-\frac{n}{q} \leq \alpha-\frac{n}{\gamma}-m,$$
$$(v)\ s-\frac{n}{q} > m-n-\alpha+\frac{n}{\gamma},$$

and of course we need $(\rho-m+|\nu|)+(\delta-|\nu|)$ to be less than or equal to β, that is, $\rho-m+\delta \leq \beta$.

(2) For $|\nu| = m$

$$W^{\alpha,\gamma}_{\rho} \times W^{s-m,q}_{\delta-m} \hookrightarrow W^{\sigma,q}_{\beta},$$
$$W^{s-m,q}_{\delta-m} \hookrightarrow W^{\sigma,q}_{\beta}.$$

Note that, $a_\nu \partial^\nu = (a_\nu - a^\infty_\nu)\partial^\nu + a^\infty_\nu \partial^\nu$. a^∞_ν is constant and $(a_\nu - a^\infty_\nu) \in W^{\alpha,\gamma}_{\rho}$, so it should be clear why we need the above embeddings to be true. By using the multiplication lemma it turns out that the only extra assumption that we need for the first embedding to be true is that $\sigma \leq s-m$ and then the only extra assumption that we need for the second embedding to be true is that $\beta \geq \delta-m$.

To complete the proof we just need to note that if $A_\infty = 0$ then we do not need to have the embedding $W^{s-m,q}_{\delta-m} \hookrightarrow W^{\sigma,q}_{\beta}$ and so β can be any number larger than or equal to $\delta-m+\rho$. □

Remark B.3. *In the above proof we implicitly assumed that the following statement is true: If $A : \Gamma(E) \to \Gamma(E)$ is a partial differential operator whose representation in each local chart is continuous from $W^{s,q}_{\delta}$ to $W^{\sigma,q}_{\beta}$, then A is a continuous operator from $W^{s,q}_{\delta}(E)$ to $W^{\sigma,q}_{\beta}(E)$.*

Example: If the metric of an asymptotically flat manifold is of class $W^{\alpha,\gamma}_{\rho}$ with $\alpha\gamma > n$ and $\rho < 0$, then the Laplacian and conformal Laplacian are elliptic operators in class $D^{\alpha,\gamma}_{2,\rho}(M \times \mathbb{R})$; vector Laplacian is an elliptic operator in the class $D^{\alpha,\gamma}_{2,\rho}(TM)$.

Duality Pairing. Let $\bar{h}$ denote the Euclidean metric on $\mathbb{R}^n$. Let $\sigma, \delta \in \mathbb{R}$ and $q \in (1,\infty)$. We denote the duality pairing $W^{-\sigma,q'}_{-n-\delta}(\mathbb{R}^n) \times W^{\sigma,q}_{\delta}(\mathbb{R}^n) \to \mathbb{R}$ by $\langle \cdot,\cdot \rangle_{W^{-\sigma,q'}_{-n-\delta} \times W^{\sigma,q}_{\delta}}$ or just $\langle \cdot,\cdot \rangle_{(\mathbb{R}^n,\bar{h})}$ if the spaces are clear from the context. Clearly the duality pairing is a continuous bilinear map. The restriction of this map to $C^\infty_c(\mathbb{R}^n) \times C^\infty_c(\mathbb{R}^n)$ is the L^2 inner product:

$$\forall\ u, v \in C^\infty_c(\mathbb{R}^n) \quad \langle u, v \rangle_{(\mathbb{R}^n,\bar{h})} = \int_{\mathbb{R}^n} uv dx.$$

Now suppose (M^n, h) is an AF manifold of class $W^{\alpha,\gamma}_{\rho}$ where $\rho < 0$ and $\gamma \in (1,\infty)$. Our claim is that $(W^{\sigma,q}_{\delta}(M))^*$ can be identified with $W^{-\sigma,q'}_{-n-\delta}(M)$. This identification can be done in at least two ways which we describe below:

- **First Method:** By using the corresponding AF atlas and the subordinate partition of unity that was used in the Definition A.7 one can construct a smooth metric $\hat{h}$ such that $(M, \hat{h})$ is of class $W^{\alpha,\gamma}_{\rho}$. Recall that our definition of Sobolev spaces on M is independent of the underlying metric. The bilinear map $\langle \cdot,\cdot \rangle_{(M,\hat{h})} : C^\infty_c(M) \times C^\infty_c(M) \to \mathbb{R}$ which is defined by

$$\langle u, v \rangle_{(M,\hat{h})} = \int_M uv dV_{\hat{h}}$$

can be uniquely extended to a continuous bilinear form

$$\langle \cdot,\cdot \rangle_{(M,\hat{h})} : W^{-\sigma,q'}_{-n-\delta}(M) \times W^{\sigma,q}_{\delta}(M) \to \mathbb{R}.$$

The above bilinear map induces a topological isomorphism $(W^{\sigma,q}_{\delta}(M))^* = W^{-\sigma,q'}_{-n-\delta}(M)$; also if v has compact support in U_j (domain of a coordinate chart in the AF atlas), then

$$\langle u, v \rangle_{(M,\hat{h})} = \langle u \circ \phi_j^{-1}, \sqrt{\det \hat{h}}\, v \circ \phi_j^{-1} \rangle_{(\mathbb{R}^n,\bar{h})}.$$

Note that in the above, $u \circ \phi_j^{-1}$ represents any extension of $u \circ \phi_j^{-1}$ from $W^{-\sigma,q'}_{-n-\delta}(\phi_j(U_j))$ to $W^{-\sigma,q'}_{-n-\delta}(\mathbb{R}^n)$. Also $v \circ \phi_j^{-1}$ represents the extension of $v \circ \phi_j^{-1} \in W^{\sigma,q}_{\delta}(\phi_j(U_j))$ by zero. Since v has compact support, we know that $\sqrt{\det \hat{h}}\, v \circ \phi_j^{-1} \in W^{\sigma,q}_{\delta}(\mathbb{R}^n)$.

Similarly there exists a continuous bilinear form $\langle \cdot,\cdot \rangle_{(M,\hat{h})} : W^{-\sigma,q'}_{-n-\delta}(TM) \times W^{\sigma,q}_{\delta}(TM) \to \mathbb{R}$ whose restriction to $C^\infty_c(TM) \times C^\infty_c(TM)$ is

$$(Y, X) \mapsto \int_M \hat{h}(Y, X) dV_{\hat{h}}.$$

This map induces an isomorphism $(W^{\sigma,q}_{\delta}(TM))^* = W^{-\sigma,q'}_{-n-\delta}(TM)$; If $X \in W^{\sigma,q}_{\delta}(TM)$, $Y \in W^{-\sigma,q'}_{-n-\delta}(TM)$ and X has compact support in U_j then

$$\langle Y, X \rangle_{(M,\hat{h})} = \sum_{l,p} \langle Y_l \circ \phi_j^{-1}, \sqrt{\det \hat{h}}\ \hat{h}^{lp} X_p \circ \phi_j^{-1} \rangle_{(\mathbb{R}^n,\bar{h})}.$$

The disadvantage of this method is that the restriction of the bilinear form that was constructed above to C_c^∞ is $\int_M uvdV_{\hat{h}}$ instead of $\int_M uvdV_h$. We prefer to construct the isomorphism using the rough metric instead of $\hat{h}$. It turns out that this can be done for a limited range of σ and q.

- **Second Method:** Suppose $\alpha\gamma > n$. Then there exists a continuous function f such that $dV_h = fdV_{\hat{h}}$ and $f - \varsigma \in W_\rho^{\alpha,\gamma}$ for some constant $\varsigma > 0$ [38, 8]. Formally we can write
$$\langle u, v\rangle_{(M,h)} = \int_M uvdV_h = \int_M uvfdV_{\hat{h}} = \int_M ufvdV_{\hat{h}} = \langle u, fv\rangle_{(M,\hat{h})}.$$
This motivates the following definition:
$$\forall u \in W_{-n-\delta}^{-\sigma,q'}\ \forall v \in W_\delta^{\sigma,q} \quad \langle u, v\rangle_{(M,h)} := \langle u, fv\rangle_{(M,\hat{h})}.$$
Of course for the above definition to make sense we need to make sure that $fv \in W_\delta^{\sigma,q}$. Note that $f - \varsigma \in W_\rho^{\alpha,\gamma}$ and so by Lemma A.28 this holds provided
$$\sigma \in [-\alpha, \alpha] \ \ (\sigma \neq \pm\alpha \text{ if } \alpha \notin \mathbb{N}_0 \text{ and } q < \gamma), \quad \sigma - \frac{n}{q} \in [-n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}].$$
It is easy to see that (since $\alpha\gamma > n$) if $\sigma \in [0, \alpha]$ and $q = \gamma$ then the above conditions hold true. Clearly the restriction of $\langle\cdot,\cdot\rangle_{(M,h)}$ to $C_c^\infty \times C_c^\infty$ is given by $\langle u, v\rangle_{(M,h)} = \int_M uvdV_h$. This shows that this bilinear form does not depend on the choice of $\hat{h}$. The above pairing make sense even if $u \in W_{loc}^{-\sigma,q'}$ and $v \in W_{loc}^{\sigma,q}$ provided at least one of u or v has compact support.
  Similarly for vector fields X and Y formally we may write
$$\begin{aligned}\langle Y, X\rangle_{(M,h)} &= \int_M h(Y,X)dV_h = \int_M h_{bc}X^cY^bfdV_{\hat{h}}\\ &= \int_M \hat{h}_{ad}(f\hat{h}^{ab}h_{bc}X^c)Y^ddV_{\hat{h}} \qquad (Y^b = \delta_d^bY^d = \hat{h}_{ad}\hat{h}^{ab}Y^d)\\ &= \int_M \hat{h}(Y, X_*)dV_{\hat{h}} = \langle Y, X_*\rangle_{(M,\hat{h})} \quad (X_*^a := f\hat{h}^{ab}h_{bc}X^c).\end{aligned}$$
This motivates the following definition:
$$\forall\ Y \in \mathbf{W}_{-n-\delta}^{-\sigma,q'}\ \forall X \in \mathbf{W}_\delta^{\sigma,q} \quad \langle Y, X\rangle_{(M,h)} := \langle Y, X_*\rangle_{(M,\hat{h})}, \quad (\mathbf{W}_\delta^{\sigma,q} := W_\delta^{\sigma,q}(TM))$$
where $X_*^a := f\hat{h}^{ab}h_{bc}X^c$. Again one can check that the above definition makes sense provided $\sigma \in [-\alpha, \alpha]$ ($\sigma \neq \pm\alpha$ if $\alpha \notin \mathbb{N}_0$ and $q < \gamma$), $\sigma - \frac{n}{q} \in [-n-\alpha+\frac{n}{\gamma}, \alpha-\frac{n}{\gamma}]$. As an example, if $n = 3$ and $\alpha > 1$ (and of course $\alpha > \frac{3}{\gamma}$), then the duality pairing of $W_{-3-\delta}^{-1,2}$ and $W_\delta^{1,2}$ is well defined:
$$1 \in (-\alpha, \alpha), \quad 1 - \frac{3}{2} \in [-3 - \alpha + \frac{3}{\gamma}, \alpha - \frac{3}{\gamma}] \quad (\text{because } \alpha > \frac{3}{\gamma}).$$

Remark B.4. ***Order on*** $W_\delta^{-\sigma,\gamma}(M)$ ***for*** $\sigma \in (-\infty, \alpha]$

As before suppose (M^n, h) is an AF manifold of class $W_\rho^{\alpha,\gamma}$ where $\rho < 0$, $\gamma \in (1,\infty)$, and $\alpha\gamma > n$.

- *If $\sigma \leq 0$, then $W_\delta^{-\sigma,q} \hookrightarrow L_\delta^q$ and so the elements of $W_\delta^{-\sigma,q}$ are ordinary functions (or more precisely, equivalence classes of ordinary functions). In this case we define an order on $W_\delta^{-\sigma,q}$ as follows: the functions $u, v \in W_\delta^{-\sigma,q}$ satisfy $u \geq v$ if and only if $u(x) - v(x) \geq 0$ for almost all $x \in M$.*

- *If $\sigma \in (0,\alpha]$, then it is easy to check that the duality pairing $\langle \cdot,\cdot \rangle_{(M,\hat{h})} : W_\delta^{-\sigma,\gamma}(M) \times W_{-n-\delta}^{\sigma,\gamma'}(M) \to \mathbb{R}$ is well-defined. We define an order on $W_\delta^{-\sigma,\gamma}$ as follows: the functions $u,v \in W_\delta^{-\sigma,\gamma}$ satisfy $u \geq v$ if and only if $\langle u-v,\xi\rangle_{(M,h)} \geq 0$ for all $\xi \in C_c^\infty(M)$ with $\xi \geq 0$. Notice that if u and v are ordinary functions in $W_\delta^{-\sigma,\gamma}(M)$, then if follows from the definition that $u \geq v$ if and only if $u(x) \geq v(x)$ a.e..*

  According to the above items, if $\alpha \geq 1$ we have a well-defined order on $W_\delta^{\alpha-2,\gamma}(M)$ and in particular if u is an ordinary function in $W_\delta^{\alpha-2,\gamma}(M)$, then $u \geq 0$ if and only if $u(x) \geq 0$ for almost all x.

In what follows, we state and prove Lemma B.5, Lemma B.6, Proposition B.7, and Lemma B.9 for $\mathbb{R}^n$. The proofs can be easily extended to AF manifolds.

Lemma B.5. *Let the following assumptions hold:*

- *$A \in D_m^{\alpha,\gamma}$ where $\gamma \in (1,\infty)$ and $\alpha - \frac{n}{\gamma} > \max\{0, \frac{m-n}{2}\}$; A is elliptic.*
- *$q \in (1,\infty)$, $s \in (m-\alpha,\alpha]$ (if $s = \alpha \notin \mathbb{N}_0$, then let $q \in [\gamma,\infty)$).*
- *$s - \frac{n}{q} \in (m-n-\alpha+\frac{n}{\gamma}, \alpha - \frac{n}{\gamma}]$.*

Then: If U and V are bounded open sets with $U \subset\subset V$, then there exists $\tilde{s} < s$ such that for all $u \in W^{s,q}$

$$\|u\|_{s,q,U} \preceq \|Au\|_{s-m,q,V} + \|u\|_{\tilde{s},q,V}. \tag{B.1}$$

Proof. **(Lemma B.5)** The proof of the interior regularity lemma in [32] (Lemma A.25), with obvious changes, goes through for the above claim as well. The approach of the proof is similar to our proof for Proposition B.7. Since the claim is about unweighted Sobolev spaces we do not repeat that argument here. □

Lemma B.6. *Suppose A is a constant coefficient elliptic operator that has only derivatives of order m with $m < n$ on $\mathbb{R}^n$ (clearly $A \in D_{m,\rho}^{\alpha,\gamma}$ for all possible α,γ, and $\rho < 0$ because one can take $A_\infty = A$ and so $A_\infty - A = 0$; of course for our definition of ellipticity to work, $\alpha\gamma$ must be larger than n). Then for $s \in \mathbb{R}$, $p \in (1,\infty)$, and $\delta \in (m-n,0)$, $A : W_\delta^{s,p} \to W_{\delta-m}^{s-m,p}$ is an isomorphism.*

Proof. **(Lemma B.6)** We closely follow and extend the proof that is given for the special $p = 2$ in [38] [Lemma 4.8]. Let $A_{s,p,\delta}$ denote the operator A acting on $W_\delta^{s,p}$. We consider three cases $s \geq m$, $s \in (-\infty,0]$, and $s \in (0,m)$.

- **Case 1: $s \geq m$.**
  For $s \in \mathbb{N}$ and $s \geq m$, the claim follows from the argument in [37]. If $s \notin \mathbb{N}$, let $k = [s]$, $\theta = s-k$. We know that $A_{k,p,\delta}$ and $A_{k+1,p,\delta}$ have inverses and in fact
  $$A_{k,p,\delta}^{-1} : W_{\delta-m}^{k-m,p} \to W_\delta^{k,p},$$
  $$A_{k+1,p,\delta}^{-1} : W_{\delta-m}^{k+1-m,p} \to W_\delta^{k+1,p},$$
  are continuous maps. Note that
  $$W_\delta^{k+1,p} \hookrightarrow W_\delta^{k,p}, \quad W_{\delta-m}^{k+1-m,p} \hookrightarrow W_{\delta-m}^{k-m,p},$$
  $$(W_\delta^{k,p}, W_\delta^{k+1,p})_{\theta,p} = W_\delta^{s,p}, \quad (W_{\delta-m}^{k-m,p}, W_{\delta-m}^{k+1-m,p})_{\theta,p} = W_{\delta-m}^{s-m,p}.$$
  So by interpolation we get a continuous operator $T : W_{\delta-m}^{s-m,p} \to W_\delta^{s,p}$ which must be the restriction of $A_{k,p,\delta}^{-1}$ to $W_{\delta-m}^{s-m,p}$. Now for all $u \in W_\delta^{s,p}$ we have
  $$\begin{aligned} u \in W_\delta^{s,p} \hookrightarrow W_\delta^{k,p} \Rightarrow A_{s,p,\delta}u = A_{k,p,\delta}u \Rightarrow T(A_{s,p,\delta}u) &= T(A_{k,p,\delta}u) \\ &= A_{k,p,\delta}^{-1}(A_{k,p,\delta}u) = u. \end{aligned}$$

Similarly $A_{s,p,\delta}Tu = u$. It follows that $T = A^{-1}_{s,p,\delta}$.

- **Case 2:** $s \leq 0$**.**
We want to show that $A_{s,p,\delta} : W^{s,p}_{\delta} \to W^{s-m,p}_{\delta-m}$ is an isomorphism. We note that since $A_{s,p,\delta}$ is a homogeneous constant coefficient elliptic operator, its adjoint $(A_{s,p,\delta})^* : W^{-s+m,p'}_{-\delta-n+m} \to W^{-s,p'}_{-\delta-n}$ is also a homogeneous constant coefficient elliptic operator. So by what was proved in the previous case we know that if $-s+m \geq m$ and $-\delta-n+m \in (m-n,0)$ (which are true because by assumption $s \leq 0$ and $\delta \in (m-n,0)$) then $(A_{s,p,\delta})^*$ is an isomorphism. Now for $u \in W^{s-m,p}_{\delta-m}$ define the distribution Tu by
$$\langle Tu, \varphi\rangle = \langle u, ((A_{s,p,\delta})^*)^{-1}\varphi\rangle_{W^{s-m,p}_{\delta-m}\times(W^{s-m,p}_{\delta-m})^*}$$
$$\text{(note that } ((A_{s,p,\delta})^*)^{-1} : W^{-s,p'}_{-\delta-n} \to (W^{s-m,p}_{\delta-m})^*\text{)},$$
for all $\varphi \in C^\infty_c$. We claim that T is the inverse of $A_{s,p,\delta}$. To this end first we show that T sends $W^{s-m,p}_{\delta-m}$ to $W^{s,p}_{\delta}$ and then we show that the composition of T and $A_{s,p,\delta}$ is the identity map.
Suppose $u \in W^{s-m,p}_{\delta-m}$. In order to prove that $Tu \in W^{s,p}_{\delta}$ it is enough to show that
$$\|Tu\|_{s,p,\delta} = \sup_{\varphi\in C^\infty_c} \frac{|\langle Tu,\varphi\rangle|}{\|\varphi\|_{-s,p',-\delta-n}} < \infty \quad \text{(we are interpreting } W^{s,p}_{\delta} \text{ as } (W^{-s,p'}_{-\delta-n})^*\text{)}$$
We have
$$|\langle Tu,\varphi\rangle| \leq \|u\|_{s-m,p,\delta-m}\|((A_{s,p,\delta})^*)^{-1}\varphi\|_{(W^{s-m,p}_{\delta-m})^*}$$
$$\leq \|u\|_{s-m,p,\delta-m}\|((A_{s,p,\delta})^*)^{-1}\|_{op}\|\varphi\|_{-s,p',-\delta-n}$$
Therefore
$$\|Tu\|_{s,p,\delta} \leq \|u\|_{s-m,p,\delta-m}\|((A_{s,p,\delta})^*)^{-1}\|_{op} < \infty.$$
This implies that T sends $W^{s-m,p}_{\delta-m}$ to $W^{s,p}_{\delta}$. Now note that for all $u \in W^{s,p}_{\delta}$, $\varphi \in C^\infty_c$
$$\langle TA_{s,p,\delta}u,\varphi\rangle = \langle A_{s,p,\delta}u, ((A_{s,p,\delta})^*)^{-1}\varphi\rangle = \langle u, (A_{s,p,\delta})^*((A_{s,p,\delta})^*)^{-1}\varphi\rangle = \langle u,\varphi\rangle.$$
This means $TA_{s,p,\delta}u = u$. Similarly $A_{s,p,\delta}Tu = u$.

- **Case 3:** $s \in (0,m)$**.**
By what was proved in the previous cases we know that $A_{0,p,\delta}$ and $A_{m,p,\delta}$ are invertible. In fact
$$A^{-1}_{0,p,\delta} : W^{-m,p}_{\delta-m} \to W^{0,p}_{\delta},$$
$$A^{-1}_{m,p,\delta} : W^{0,p}_{\delta-m} \to W^{m,p}_{\delta},$$
are continuous maps. Let $\theta = \frac{s}{m}$. Note that
$$W^{m,p}_{\delta} \hookrightarrow W^{0,p}_{\delta}, \quad W^{0,p}_{\delta-m} \hookrightarrow W^{-m,p}_{\delta-m},$$
$$(W^{0,p}_{\delta}, W^{m,p}_{\delta})_{\theta,p} = W^{s,p}_{\delta}, \quad (W^{-m,p}_{\delta-m}, W^{0,p}_{\delta-m})_{\theta,p} = W^{s-m,p}_{\delta-m} \quad \text{if } s\notin\mathbb{N}, \quad \text{(real interpolation)}$$
$$[W^{0,p}_{\delta}, W^{m,p}_{\delta}]_{\theta} = W^{s,p}_{\delta}, \quad [W^{-m,p}_{\delta-m}, W^{0,p}_{\delta-m}]_{\theta} = W^{s-m,p}_{\delta-m} \quad \text{if } s\in\mathbb{N}. \quad \text{(complex interpolation)}$$
So by interpolation we get a continuous operator $T : W^{s-m,p}_{\delta-m} \to W^{s,p}_{\delta}$ which must be the restriction of $A^{-1}_{0,p,\delta}$ to $W^{s-m,p}_{\delta-m}$. Now for all $u \in W^{s,p}_{\delta}$ we have
$$u \in W^{s,p}_{\delta} \hookrightarrow W^{0,p}_{\delta} \Rightarrow A_{s,p,\delta}u = A_{0,p,\delta}u \Rightarrow T(A_{s,p,\delta}u) = T(A_{0,p,\delta}u) = A^{-1}_{0,p,\delta}(A_{0,p,\delta}u) = u.$$
Similarly $A_{s,p,\delta}Tu = u$. It follows that $T = A^{-1}_{s,p,\delta}$.

□

Proposition B.7. *Let the following assumptions hold:*

- $A \in D^{\alpha,\gamma}_{m,\rho}$ *where* $\gamma \in (1,\infty)$, $\alpha > \frac{n}{\gamma}$, $\rho < 0$, *and* $m < n$; *A is elliptic.*
- $q \in (1,\infty)$, $s \in (m-\alpha, \alpha]$ *(if* $s = \alpha \notin \mathbb{N}_0$, *then let* $q \in [\gamma, \infty)$*).*
- $s - \frac{n}{q} \in (m - n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}]$.
- $\delta \in (m-n, 0)$.

In particular note that if the elliptic operator $A \in D^{\alpha,\gamma}_{m,\rho}$, *is given, then* $s := \alpha$ *and* $q := \gamma$ *satisfy the desired conditions.*

Then: If $t < s$ *and* $\delta' > \delta$, *then for every* $u \in W^{s,q}_{\delta}$ *we have*

$$\|u\|_{s,q,\delta} \preceq \|Au\|_{s-m,q,\delta-m} + \|u\|_{t,q,\delta'}$$

Moreover $A : W^{s,q}_{\delta} \to W^{s-m,q}_{\delta-m}$ *is semi-Fredholm.*

Proof. **(Proposition B.7)** The approach of the proof is standard. Here we will closely follow the proof that is given for the case $q = 2$ in [38] [Lemma 4.9]. In the proof we use the following facts (for these facts $s, \delta \in \mathbb{R}$ and $p \in (1,\infty)$):

- **Fact 1:**(see Lemma A.18) If $f \in C^{\infty}_{c}(\mathbb{R}^n)$ and $u \in W^{s,p}(\mathbb{R}^n)$, then $fu \in W^{s,p}(\mathbb{R}^n)$ and moreover $\|fu\|_{s,p} \preceq \|u\|_{s,p}$ (the implicit constant may depend on f but is independent of u).
- **Fact 2:**(see Lemma A.19) Let $\chi \in C^{\infty}_{c}(\mathbb{R}^n)$ be a cutoff function equal to 1 on B_1 and equal to 0 outside of B_2. Let $\tilde{\chi}(x) = 1 - \chi(x)$ and for all $\epsilon > 0$ define $\chi_{\epsilon}(x) = \chi(\frac{x}{\epsilon})$, $\tilde{\chi}_{\epsilon}(x) = \tilde{\chi}(\frac{x}{\epsilon})$. Then we have $\|\chi_{\epsilon} u\|_{s,p,\delta} \preceq \|u\|_{s,p,\delta}$ and $\|\tilde{\chi}_{\epsilon} u\|_{s,p,\delta} \preceq \|u\|_{s,p,\delta}$.
- **Fact 3:** Let $u \in W^{s,p}_{\delta}(\mathbb{R}^n)$. Also let Ω be an open bounded subset of $\mathbb{R}^n$. Then
  - $u \in W^{s,p}(\Omega)$ and $\|u\|_{W^{s,p}(\Omega)} \preceq \|u\|_{W^{s,p}_{\delta}(\mathbb{R}^n)}$.
  - If $\operatorname{supp} u \subseteq \Omega$, then $u \in W^{s,p}(\mathbb{R}^n)$ and $\|u\|_{W^{s,p}(\Omega)} \simeq \|u\|_{W^{s,p}_{\delta}(\mathbb{R}^n)}$.

  If $s \in \mathbb{N}_0$, the above items follow from the fact that weights are of the form $\langle x \rangle^a$ and so they attain their maximum and minimum on any compact subset of $\mathbb{R}^n$. If s is not an integer, they can be proved by interpolation.
- **Fact 4:** Suppose $f \in W^{s,p}_{\delta}$ with $\delta > 0$ and f vanishes in a neighborhood of the origin. Then

$$\lim_{i \to \infty} \|S_{2^{-i}} f\|_{s,p,\delta} = 0.$$

  The reason is as follows: by assumption there exists $l \in \mathbb{N}$ such that $f = 0$ on $B_{2^{-l}}$. So if $\hat{l} \in \mathbb{Z}$ and $\hat{l} < -l - 1$ then $S_{2^{\hat{l}}} f = 0$ on B_2. Indeed,

$$x \in B_2 \;\Rightarrow\; |2^{\hat{l}} x| < 2^{\hat{l}+1} < 2^{-l} \;\Rightarrow\; f(2^{\hat{l}} x) = 0 \;\Rightarrow\; S_{2^{\hat{l}}} f(x) = 0.$$

So for $i > l+2$ we can write

$$\begin{aligned}
\|S_{2^{-i}}f\|^p_{s,p,\delta} &= \sum_{j=0}^{\infty} 2^{-p\delta j}\|S_{2^j}(\phi_j S_{2^{-i}}f)\|^p_{W^{s,p}(\mathbb{R}^n)} \\
&= \sum_{j=0}^{\infty} 2^{-p\delta j}\|\phi S_{2^{j-i}}f\|^p_{W^{s,p}(B_2)} \qquad (S_{2^j}\phi_j = \phi,\ \operatorname{supp}\phi \subseteq B_2) \\
&= \sum_{j=i-l-1}^{\infty} 2^{-p\delta j}\|\phi S_{2^{j-i}}f\|^p_{W^{s,p}(B_2)} \quad (S_{2^{j-i}}f = 0 \text{ on } B_2 \text{ if } j-i < -l-1) \\
&= \sum_{\hat{j}=1}^{\infty} 2^{-p\delta(\hat{j}+i-l-2)}\|\phi S_{2^{\hat{j}-l-2}}f\|^p_{W^{s,p}(B_2)} \quad (\hat{j} := j-(i-l)+2) \\
&= 2^{-p\delta(i-l-2)}\sum_{\hat{j}=1}^{\infty} 2^{-p\delta\hat{j}}\|S_{2^{\hat{j}}}(\phi_{\hat{j}} S_{2^{-l-2}}f)\|^p_{W^{s,p}(\mathbb{R}^n)} \\
&= 2^{-p\delta(i-l-2)}\|S_{2^{-l-2}}f\|^p_{s,p,\delta} \preceq 2^{-p\delta(i-l-2)}\|f\|^p_{s,p,\delta}.
\end{aligned}$$

It follows that $\lim_{i\to\infty}\|S_{2^{-i}}f\|_{s,p,\delta} = 0$.

- **Fact 5:** [**Equivalence Lemma**][53] Let E_1 be a Banach space, E_2, E_3 normed spaces, and let $A \in L(E_1, E_2)$, $B \in L(E_1, E_3)$ such that one has:
  - $\|u\|_1 \lesssim \|Au\|_2 + \|Bu\|_3$.
  - B is compact.

  Then $ker A$ is finite dimensional and the range of A is closed, i.e., A is semi-Fredholm.

Now let's start the proof. Let $A = A_\infty + R$ where A_∞ is the principal part of A at infinity. Let r be a fixed dyadic integer to be selected later and let $u_r = \tilde{\chi}_r u$. By Lemma B.6 we have

$$\|u_r\|_{s,q,\delta} \preceq \|A_\infty u_r\|_{s-m,q,\delta-m} \le \|Au_r\|_{s-m,q,\delta-m} + \|Ru_r\|_{s-m,q,\delta-m}.$$

The implicit constant in the above inequality does not depend on r. Now $R \in D^{\alpha,\gamma}_{m,\rho}$ has vanishing principal part at infinity. Therefore by Theorem B.2 we can consider R as a continuous operator from $W^{s,q}_{\delta-\rho}$ to $W^{s-m,q}_{(\delta-\rho-m)+\rho} = W^{s-m,q}_{\delta-m}$. Also since $\rho < 0$ we have $u_r \in W^{s,q}_\delta \hookrightarrow W^{s,q}_{\delta-\rho}$. Consequently

$$\|Ru_r\|_{s-m,q,\delta-m} \preceq \|R\|_{op}\|u_r\|_{s,q,\delta-\rho} = \|R\|_{op}\|\tilde{\chi}_{\frac{r}{2}}u_r\|_{s,q,\delta-\rho} \quad (\text{note that } \tilde{\chi}_{\frac{r}{2}}u_r = u_r)$$

Now it is easy to check that $W^{\alpha,q}_{-\rho} \times W^{s,q}_\delta \hookrightarrow W^{s,q}_{\delta-\rho}$. Therefore

$$\|Ru_r\|_{s-m,q,\delta-m} \preceq \|R\|_{op}\|\tilde{\chi}_{\frac{r}{2}}\|_{\alpha,q,-\rho}\|u_r\|_{s,q,\delta}.$$

By **Fact 4**, $\lim_{i\to\infty}\|\tilde{\chi}_{2^i}\|_{\alpha,q,-\rho} \to 0$. Thus we can choose the fixed dyadic number r large enough so that

$$\|R\|_{op}\|\tilde{\chi}_{\frac{r}{2}}\|_{\alpha,q,-\rho} < \frac{1}{2},$$

and so we get

$$\|u_r\|_{s,q,\delta} \preceq \|Au_r\|_{s-m,q,\delta-m} + \frac{1}{2}\|u_r\|_{s,q,\delta}.$$

Hence

$$\|u_r\|_{s,q,\delta} \preceq \|Au_r\|_{s-m,q,\delta-m} \le \|\tilde{\chi}_r Au\|_{s-m,q,\delta-m} + \|[A,\tilde{\chi}_r]u\|_{s-m,q,\delta-m}.$$

By **Fact 2**, $\|\tilde{\chi}_r Au\|_{s-m,q,\delta-m} \preceq \|Au\|_{s-m,q,\delta-m}$. Also one can easily show that $[A,\tilde{\chi}_r]u$ has support in B_{2r} and so by **Fact 3**, $\|[A,\tilde{\chi}_r]u\|_{s-m,q,\delta-m} \simeq \|[A,\tilde{\chi}_r]u\|_{W^{s-m,q}(B_{2r})}$. On the bounded domain B_{2r}, $[A,\tilde{\chi}_r] \in D^{\alpha,\gamma}_{m-1}$, so $[A,\tilde{\chi}_r] : W^{s-1,q}(B_{2r}) \to W^{s-m,q}(B_{2r})$ is continuous. Consequently

$$\|[A,\tilde{\chi}_r]u\|_{W^{s-m,q}(B_{2r})} \preceq \|u\|_{W^{s-1,q}(B_{2r})} \preceq \|u\|_{W^{s,q}(B_{2r})}.$$

Thus

$$\|u_r\|_{s,q,\delta} \preceq \|Au\|_{s-m,q,\delta-m} + \|u\|_{W^{s,q}(B_{2r})}.$$

Now we can write

$$\begin{aligned}
\|u\|_{s,q,\delta} &= \|u_r + \chi_r u\|_{s,q,\delta} \leq \|u_r\|_{s,q,\delta} + \|\chi_r u\|_{s,q,\delta} \\
&\preceq \|u_r\|_{s,q,\delta} + \|\chi_r u\|_{W^{s,q}(B_{2r})} \quad (\chi_r u \text{ has support in } B_{2r},\ \textbf{Fact 3}) \\
&\preceq \|u_r\|_{s,q,\delta} + \|u\|_{W^{s,q}(B_{2r})} \quad (\textbf{Fact 1}) \\
&\preceq \|Au\|_{s-m,q,\delta-m} + \|u\|_{W^{s,q}(B_{2r})}.
\end{aligned}$$

From interior regularity estimate for elliptic operators on unweighted Sobolev spaces [Lemma B.5] we know there exists $\tilde{s} < s$ such that

$$\|u\|_{W^{s,q}(B_{2r})} \preceq \|Au\|_{W^{s-m,q}(B_{3r})} + \|u\|_{W^{\tilde{s},q}(B_{3r})}, \tag{B.2}$$

and by **Fact 3**

$$\|Au\|_{W^{s-m,q}(B_{3r})} \preceq \|Au\|_{s-m,q,\delta-m}.$$

It follows that

$$\|u\|_{s,q,\delta} \preceq \|Au\|_{s-m,q,\delta-m} + \|u\|_{W^{\tilde{s},q}(B_{3r})}.$$

But by **Fact 3**, for any $\delta' \in \mathbb{R}$ we have $\|u\|_{W^{\tilde{s},q}(B_{3r})} \preceq \|u\|_{\tilde{s},q,\delta'}$. This implies

$$\|u\|_{s,q,\delta} \preceq \|Au\|_{s-m,q,\delta-m} + \|u\|_{\tilde{s},q,\delta'}. \tag{B.3}$$

Now, if $t < s$ then either $t \geq \tilde{s}$ or $t < \tilde{s}$. If $t \geq \tilde{s}$ then $W^{t,q}_{\delta'} \hookrightarrow W^{\tilde{s},q}_{\delta'}$ and so $\|u\|_{\tilde{s},q,\delta'} \preceq \|u\|_{t,q,\delta'}$. If $t < \tilde{s}$, then for $\delta' > \delta$ we have $W^{s,q}_{\delta} \hookrightarrow W^{\tilde{s},q}_{\delta'} \hookrightarrow W^{t,q}_{\delta'}$ where the first embedding is compact and the second is continuous. Therefore, by Ehrling's lemma, for all $\epsilon > 0$ there exists $C(\epsilon)$ such that

$$\|u\|_{\tilde{s},q,\delta'} \leq \epsilon\|u\|_{s,q,\delta} + C(\epsilon)\|u\|_{t,q,\delta'}$$

In particular the above inequality holds for $\epsilon = \frac{1}{2}$. Combining this with (B.3) we can conclude that for all $t < s$ and $\delta' > \delta$

$$\|u\|_{s,q,\delta} \preceq \|Au\|_{s-m,q,\delta-m} + \|u\|_{t,q,\delta'}.$$

It remains to show that $A : W^{s,q}_{\delta} \to W^{s-m,q}_{\delta-m}$ is semi-Fredholm. Pick any δ' strictly larger than δ. By assumption $s > m - \alpha$, so we have $\|u\|_{s,q,\delta} \preceq \|Au\|_{s-m,q,\delta-m} + \|u\|_{m-\alpha,q,\delta'}$. Also $W^{s,q}_{\delta} \hookrightarrow W^{m-\alpha,q}_{\delta'}$ is compact. Hence by the estimate that was proved above and **Fact 5**, $A : W^{s,q}_{\delta} \to W^{s-m,q}_{\delta-m}$ is semi-Fredholm. □

Remark B.8. *The proof of Proposition B.7 in fact shows that if $u \in W^{t,q}_{\delta'}$ for some $t < s$ and $\delta' > \delta$ and if $Au \in W^{s-m,q}_{\delta-m}$ then $u \in W^{s,q}_{\delta}$.*

Lemma B.9. *Let the following assumptions hold:*

- $A \in D^{\alpha,\gamma}_{m,\rho}$, $\gamma \in (1,\infty)$, $\alpha > \frac{n}{\gamma}$, $\rho < 0$, *and* $m < n$. A *is elliptic.*
- $e \in (m-\alpha, \alpha]$ *(if $e = \alpha \notin \mathbb{N}_0$, then let $q \in [\gamma, \infty)$).*
- $e - \frac{n}{q} \in (m - n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}]$.

Then: If $u \in W^{e,q}_{\beta}$ for some $\beta < 0$ satisfies $Au = 0$, then $u \in W^{e,q}_{\beta'}$ for all $\beta' \in (m-n, 0)$.

Proof. **(Lemma B.9)** Following the proof of [41] [Lemma 3.8], let $A = A_\infty + R$ where A_∞ is the principal part of A at infinity. R has vanishing principal part at infinity and therefore by Theorem B.2, $Ru \in W^{e-m,q}_{\beta-m+\rho}$. Consequently $A_\infty u = -Ru \in W^{e-m,q}_{\beta-m+\rho}$. Now we may consider two cases:

- If $\beta+\rho \leq m-n$, then $\beta-m+\rho \leq -n$ and so $W^{e-m,q}_{\beta-m+\rho} \hookrightarrow W^{e-m,q}_{\eta}$ for all $\eta \geq -n$. Consequently $A_\infty u \in W^{e-m,q}_{\eta}$ for all $\eta \geq -n$. Since $A_\infty : W^{e,q}_{\beta'} \to W^{e-m,q}_{\beta'-m}$ is an isomorphism for all $\beta' \in (m-n, 0)$ we conclude that $u \in W^{e,q}_{\beta'}$ for all $\beta' \in (m-n, 0)$.
- If $\beta + \rho > m - n$ then $A_\infty : W^{e,q}_{\beta+\rho} \to W^{e-m,q}_{\beta-m+\rho}$ is an isomorphism. and therefore $u \in W^{e,q}_{\beta+\rho}$ which implies $u \in W^{e,q}_{\beta'}$ for all $\beta' \in (\beta+\rho, 0)$

Combining the above observations, we can conclude that $u \in W^{e,q}_{\beta'}$ for every $\beta' \in (\max(m-n, \beta+\rho), 0)$.

Now clearly for some $N \in \mathbb{N}$ we have $\beta + N\rho < m - n$ and therefore by iteration we get $u \in W^{e,q}_{\beta'}$ for every $\beta' \in (m-n, 0)$. □

Remark B.10. *In contrast to the notation that was introduced in the main text, in this Appendix and in particular in the statement of lemmas B.11 and B.12 we do not use the notation A_L for the Laplace operator when it acts on weighted Sobolev spaces.*

Lemma B.11 (Maximum principle)**.** *Suppose (M^n, h) is an AF manifold of class $W^{s,p}_\delta$ where $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $p \in (1, \infty)$, and $\delta < 0$. Also suppose $a \in W^{s-2,p}_{\eta-2}$, $\eta \in \mathbb{R}$, $\eta < 0$. Suppose that $a \geq 0$.*

- *(a) If $u \in W^{s,p}_\rho$ for some $\rho < 0$ satisfies*
$$-\Delta_h u + au \leq 0$$
then $u \leq 0$. In particular, if $-\Delta_h u + au = 0$, then applying this result to u and $-u$ shows that $u = 0$.
- *(b) Suppose that $u \in W^{s,p}_\rho$ is nonpositive and satisfies*
$$-\Delta_h u \leq 0.$$
If $u(x) = 0$ at some point $x \in M$, then u vanishes identically.

Proof. **(Lemma B.11)** For (a), we combine the proof that is given in [32] for the case of closed manifolds and the proof that is given in [38] for the case where $p = 2$. Fix $\epsilon > 0$. By assumptions $u \in W^{s,p}_\rho \hookrightarrow C^0_\rho$ and therefore u goes to zero at infinity. Therefore if we let $v = (u-\epsilon)^+ := \max\{u-\epsilon, 0\}$, then v is compactly supported. Note that if $f \in W^{1,q}_{loc}$ then $f^+ \in W^{1,q}_{loc}$ [32] and so we have

$$u \in W^{s,p}_\rho \hookrightarrow W^{1,n}_\rho \Rightarrow u \in W^{1,n}_{loc} \Rightarrow u - \epsilon \in W^{1,n}_{loc} \Rightarrow v \in W^{1,n}_{loc} \Rightarrow v \in W^{1,n},$$

since v has compact support. Now $u \in W^{s,p}_{loc}$ and so $u \in W^{s,p}$ in the support of v. By the multiplication lemma $W^{s,p} \times W^{1,n} \hookrightarrow W^{1,n}$, therefore uv is a nonnegative, compactly supported element of $W^{1,n}$. Since $W^{1,n} \hookrightarrow (W^{s-2.p})^*$ and $a \in W^{s-2,p}_{\eta-2} \subseteq W^{s-2,p}_{loc}$, we can apply a to uv and since $a \geq 0$ and $uv \geq 0$ we have $\langle a, uv\rangle_{(M,h)} \geq 0$. Hence

$$0 \geq -\langle a, uv\rangle \geq \langle -\Delta_h u, v\rangle = \langle \nabla u, \nabla v\rangle = \langle \nabla v, \nabla v\rangle.$$

It follows that v is constant with compact support which means $v \equiv 0$. Note that $v \equiv 0$ if and only if $u - \epsilon \leq 0$. So $u \leq \epsilon$ for all $\epsilon > 0$. This shows $u \leq 0$.

For (b), the proof is based on the weak Harnack inequality. The exact same proof as the one that is given in [32] for closed manifolds, works for the above setting as well. □

Lemma B.12 (Elliptic estimate for Laplacian)**.** *Suppose (M^n, h) is an AF manifold of class $W^{\alpha,\gamma}_{\rho}$, $\rho < 0$.*

1. *(1) $-\Delta_h \in D^{\alpha,\gamma}_{2,\rho}$.*
2. *(2) For all $\delta \in \mathbb{R}$, $\sigma \in [2-\alpha, \alpha]$, $-\Delta_h : W^{\sigma,\gamma}_{\delta} \to W^{\sigma-2,\gamma}_{\delta-2}$ is a continuous elliptic operator.*
3. *(3) For all $\delta \in (2-n, 0)$, $\sigma \in (2-\alpha, \alpha]$, $-\Delta_h : W^{\sigma,\gamma}_{\delta} \to W^{\sigma-2,\gamma}_{\delta-2}$ is semi-Fredholm and satisfies the following elliptic estimate:*
$$\|u\|_{W^{\sigma,\gamma}_{\delta}} \lesssim \|-\Delta_h u\|_{W^{\sigma-2,\gamma}_{\delta-2}} + \|u\|_{W^{2-\sigma,\gamma}_{\delta'}}.$$
where δ' can be any real number larger than δ.
4. *(4) For all $\delta \in (2-n, 0)$, $-\Delta_h : W^{\alpha,\gamma}_{\delta} \to W^{\alpha-2,\gamma}_{\delta-2}$ is an isomorphism. In particular*
$$\|u\|_{W^{\alpha,\gamma}_{\delta}} \lesssim \|-\Delta_h u\|_{W^{\alpha-2,\gamma}_{\delta-2}}$$

Proof. **(Lemma B.12)** Item 1 is a direct consequence of the multiplication lemma and the expression of Laplacian in local coordinates. Item 2 is a direct consequence of item 1 and Theorem B.2. Item 3 is a direct consequence of item 1 and Proposition B.7. For the last item we can proceed as follows:

By item 3, $-\Delta_h : W^{\alpha,p}_{\delta} \to W^{\alpha-2,p}_{\delta-2}$ is semi-Fredholm. On the other hand, Laplacian of the rough metric can be approximated by the Laplacian of smooth metrics and it is well known that Laplacian of a smooth metric is Fredholm of index zero. Therefore since the index of a semi-Fredholm map is locally constant, it follows that $-\Delta_h$ is Fredholm with index zero. Now maximum principle (Lemma B.11) implies that the kernel of $-\Delta_h : W^{\alpha,p}_{\delta} \to W^{\alpha-2,p}_{\delta-2}$ is trivial. An injective operator of index zero is surjective as well. Consequently $-\Delta_h : W^{\alpha,p}_{\delta} \to W^{\alpha-2,p}_{\delta-2}$ is a continuous bijective operator. Therefore by the open mapping theorem, $-\Delta_h : W^{\alpha,p}_{\delta} \to W^{\alpha-2,p}_{\delta-2}$ is an isomorphism of Banach spaces. In particular the inverse is continuous and so $\|u\|_{W^{\alpha,\gamma}_{\delta}} \lesssim \|-\Delta_h u\|_{W^{\alpha-2,\gamma}_{\delta-2}}$. □

Lemma B.13. *Let the following assumptions hold:*

- $\eta \in \mathbb{R}$, $\delta \in (-\infty, 0)$.
- $p \in (1, \infty)$, $\alpha \in (\frac{n}{p}, \infty) \cap (1, \infty)$.
- $\sigma \in (2-\alpha, \alpha] \cap (\frac{2n}{p} - n + 2 - \alpha, \infty)$.
- $a(x) \in W^{\alpha-2,p}_{\eta-2}$.

Then: For all $\nu > \delta + \eta - 2$, the operator $K : W^{\sigma,p}_{\delta}(\mathbb{R}^n) \to W^{\sigma-2,p}_{\nu}(\mathbb{R}^n)$ defined by $K(\psi) = a\psi$ is compact. (In particular, we can set $\nu = \eta - 2$ and for $n \geq 3$ we can set $\sigma = \alpha$.)

Proof. **(Lemma B.13)** $W^{\sigma,p}_{\delta}$ is a reflexive Banach space; so in order to show that K is a compact operator, we just need to prove that it is completely continuous, that is, we need to show if $\psi_n \to \psi$ weakly in $W^{\sigma,p}_{\delta}$, then $K\psi_n \to K\psi$ strongly in $W^{\sigma-2,p}_{\nu}$. Let

$$\beta = \min\{\alpha - \frac{n}{p}, \sigma - (2-\alpha), 1, \sigma - n(\frac{2}{p} - 1) - 2 + \alpha\}.$$
$$\theta = \sigma - \frac{1}{2}\beta.$$
$$\delta' = \delta + \frac{1}{2}[\nu - (\delta + \eta - 2)].$$

- **Step 1:** It follows from the assumptions that $\beta > 0$ and so $\theta < \sigma$. Also clearly $\delta' > \delta$. Thus we can conclude that $W^{\sigma,p}_{\delta} \hookrightarrow W^{\theta,p}_{\delta'}$ is compact. Therefore $\psi_n \to \psi$ strongly in $W^{\theta,p}_{\delta'}$.
- **Step 2:** Now we prove that
$$W^{\alpha-2,p}_{\eta-2} \times W^{\theta,p}_{\delta'} \hookrightarrow W^{\sigma-2,p}_{\nu}.$$
According to the multiplication lemma we need to check the following conditions
  (i) $\alpha - 2 \geq \sigma - 2$. True because $\alpha \geq \sigma$.
  $\theta \geq \sigma - 2$. True because $\beta \leq 1$ and so
$$\theta = \sigma - \frac{1}{2}\beta \geq \sigma - \frac{1}{2} \geq \sigma - 2.$$
  (ii) $\alpha - 2 + \theta \geq 0$. True because $\beta \leq \sigma - (2 - \alpha)$ and so
$$\theta = \sigma - \frac{1}{2}\beta \geq \sigma - \beta \geq \sigma - (\sigma - (2 - \alpha)) = 2 - \alpha.$$
  (iii) $(\alpha - 2) - (\sigma - 2) \geq 0$. True because $\alpha \geq \sigma$.
  $\theta - (\sigma - 2) \geq 0$. This one is the same as the second item.
  (iv) $(\alpha-2)+\theta-(\sigma-2) > n(\frac{1}{p}+\frac{1}{p}-\frac{1}{p})$. That is, we need to show $\theta > \sigma-(\alpha-\frac{n}{p})$. This is true because $\beta \leq \alpha - \frac{n}{p}$ and so
$$\theta = \sigma - \frac{1}{2}\beta \geq \sigma - \frac{1}{2}(\alpha - \frac{n}{p}) > \sigma - (\alpha - \frac{n}{p}).$$
  (v) $(\alpha - 2) + \theta > n(\frac{1}{p} + \frac{1}{p} - 1)$. This is true because $\beta \leq \sigma - n(\frac{2}{p} - 1) - 2 + \alpha$ and so
$$\theta = \sigma - \frac{1}{2}\beta > \sigma - \beta \geq \sigma - [\sigma - n(\frac{2}{p} - 1) - 2 + \alpha] = n(\frac{2}{p} - 1) + 2 - \alpha.$$
The numbering of the above items agrees with the numbering of the conditions in the multiplication lemma. Also note that $(\eta - 2) + \delta' \leq \nu$ by the definition of δ'.
- **Step 3:** By what was proved in **Step 2** we have
$$\|a(\psi_n - \psi)\|_{W^{\sigma-2,p}_{\nu}} \preceq \|a\|_{W^{\alpha-2,p}_{\eta-2}} \|\psi_n - \psi\|_{W^{\theta,p}_{\delta'}}.$$
But by **Step 1**, the right hand side goes to zero, which means $a\psi_n \to a\psi$ strongly in $W^{\sigma-2,p}_{\nu}$.

□

Lemma B.14 (Ehrling's lemma)**.** [50] *Let X, Y and Z be Banach spaces. Assume that X is compactly embedded in Y and Y is continuously embedded in Z. Then for every $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that*
$$\|x\|_Y \leq \epsilon \|x\|_X + c(\epsilon)\|x\|_Z$$

APPENDIX C. ARTIFICIAL CONFORMAL COVARIANCE OF THE HAMILTONIAN CONSTRAINT

Here we develop several results we need involving properties of the Hamiltonian constraint under a conformal change. We closely follow the argument in [32] for closed manifolds.

Let (M,h) be a 3-dimensional AF manifold of class $W^{s,p}_{\delta}$ where $p\in(1,\infty)$, $s\in(\frac{3}{p},\infty)\cap[1,\infty)$, and $\delta<0$. Suppose $\beta<0$. For $\psi\in W^{s,p}_{\delta}$ and $a_\tau,a_\rho,a_W\in W^{s-2,p}_{\beta-2}$, let

$$H(\psi,a_W,a_\tau,a_\rho):=-\Delta_h\psi+a_{R_h}(\psi+\mu)+a_\tau(\psi+\mu)^5-a_W(\psi+\mu)^{-7}-a_\rho(\psi+\mu)^{-3}$$

where μ is a fixed positive constant, $a_{R_h}=\frac{R_h}{8}$, and $R_h\in W^{s-2,p}_{\delta-2}$ is the scalar curvature of the metric h. Note that the Hamiltonian constraint can be represented by the equation $H=0$.

Now let $\tilde{h}=(\xi+1)^4h$ where $\xi\in W^{s,p}_{\delta}$ is a fixed function with $\xi>-1$. According to the discussion right after Definition 3.1 we know that $(M,\tilde{h})$ is also AF of class $W^{s,p}_{\delta}$. Define

$$\tilde{H}(\psi,a_W,a_\tau,a_\rho):=-\Delta_{\tilde{h}}\psi+a_{R_{\tilde{h}}}(\psi+\mu)+a_\tau(\psi+\mu)^5-\tilde{a}_W(\psi+\mu)^{-7}-\tilde{a}_\rho(\psi+\mu)^{-3}$$

where $\tilde{a}_W:=(\xi+1)^{-12}a_W$ and $\tilde{a}_\rho:=(\xi+1)^{-8}a_\rho$. Note that it follows from Lemma A.28 that $\tilde{a}_W$ and $\tilde{a}_\rho$ are in $W^{s-2,p}_{\beta-2}$.

Proposition C.1. *For all $\psi\in W^{s,p}_{\delta}$*

$$\tilde{H}(\psi,a_W,a_\tau,a_\rho)=(\xi+1)^{-5}H((\xi+1)\psi+\mu\,\xi,a_W,a_\tau,a_\rho).$$

Proof. **(Proposition C.1)** Let $\theta=\xi+1$. Then we have

$$R_{\tilde{h}}=(-8\Delta_h\theta+R_h\theta)\theta^{-5},$$
$$\Delta_h(\theta\psi+\mu(\theta-1))=\Delta_h(\theta\psi)+\mu\Delta_h\theta=(\Delta_h\theta)\psi+\theta\Delta_h\psi+2\langle\nabla\psi,\nabla\theta\rangle_h+\mu\Delta_h\theta,$$
$$\Delta_{\tilde{h}}\psi=\theta^{-4}\Delta_h\psi+2\theta^{-5}\langle\nabla\psi,\nabla\theta\rangle_h.$$

Therefore we can write

$$\begin{aligned}
(\xi+1)^{-5}&H((\xi+1)\psi+\mu\,\xi,a_W,a_\tau,a_\rho)=\theta^{-5}H(\theta\psi+\mu\theta-\mu,a_W,a_\tau,a_\rho)\\
&=\theta^{-5}[-\Delta_h(\theta\psi+\mu\theta-\mu)+\frac{1}{8}R_h(\theta\psi+\theta\mu)+a_\tau(\theta\psi+\theta\mu)^5\\
&\qquad-a_W(\theta\psi+\theta\mu)^{-7}-a_\rho(\theta\psi+\theta\mu)^{-3}]\\
&=\theta^{-5}[(-\Delta_h\theta)\psi-\theta\Delta_h\psi-2\langle\nabla\psi,\nabla\theta\rangle_h-\mu\Delta_h\theta+\frac{1}{8}R_h\theta(\psi+\mu)\\
&\qquad+a_\tau\theta^5(\psi+\mu)^5-a_W\theta^{-7}(\psi+\mu)^{-7}-a_\rho\theta^{-3}(\psi+\mu)^{-3}]\\
&=\big[-\theta^{-4}\Delta_h\psi-2\theta^{-5}\langle\nabla\psi,\nabla\theta\rangle_h\big]+\big[-\theta^{-5}(\Delta_h\theta)\psi-\mu\theta^{-5}\Delta_h\theta\\
&\qquad+\frac{1}{8}R_h\theta^{-4}(\psi+\mu)\big]+a_\tau(\psi+\mu)^5-a_W\theta^{-12}(\psi+\mu)^{-7}-a_\rho\theta^{-8}(\psi+\mu)^{-3}\\
&=-\Delta_{\tilde{h}}\psi+\frac{1}{8}R_{\tilde{h}}(\psi+\mu)+a_\tau(\psi+\mu)^5-\tilde{a}_W(\psi+\mu)^{-7}-\tilde{a}_\rho(\psi+\mu)^{-3}\\
&=\tilde{H}(\psi,a_W,a_\tau,a_\rho).
\end{aligned}$$

□

We have the following important corollary:

Corollary C.2. *Assume the above setting. Then we have*

$$\tilde{H}(\tilde{\psi},a_W,a_\tau,a_\rho)=0\iff H((\xi+1)\tilde{\psi}+\mu\,\xi,a_W,a_\tau,a_\rho)=0,$$
$$\tilde{H}(\tilde{\psi},a_W,a_\tau,a_\rho)\geq0\iff H((\xi+1)\tilde{\psi}+\mu\,\xi,a_W,a_\tau,a_\rho)\geq0,$$
$$\tilde{H}(\tilde{\psi},a_W,a_\tau,a_\rho)\leq0\iff H((\xi+1)\tilde{\psi}+\mu\,\xi,a_W,a_\tau,a_\rho)\leq0.$$

In particular, if $\tilde{\psi}_+$ and $\tilde{\psi}_-$ are sub and supersolutions for the equation $\tilde{H} = 0$, then $\psi_+ := (\xi+1)\tilde{\psi}_+ + \mu\,\xi$ and $\psi_- := (\xi+1)\tilde{\psi}_- + \mu\,\xi$ are sub and supersolutions for the equation $H = 0$.

APPENDIX D. METRICS IN THE POSITIVE YAMABE CLASS

Here we collect some facts regarding the Yamabe invariant in the case of AF manifolds.

Let (M,h) be a 3-dimensional AF manifold of class $W^{s,p}_\delta$ where $p \in (1,\infty)$, $s \in (\frac{3}{p},\infty)\cap[1,\infty)$, and $-1<\delta<0$. We define the Yamabe invariant as follows: [41, 8]

$$\lambda_h = \inf_{f\in C_c^\infty(M), f\not\equiv 0} \frac{\int_M 8|\nabla f|^2 dV_h + \langle R_h, f^2\rangle_{(M,h)}}{\|f\|^2_{L^6}}$$

We say h is in the positive Yamabe class if and only if $\lambda_h > 0$. Contrary to what we have for closed manifolds(e.g [32]), as it is discussed in [41] and [8] we have

$\lambda_h > 0$ **if and only if there exists a conformal factor** $\eta > 0$ **such that** $\eta - 1 \in W^{s,p}_\delta$ **and** $(M, \eta^4 h)$ **is scalar flat.**

It is interesting to notice that if $\lambda_h > 0$, then h is also conformal to a metric with **continuous positive** scalar curvature.

Proposition D.1. *Let (M,h) be a 3-dimensional AF manifold of class $W^{s,p}_\delta$ where $p \in (1,\infty)$, $s \in (\frac{3}{p},\infty)\cap[1,\infty)$, and $-1<\delta<0$. If h belongs to the positive Yamabe class, then there exist $\chi \in W^{s,p}_\delta$ such that if we set $\hat{h} = (1+\chi)^4 h$, then $R_{\hat{h}}$ is continuous and positive.*

Proof. **(Proposition D.1)** If h is in the positive Yamabe class, then there exists $\eta \in W^{s,p}_\delta$, $\eta > -1$ such that $R_{\tilde{h}} = 0$ where $\tilde{h} = (1+\eta)^4 h$. Let f be a smooth positive function in $W^{s-2,p}_{\delta-2}$. By Lemma B.12 there exists a unique function $v \in W^{s,p}_\delta$ such that $-8\Delta_{\tilde{h}} v = f$. By the maximum principle (Lemma B.11) v is positive. Now define $\hat{h} = (1+v)^4 \tilde{h}$. We have

$$R_{\hat{h}} = \big(-8\Delta_{\tilde{h}} v + R_{\tilde{h}}(1+v)\big)(1+v)^{-5} = 8f(1+v)^{-5}.$$

Since f and v are both continuous and positive we can conclude that $R_{\hat{h}}$ is continuous and positive. If we set $\chi = v + \eta + \eta v$, then $\chi \in W^{s,p}_\delta$ and

$$\hat{h} = (1+v)^4(1+\eta)^4 h = (1+\chi)^4 h.$$

Note that since $v > 0$ and $\eta > -1$ we have $\chi > -1$. □

APPENDIX E. ANALYSIS OF THE LCBY EQUATIONS IN BESSEL POTENTIAL SPACES

As it was pointed out in Appendix A, Sobolev-Slobodeckij spaces are not the only option that we have if we wish to work with noninteger order Sobolev spaces. Another option is to consider the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ and then define the weighted spaces based on Bessel potential spaces. Bessel potential spaces agree with Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ when s is an integer and therefore they can be considered as an extension of integer order Sobolev spaces. There are two main advantages in working with Bessel potential spaces (and the corresponding weighted versions) in comparison with Sobolev-Slobodeckij spaces: First, Bessel potential spaces have better interpolation properties; second we have a better (stronger) multiplication lemma for Bessel potential spaces.

Theorem E.1 (Complex Interpolation)**.** [56] *Suppose* $\theta \in (0,1)$*,* $0 \leq s_0, s_1 < \infty$*, and* $1 < p_0, p_1 < \infty$*. If*

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

then $H^{s,p}(\mathbb{R}^n) = [H^{s_0,p_0}(\mathbb{R}^n), H^{s_1,p_1}(\mathbb{R}^n)]_\theta$*.*

Lemma E.2. *Let* $s_i \geq s$ *with* $s_1 + s_2 \geq 0$*, and* $1 < p, p_i < \infty$ *(*$i = 1, 2$*) be real numbers satisfying*

$$s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p}), \quad s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0,$$

In case $s < 0$ *let*

$$s_1 + s_2 > n(\frac{1}{p_1} + \frac{1}{p_2} - 1) \quad \textit{(equality is allowed if } min(s_1, s_2) < 0\textit{)}.$$

Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$H^{s_1,p_1}(\mathbb{R}^n) \times H^{s_2,p_2}(\mathbb{R}^n) \to H^{s,p}(\mathbb{R}^n).$$

We will prove the above lemma later in this Appendix.

Remark E.3. *We make the following observations.*

- *Note that in the above multiplication lemma there is no restriction on the values of* p_1 *and* p_2 *with respect to* p*. That is, it is allowed for* p_1 *or* p_2 *to be greater than* p*.*
- *Note that contrary to what we had for Sobolev-Slobodeckij spaces, the complex interpolation works regardless of whether exponents are integer or noninteger. This feature is crucial because complex interpolation works much better for interpolation of bilinear forms. This is one of the reasons that we have a stronger multiplication lemma for Bessel potential spaces.*

Let us denote the weighted spaces based on $H^{s,p}$ by $H^{s,p}_\delta$ (rather than $W^{s,p}_\delta$). Our spaces $H^{s,p}_\delta(\mathbb{R}^n)$ correspond with the spaces $h^s_{p,ps-p\delta-n}(\mathbb{R}^n)$ in [54, 55].

Theorem E.4 (Complex Interpolation, Weighted Spaces)**.** [54, 55] *Suppose* $\theta \in (0,1)$*. If*

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \delta = (1-\theta)\delta_0 + \theta\delta_1$$

then $H^{s,p}_\delta(\mathbb{R}^n) = [H^{s_0,p_0}_{\delta_0}(\mathbb{R}^n), H^{s_1,p_1}_{\delta_1}(\mathbb{R}^n)]_\theta$*.*

The corresponding weighted version of the multiplication lemma can be proved using the exact same argument as the one that we used for weighted Sobolev-Slobodeckij spaces.

Lemma E.5 (Multiplication Lemma, Weighted Bessel potential spaces)**.** *Assume* s, s_1, s_2 *and* $1 < p, p_1, p_2 < \infty$ *are real numbers satisfying*

(i) $s_i \geq s \quad (i = 1, 2)$*,*
(ii) $s_1 + s_2 \geq 0$*,*
(iii) $s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p}) \quad (i = 1, 2)$*,*
(iv) $s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0$*.*

In case $s < 0$*, in addition let*

(v) $s_1+s_2>n(\frac{1}{p_1}+\frac{1}{p_2}-1)$ *(equality is allowed if* $min(s_1,s_2)<0$*).*

Then for all $\delta_1,\delta_2\in\mathbb{R}$*, the pointwise multiplication of functions extends uniquely to a continuous bilinear map*

$$H^{s_1,p_1}_{\delta_1}(\mathbb{R}^n)\times H^{s_2,p_2}_{\delta_2}(\mathbb{R}^n)\to H^{s,p}_{\delta_1+\delta_2}(\mathbb{R}^n).$$

Again notice that p_1 and p_2 do NOT need to be less than or equal to p. This extra degree of freedom that we have for multiplication in Bessel potential spaces allows us to remove the restrictions of the type "$p=q$ if $e=s\notin\mathbb{N}_0$" in all the statements of the main text. Consequently we will have a stronger existence theorem as follows:

Theorem E.6. *Let* (M,h) *be a* 3*-dimensional AF Riemannian manifold of class* $H^{s,p}_{\delta}$ *where* $p\in(1,\infty)$*,* $s\in(1+\frac{3}{p},\infty)$ *and* $-1<\delta<0$ *are given. Suppose* h *admits no nontrivial conformal Killing field and is in the positive Yamabe class. Let* $\beta\in(-1,\delta]$*. Select* q *and* e *to satisfy:*

$$\frac{1}{q}\in(0,1)\cap(0,\frac{s-1}{3})\cap[\frac{3-p}{3p},\frac{3+p}{3p}],$$
$$e\in(1+\frac{3}{q},\infty)\cap[s-1,s]\cap[\frac{3}{q}+s-\frac{3}{p}-1,\frac{3}{q}+s-\frac{3}{p}].$$

Assume that the data satisfies:

- $\tau\in H^{e-1,q}_{\beta-1}$ *if* $e\geq 2$ *and* $\tau\in H^{1,z}_{\beta-1}$ *otherwise, where* $z=\frac{3q}{3+(2-e)q}$ *(note that if* $e=2$*, then* $H^{e-1,q}_{\beta-1}=H^{1,z}_{\beta-1}$*),*
- $\sigma\in H^{e-1,q}_{\beta-1}$*,*
- $\rho\in H^{s-2,p}_{\beta-2}\cap L^{\infty}_{2\beta-2}$*,* $\rho\geq 0$ *(*ρ *can be identically zero),*
- $J\in\boldsymbol{H}^{e-2,q}_{\beta-2}$*.*

If μ *is chosen to be sufficiently small and if* $\|\sigma\|_{L^{\infty}_{\beta-1}}$*,* $\|\rho\|_{L^{\infty}_{2\beta-2}}$*, and* $\|J\|_{\boldsymbol{H}^{e-2,q}_{\beta-2}}$ *are sufficiently small, then there exists* $\psi\in H^{s,p}_{\delta}$ *with* $\psi>-\mu$ *and* $W\in\boldsymbol{H}^{e,q}_{\beta}$ *solving (3.3) and (3.4).*

Our plan for the remainder of this appendix is to discuss the proof of the stronger version of multiplication lemma that was stated for Bessel potential spaces. In our proof we will make use of some of the well-known results for pointwise multiplication in Triebel-Lizorkin spaces that can be found in [51]. Just for the purpose of completeness we quickly review the definition of Besov spaces and Triebel-Lizorkin spaces and their relations to the Sobolev-Slobodeckij spaces and Bessel potential spaces.

Definition E.7. *Consider the partition of unity* $\{\varphi_j\}$ *that was introduced in the beginning of Appendix A.*

- *For* $s\in\mathbb{R}$*,* $1\leq p<\infty$*, and* $1\leq q<\infty$ *(or* $p=q=\infty$*) define the Triebel-Lizorkin space* $F^s_{p,q}(\mathbb{R}^n)$ *as follows*

$$F^s_{p,q}(\mathbb{R}^n)=\{u\in S'(\mathbb{R}^n):\|u\|_{F^s_{p,q}(\mathbb{R}^n)}=\big|\big|\|2^{sj}\mathcal{F}^{-1}(\varphi_j\mathcal{F}u)\|_{l^q}\big|\big|_{L^p(\mathbb{R}^n)}<\infty\}$$

- *For* $s\in\mathbb{R}$*,* $1\leq p<\infty$*, and* $1\leq q<\infty$ *define the Besov space* $B^s_{p,q}(\mathbb{R}^n)$ *as follows*

$$B^s_{p,q}(\mathbb{R}^n)=\{u\in S'(\mathbb{R}^n):\|u\|_{B^s_{p,q}(\mathbb{R}^n)}=\big|\big|\|2^{sj}\mathcal{F}^{-1}(\varphi_j\mathcal{F}u)\|_{L^p(\mathbb{R}^n)}\big|\big|_{l^q}<\infty\}$$

We have the following relations [56, 52]:

$$\begin{aligned}
L^p &= F^0_{p,2}, \quad 1 < p < \infty, \\
B^s_{p,p} &= F^s_{p,p}, \quad s \in \mathbb{R}, \quad 1 < p < \infty, \\
H^{s,p} &= F^s_{p,2}, \quad s \in \mathbb{R}, \quad 1 < p < \infty, \\
W^{k,p} &= H^{k,p} = F^k_{p,2}, \quad k \in \mathbb{Z}, \quad 1 < p < \infty, \\
W^{s,p} &= B^s_{p,p} = F^s_{p,p}, \quad s \in \mathbb{R} \setminus \mathbb{Z}, \quad 1 < p < \infty,
\end{aligned}$$

If $k \in \mathbb{N}$, then $B^k_{p,p} \hookrightarrow W^{k,p}$ for $1 \leq p \leq 2$ and $W^{k,p} \hookrightarrow B^k_{p,p}$ for $p \geq 2$.

With these definitions and notation at our disposal, we now give an abbreviated proof of the key multiplication Lemma E.2 that was stated earlier. The new complete proof for all cases may be found in [6].

Proof. **(Lemma E.2)** We prove the lemma for the case $s \geq 0$. The case $s < 0$ can be proved by using a duality argument that can be found in [6]. We may consider three cases:

- **Case 1:** $p_1, p_2 \leq p$**:** This case is proved in [6]. The proof in [6] is based on complex interpolation.
- **Case 2:** $p \leq \min\{p_1, p_2\}$**:** In what follows we will discuss the proof of this case. For now let's assume the lemma holds true in this case.
- **Case 3:** $p_1 > p$, $p_2 \leq p$ **or** $p_2 > p$, $p_1 \leq p$**:** Here we prove the case where $p_1 > p$, $p_2 \leq p$. The proof of the other case is completely analogous. We have

  $$H^{s_1,p_1} \times H^{s_2,p_2} \hookrightarrow H^{s_1,p_1} \times H^{s_2 - \frac{n}{p_2} + \frac{n}{p}, p} \hookrightarrow H^{s,p}.$$

  Note that by assumption $s_2 - \frac{n}{p_2} \geq s - \frac{n}{p}$ and so $s_2 - \frac{n}{p_2} + \frac{n}{p} \geq s \geq 0$. The first embedding is true because $H^{s_2,p_2} \hookrightarrow H^{s_2 - \frac{n}{p_2} + \frac{n}{p}, p}$ (one can easily check that the conditions of Theorem A.11, which is also valid for Bessel potential spaces, are satisfied). Also as a direct consequence of the claim of **Case 2**, the second embedding holds true (note that $p \leq \min\{p, p_1\}$).

So it remains to prove the claim of **Case 2**, that is the case where $p \leq \min\{p_1, p_2\}$. Of course if both p_1 and p_2 are equal to p, then the claim follows from case 1; so we may assume at least one of p_1 or p_2 is strictly larger than p. To prove **Case 2** we proceed as follows:

- **Step 1:** In this step we consider the case where $s_1 = s_2 = s$. Note that by assumption $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \geq 0$. If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, then let $k = \lfloor m \rfloor$. We have ([61])

  $$\begin{aligned}
  H^{k+1,p_1} \times H^{k+1,p_2} &\hookrightarrow H^{k+1,p}, \\
  H^{k,p_1} \times H^{k,p_2} &\hookrightarrow H^{k,p},
  \end{aligned}$$

  so by complex interpolation we get

  $$H^{s,p_1} \times H^{s,p_2} \hookrightarrow H^{s,p}.$$

  As a direct consequence of Theorem 2 in page 239 of [51], the above embedding remains valid if $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{p}$ and $p_1, p_2 > p$. What if $p_2 = p$ or $p_1 = p$? Here we consider the case where $p_2 = p$ (and so $p_1 > p$). The proof of the other case is completely analogous. Note that by assumption $s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) = \frac{n}{p_1}$; under this assumption we need to prove the following:

$$H^{s,p_1}\times H^{s,p}\hookrightarrow H^{s,p}.$$

If $s\neq\frac{n}{p}$, the above embedding follows from Theorem 1 in page 176 and Theorem 2 in page 177 of [51]. Now if $s=\frac{n}{p}$, we set $\epsilon=\frac{n}{p}-\frac{n}{p_1}$ and then since the claim is true for $s\neq\frac{n}{p}$ we have

$$H^{\frac{n}{p}-\frac{\epsilon}{2},p_1}\times H^{\frac{n}{p}-\frac{\epsilon}{2},p}\hookrightarrow H^{\frac{n}{p}-\frac{\epsilon}{2},p},$$
$$H^{\frac{n}{p}+\frac{\epsilon}{2},p_1}\times H^{\frac{n}{p}+\frac{\epsilon}{2},p}\hookrightarrow H^{\frac{n}{p}+\frac{\epsilon}{2},p},$$

so the result follows from complex interpolation.

- **Step 2:** Let $t_1,t_2\in[0,\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p}]$ and suppose $\epsilon>0$ is such that $t_1+t_2-(\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p})-\epsilon\geq 0$. Then as a direct consequence of the Corollary that is stated in page 189 of [51] we have

$$H^{t_1,p_1}\times H^{t_2,p_2}\hookrightarrow H^{t_1+t_2-(\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p})-\epsilon,p}$$

- **Step 3:** Note that by **Step 1**, if $b>\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p}$, then

$$H^{b,p_1}\times H^{b,p_2}\hookrightarrow H^{b,p}.$$

Also if we let $\frac{1}{r}=\frac{1}{p}-\frac{1}{p_2}$, then $H^{b,p_1}\hookrightarrow L^r$ and so by Holder's inequality

$$H^{b,p_1}\times H^{0,p_2}\hookrightarrow H^{0,p}.$$

By complex interpolation we get

$$\forall t\in[0,b]\quad H^{b,p_1}\times H^{t,p_2}\hookrightarrow H^{t,p}.$$

Therefore

$$\forall\epsilon>0\;\forall t\in[0,\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p}]\quad H^{\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p}+\epsilon,p_1}\times H^{t,p_2}\hookrightarrow H^{t,p}.$$

- **Step 4:** In this step we prove the claim of **Case 2** in its general form. Without loss of generality we may assume $s_1=\max\{s_1,s_2\}$, so $s_2\in[0,s_1]$. If $s_1>\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p}$, then by what was proved in **Step 3** we have

$$H^{s_1,p_1}\times H^{s_2,p_2}\hookrightarrow H^{s_2,p}\hookrightarrow H^{s,p}.$$

In case $s_1\leq\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p}$ (that is, if $s_1,s_2\in[0,\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p}]$), choose $\epsilon>0$ such that $s_1+s_2-(\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p})-\epsilon>s\geq 0$. Then by **Step 2** we have:

$$H^{s_1,p_1}\times H^{s_2,p_2}\hookrightarrow H^{s_1+s_2-(\frac{n}{p_1}+\frac{n}{p_2}-\frac{n}{p})-\epsilon,p}\hookrightarrow H^{s,p}.$$

□

Appendix F. An Alternative Weak Formulation of the LCBY Equations

In Section 3 we described a setting where the constraint equations make sense with rough data. Here we describe a second framework in which rough data is allowed. Recall that according to our preliminary discussion in Section 3, we have already imposed the following restrictions:

$$p\in(1,\infty),\quad s\in(\frac{3}{p},\infty)\cap[1,\infty),\quad \delta<0.$$

Framework 2:
In this framework we seek W in $\mathbf{W}^{1,2r}_{\beta}$ where $r\geq 1$ and $\beta<0$. For the momentum constraint to make sense we need to ensure that

(1) it is possible to extend the operator $-\Delta_L : \mathbf{C}^\infty \to \mathbf{C}^\infty$ to an operator $\mathcal{A}_L : \mathbf{W}^{1,2r}_\beta \to \mathbf{W}^{-1,2r}_{\beta-2}$.

(2) $b_\tau(\psi+\mu)^6 + b_J \in \mathbf{W}^{-1,2r}_{\beta-2}$.

Note that Δ_L belongs to the class $D^{s,p}_{2,\delta}$. Therefore by Theorem B.2 we just need to check the following conditions (numbering corresponds to conditions in Theorem B.2):

(i)	$2r \in (1,\infty)$,	(since $r \geq 1$)
(ii)	$1 > 2-s$,	(enough to assume $s > 1$)
(iii)	$-1 < \min(1,s) - 2$,	(enough to assume $s > 1$)
(iv)	$-1 < 1 - 2 + s - \frac{3}{p}$,	(since $\frac{3}{p} < s$)
(v)	$-1 - \frac{3}{2r} \leq s - \frac{3}{p} - 2$,	(so need to assume $1 - \frac{3}{2r} \leq s - \frac{3}{p}$)
(vi)	$1 - \frac{3}{2r} > 2 - 3 - s + \frac{3}{p}$.	(since $r \geq 1$ and so $1 - \frac{3}{2r} \geq \frac{-1}{2} > -1 - (s - \frac{3}{p})$)

So the only extra assumptions that we need to make is that $1 - \frac{3}{2r} \leq s - \frac{3}{p}$ and $s > 1$. Also in order to ensure that the second condition holds true it is enough to assume τ is given in $L^{2r}_{\beta-1}$ and J is given in $\mathbf{W}^{-1,2r}_{\beta-2}$. Indeed, note that $\tau \in L^{2r}_{\beta-1}$ implies $b_\tau \in \mathbf{W}^{-1,2r}_{\beta-2}$. Since $\psi \in W^{s,p}_\delta$, it follows from Lemma A.28 that $b_\tau(\psi+\mu)^6 \in \mathbf{W}^{-1,2r}_{\beta-2}$; Lemma A.28 can be applied because clearly $2r \in (1,\infty)$ and moreover

(i)	$-1 \in (-s, s)$,	(since $s > 1$)
(ii)	$-1 - \frac{3}{2r} \leq s - \frac{3}{p}$,	(since we assumed $1 - \frac{3}{2r} \leq s - \frac{3}{p}$)
	$-3 - s + \frac{3}{p} \leq -1 - \frac{3}{2r}$.	(the same as item (vi) above)

The numbering of the above items corresponds to the numbering of the conditions in Lemma A.28.

Now let's consider the Hamiltonian constraint. Note that $W \in \mathbf{W}^{1,2r}_\beta$ and so $\mathcal{L}W \in L^{2r}_{\beta-1}$. So for a_W to be well defined it is enough to assume $\sigma \in L^{2r}_{\beta-1}$. Exactly similar to our discussion for weak formulation 1, for Hamiltonian constraint to make sense it is enough to ensure that

$$f(\psi, W) = a_R(\psi+\mu) + a_\tau(\psi+\mu)^5 - a_W(\psi+\mu)^{-7} - a_\rho(\psi+\mu)^{-3} \in W^{s-2,p}_{\eta-2},$$

where $\eta = \max\{\delta, \beta\}$. One way to guarantee that the above statement holds true is to ensure that

$$a_\tau, a_\rho, a_W \in W^{s-2,p}_{\beta-2}, \quad a_R \in W^{s-2,p}_{\delta-2},$$

We claim that for the above statements to be true it is enough to make the following extra assumptions:

$$s \leq 2, \quad 1 - \frac{3}{2r} \geq \frac{1}{2}(s - \frac{3}{p}), \quad \rho \in W^{s-2,p}_{\beta-2}.$$

The details are as follows:

(1) $a_\tau = \frac{1}{12}\tau^2$.

We want to ensure $a_\tau \in W^{s-2,p}_{\beta-2}$. Note that $\tau \in L^{2r}_{\beta-1}$, so $\tau^2 \in L^r_{2\beta-2}$. Thus we want to have $L^r_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$. We will see that this embedding becomes true provided $s \leq 2$ and $1 - \frac{3}{2r} \geq \frac{1}{2}(s - \frac{3}{p})$.

We just need to check that the assumptions of Theorem A.17 are satisfied (numbering corresponds to the assumptions in Theorem A.17)

(ii) $0 \geq s - 2$ (equivalent to $s \leq 2$),

(iii) $0 - \frac{3}{r} \geq s - 2 - \frac{3}{p}$ (equivalent to $1 - \frac{3}{2r} \geq \frac{1}{2}(s - \frac{3}{p})$),

(iv) $2\beta - 2 < \beta - 2$ (true because $\beta < 0$).

(2) $a_R = \frac{R}{8}$.
We want to ensure $a_R \in W^{s-2,p}_{\delta-2}$. Note that h is an AF metric of class $W^{s,p}_{\delta}$ and R involves the second derivatives of h, so $R \in W^{s-2,p}_{\delta-2}$. We do not need to impose any extra restrictions for this one.

(3) $a_\rho = \kappa\rho/4$.
Clearly $a_\rho \in W^{s-2,p}_{\beta-2}$ iff $\rho \in W^{s-2,p}_{\beta-2}$.

(4) $a_W = [\sigma_{ab} + (\mathcal{L}W)_{ab}][\sigma^{ab} + (\mathcal{L}W)^{ab}]/8$.
We want to ensure that $a_W \in W^{s-2,p}_{\beta-2}$. Note that $\mathcal{L}W, \sigma \in L^{2r}_{\beta-1}$ and as discussed above, $L^r_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$. So $a_W = \frac{1}{8}|\sigma + \mathcal{L}W|^2 \in L^r_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$.

Remark F.1. *According to the above discussion we need $r \geq 1$ satisfy*

$$\frac{1}{2}(s - \frac{3}{p}) \leq 1 - \frac{3}{2r} \leq s - \frac{3}{p}.$$

In particular, if we choose r such that $\frac{1}{2}(s - \frac{3}{p}) = 1 - \frac{3}{2r}$, that is, if we set $r = \frac{3p}{3+(2-s)p}$, then clearly r satisfies the above inequalities and moreover since $s \leq 2$, we have $r \geq 1$.

Weak Formulation 2. *Let (M, h) be a $3D$ AF Riemannian manifold of class $W^{s,p}_{\delta}$ where $p \in (1, \infty)$, $\beta, \delta < 0$ and $s \in (\frac{3}{p}, \infty) \cap (1, 2]$. Let $r = \frac{3p}{3+(2-s)p}$. Fix source functions:*

$$\tau \in L^{2r}_{\beta-1}, \quad \sigma \in L^{2r}_{\beta-1}, \quad \rho \in W^{s-2,p}_{\beta-2}(\rho \geq 0), \quad J \in \boldsymbol{W}^{-1,2r}_{\beta-2}.$$

Let $\eta = \max\{\beta, \delta\}$. Define $f : W^{s,p}_{\delta} \times \boldsymbol{W}^{1,2r}_{\beta} \to W^{s-2,p}_{\eta-2}$ and $\boldsymbol{f} : W^{s,p}_{\delta} \to \boldsymbol{W}^{-1,2r}_{\beta-2}$ as

$$f(\psi, W) = a_R(\psi + \mu) + a_\tau(\psi + \mu)^5 - a_W(\psi + \mu)^{-7} - a_\rho(\psi + \mu)^{-3},$$
$$\boldsymbol{f}(\psi) = b_\tau(\psi + \mu)^6 + b_J.$$

Find $(\psi, W) \in W^{s,p}_{\delta} \times \boldsymbol{W}^{1,2r}_{\beta}$ such that

$$A_L\psi + f(\psi, W) = 0, \tag{F.1}$$
$$\mathcal{A}_L W + \boldsymbol{f}(\psi) = 0. \tag{F.2}$$

Remark F.2. *Consider **Weak Formulation 1**. In the case where $s \leq 2$ and $\frac{1}{q} \geq \frac{2-d}{6}$ where $d = s - \frac{3}{p}$, this formulation becomes a special case of **Weak Formulation 2**. Indeed, we just need to check that in this case $W^{e,q}_{\beta} \hookrightarrow W^{1,2r}_{\beta}$. By Theorem A.16 we need to make sure that the followings hold true:*

$$(i)\ q \leq 2r, \quad (\textit{true because } \frac{1}{q} \geq \frac{2-d}{6} = \frac{3 + (2-s)p}{6p} = \frac{1}{2r})$$

$$(ii)\ e \geq 1, \quad (\textit{true because } e > 1 + \frac{3}{q})$$

$$(iii)\ e - \frac{3}{q} \geq 1 - \frac{3}{2r}.$$

(The numbering of the above items agrees with the numbering of the assumptions in Theorem A.16.) The third condition is true because

$$e - \frac{3}{q} \geq 1 - \frac{3}{2r} \Leftrightarrow e \geq 1 + \frac{3}{q} - \frac{3 + (2-s)p}{2p} \Leftrightarrow e \geq \frac{3}{q} + \frac{d}{2},$$

and

- *if $d > 2$, then $d - 1 > \frac{d}{2}$ and so $e \geq \frac{3}{q} + d - 1 > \frac{3}{q} + \frac{d}{2}$,*

- *if* $d \leq 2$, *then* $1 \geq \frac{d}{2}$ *and so* $e > 1 + \frac{3}{q} \geq \frac{3}{q} + \frac{d}{2}$.

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