Neutron Stars: Where Gravitation Theory Meets Nuclear Physics

Lee Lindblom

Department of Physics, University of California at San Diego

Physics Colloquium, National Central University, Zhongli, Taiwan
15 April 2025

Abstract: Neutron stars are created by the gravitational collapse of massive stars that have exhausted their nuclear fuel. The bulk of the material in these stars is compressed to densities larger than those in the nuclei of normal atoms. Their gravitational fields become nearly as strong as those at the surfaces of black holes. This talk will explore how astronomical observations of these stars and an understanding of gravitational physics can be used to learn new things about nuclear physics.

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- Rough outline of my talk:
  - A brief introduction to neutron stars.
  - How does nuclear physics affect neutron-star structure?
  - How can we use that knowledge to learn about nuclear physics?
  - How accurately can nuclear physics information be determined using noisy neutron-star observational data?

Neutron Stars

- The idea of neutron stars was proposed by Baade and Zwicky in 1934 as the end products of supernova explosions.
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- Oppenheimer and Volkoff were the first to solve Einstein's equation to determine the structures of neutron stars in 1939.
- Modern nuclear-theory models of neutron-star matter are complicated mixtures of neutrons, protons, electrons, muons and other exotic particles.

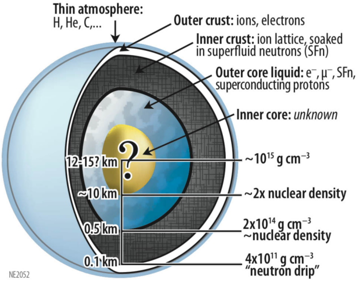

Figure from Gendreau, Arzoumanian, Okajima Proc. of SPIE, **8443**, 844313 (2012).

How do gravitational and nuclear physics determine the structures of neutron stars?

- Einstein's equations determine the structures of relativistic stars:

$$\begin{aligned}\frac{dm}{dr} &= 4\pi r^2\epsilon,\\ \frac{dp}{dr} &= -(\epsilon+p)\frac{m+4\pi r^3 p}{r(r-2m)}.\end{aligned}$$

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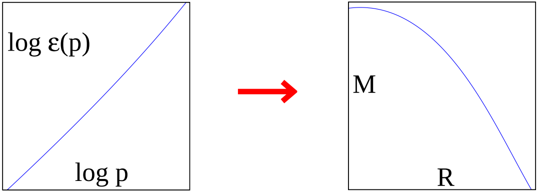
- The total mass M and radius R are determined by solving these equations for each value of p_c: $\{R(p_c), M(p_c)\}$.
- M and R are macroscopic properties of neutron stars that can (in principle) be measured by astronomical observations.

Relativistic Stellar Structure Problem (SSP)

- The relativistic stellar structure problem (SSP) can be thought of as a map from the equation of state $\epsilon = \epsilon(p)$ to a curve in the M-R parameter space $\{R(p_c), M(p_c)\}$.



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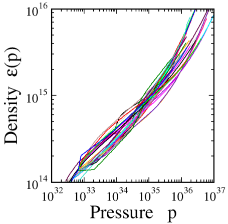

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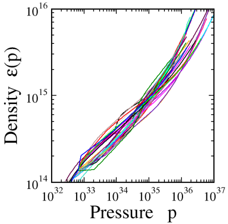
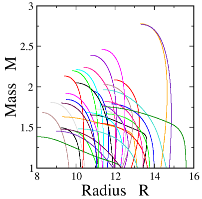
- Solving the relativistic stellar structure problem for these equations of state produces a wide range of neutron-star models.





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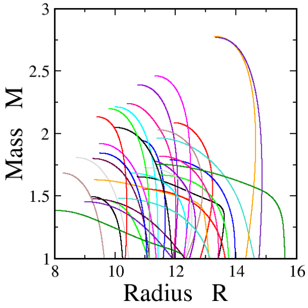

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- A more sophisticated approach would be to adjust the parameters of a particular nuclear-theory model for the equation of state by fitting the resulting neutron-star models to the observations.
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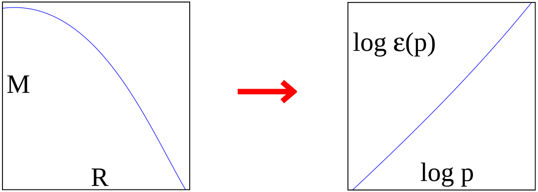
- Versions of this more sophisticated approach have been implemented by James Lattimer and collaborators, and also by Feryal Özel and collaborators.
- Can we can do better?
- Do Einstein's equations plus observations determine the neutron-star equation of state on their own without assuming anything about the nuclear-theory model?

Relativistic Inverse Stellar Structure Problem (SSP^{-1})

- The inverse stellar structure problem (SSP^{-1}) solves Einstein's equation to find the equation of state $\epsilon = \epsilon(p)$ given a curve of macroscopic observables, e.g. $\{R(\lambda), M(\lambda)\}$.
- SSP^{-1} can be thought of as the map from the M-R curve $\{R(\lambda), M(\lambda)\}$ to the equation of state $\epsilon = \epsilon(p)$.



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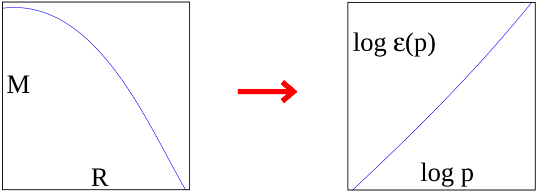
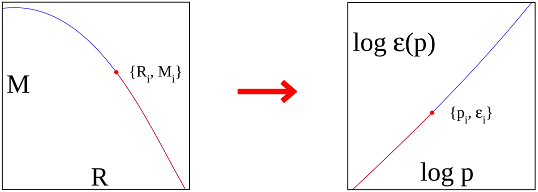


- The basic mathematical questions then become:
  - "Does this problem have a solution?"
  - "Is the solution unique?"
  - "How do we solve it?"
  - "How accurately can we solve it?"

"Formal" Solution to SSP^{-1}

- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.
- Assume the complete M-R curve is known, including the point $\{R_i, M_i\} = \{R(p_i), M(p_i)\}$.



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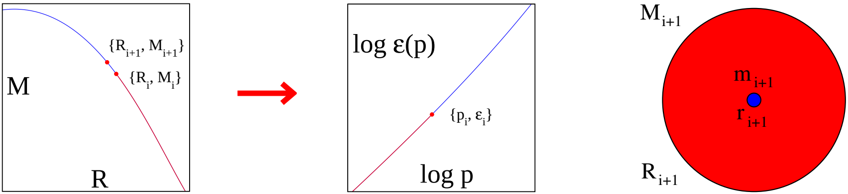

- Choose a new point on the M-R curve, $\{R_{i+1}, M_{i+1}\}$, that represents a stellar model having slightly larger central density.

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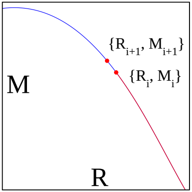

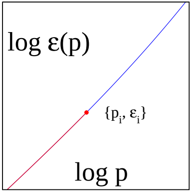

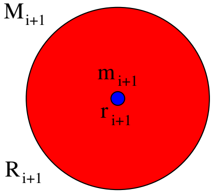

- Integrate Einstein's equations through the outer parts of the star,

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \qquad \frac{dp}{dr} = -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r - 2m)},$$

to determine the mass and radius, $\{r_{i+1}, m_{i+1}\}$, of the tiny core with large densities $\epsilon \geq \epsilon_i$.

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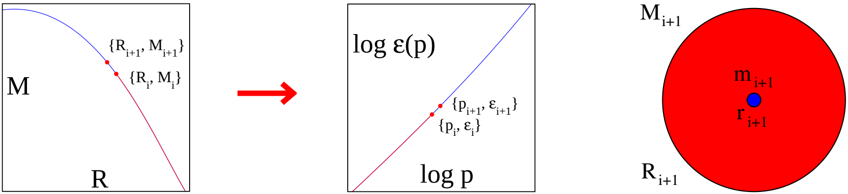

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- Determine the central density ϵ_{i+1} and pressure p_{i+1} analytically from the structure of the tiny core: $m_{i+1} = \frac{4\pi}{3}\epsilon_{i+1} r_{i+1}^3 + \mathcal{O}(r_{i+1}^5)$.

Can the Formal Solution to SSP^{-1} be Improved?

- Formal solution to the relativistic SSP^{-1} finds the equation of state, $\epsilon = \epsilon(p)$, represented as a table, $\{p_i, \epsilon_i\}$ for $i = 1, ..., N$, and an interpolation formula.
- Formal solution has several practical weaknesses:
  - Solution converges (slowly) with the number of points, as N^{-q}.
  - Each new equation of state point, $\{p_i, \epsilon_i\}$, requires the knowledge of a separate new M-R curve point, $\{R_i, M_i\}$.
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- Spectral numerical methods typically converge more rapidly, and represent functions more efficiently than finite difference methods.
- Can spectral methods provide better (*i.e.* more practical and more accurate) solutions to the SSP^{-1}?
- Can spectral methods provide interesting solutions to SSP^{-1} when only a few (*e.g.* two or three) M-R data points are available?

Solving SSP^{-1} Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \lambda_k)$, where the λ_k are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \lambda_k) = \sum_k \lambda_k \Phi_k(p)$, where the $\Phi_k(p)$ are known basis functions, *e.g.* $\Phi_k(p) = e^{ikp}$, or $\Phi_k(p) = P_k(p)$.

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- Given a set of points from the "real" M-R curve, $\{R_i, M_i\}$, choose the parameters λ_k and p_c^i that minimize the difference measure:

$$\chi^2 = \frac{1}{N_{\rm stars}} \sum_{i=1}^{N_{\rm stars}} \left\{ \left[\log\left(\frac{R(p_c^i, \lambda_k)}{R_i} \right) \right]^2 + \left[\log\left(\frac{M(p_c^i, \lambda_k)}{M_i} \right) \right]^2 \right\}$$

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- Resulting λ_k for $k = 1, ..., N_{\lambda_k}$ determines an equation of state, $\epsilon = \epsilon(p, \lambda_k)$, that provides an approximate solution of SSP^{-1}.

Causal Spectral Representations

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- The velocity function $\Upsilon(p)=\left[c^2-v^2(p)\right]/v^2(p)$ is therefore non-negative, $\Upsilon(p)\geq 0$, if and only if the fluid is causal.

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$$d\epsilon/dp=(1+\Upsilon(p))/c^2.$$

- Integrate this ode to obtain a causal spectral representation of the equation of state:

$$\epsilon(p,\lambda_k)=\epsilon_0+\frac{p-p_0}{c^2}+\frac{1}{c^2}\int_{p_0}^{p}\exp\left\{\sum_k \lambda_k\Phi_k(p')\right\}dp'.$$

Testing the Accuracy of the Spectral Representations

- The best causal representations of the equation of state, $\epsilon(p, \lambda_k)$, use basis functions $\Phi_k(p)$ made from Chebyshev polynomials.
- Test how accurately these equation of state representations work for nuclear-theory model neutron star equations of state.

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- For a given equation of state $\{\epsilon_i, p_i\}$ test the accuracy of the spectral representation by finding the spectral parameters λ_k that minimize the equation of state fitting errors, $\Delta(\lambda_k)$, defined by

$$\Delta^2(\lambda_k) = \frac{1}{N_{\rm eos}} \sum_{i=1}^{N_{\rm eos}} \left[\log\left(\frac{\epsilon(p_i, \lambda_k)}{\epsilon_i} \right) \right]^2 .$$

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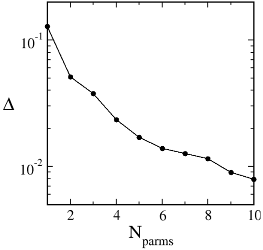
- Evaluate the average equation of state fitting errors, Δ, for a collection of 26 nuclear-theory based equations of state.



Testing the Spectral Solution of SSP^{-1}

- Test the spectral method for solving the SSP^{-1} using mock $\{M_a, R_a\}$ data computed from a known equation of state $\{\epsilon_i, p_i\}$.

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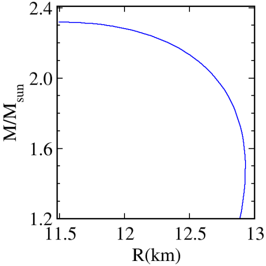
- The resulting spectral parameters λ_k determine an equation of state, $\epsilon(p, \lambda_k)$, that provides an approximate solution to SSP^{-1}.
- Evaluate the accuracy of this $\epsilon(p, \lambda_k)$ by measuring the fitting errors, Δ,

$$\Delta^2 = \frac{1}{N_{\rm EOS}}\sum_{i=1}^{N_{\rm EOS}}\left[\log\left(\frac{\epsilon(p_i,\lambda_k)}{\epsilon_i}\right)\right]^2$$

  to determine how well $\epsilon = \epsilon(p, \lambda_k)$ matches the exact tabulated equation of state $\{\epsilon_i, p_i\}$.

Testing the Spectral SSP^{-1} Solutions

- Test the solution of SSP^{-1} using mock data consisting of 10 exact $\{\tilde{M}_a, \tilde{R}_a\}$ data points from the MR curve based on the GM1L effective mean field nuclear-theory equation of state.



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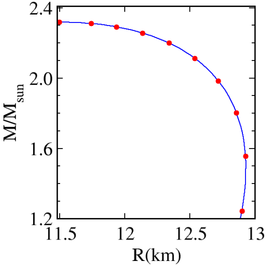

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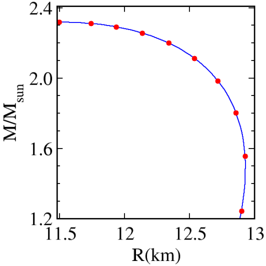

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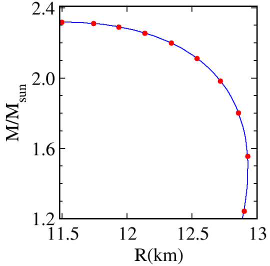
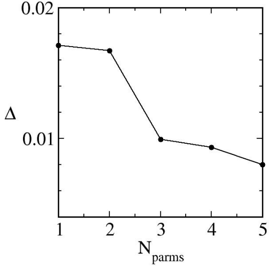


- Determine the spectral parameters λ_k that minimize the differences between the model observables $\{M(p_c^a, \lambda_k), R(p_c^a, \lambda_k)\}$ and the exact $\{\tilde{M}_a, \tilde{R}_a\}$ data.
- Evaluate the equation of state error Δ for the resulting $\epsilon(p, \lambda_k)$ based on the resulting λ_k.



Testing the Spectral SSP^{-1} Solutions II

- Next test the solution of SSP^{-1} using noisy mock mass-radius data consisting of $\{M_a, R_a\}$ data points constructed from the exact $\{\tilde{M}_a, \tilde{R}_a\}$:

$$
\begin{aligned}
M_a &= (1+\delta\mathcal{A})\tilde{M}_a, \\
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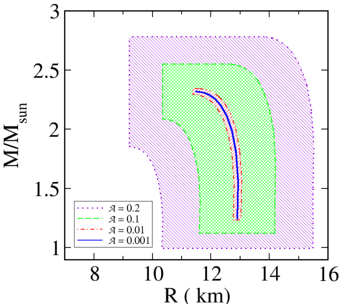
- Construct 1000 noisy $\{M_a, R_a\}$ curves for each noise amplitude $\mathcal{A} = \{0.001, 0.01, 0.1, 0.2\}$. These noisy mass-radius curves occupy the following regions of mass-radius space:



Testing the Spectral SSP^{-1} Solutions III

- Solve the SSP^{-1} to determine the best fit spectral parameters λ_k for each of the 1000 noisy $\{M_a, R_a\}$ curves in the collections with noise amplitudes $\mathcal{A} = \{0.001, 0.01, 0.1, 0.2\}$.

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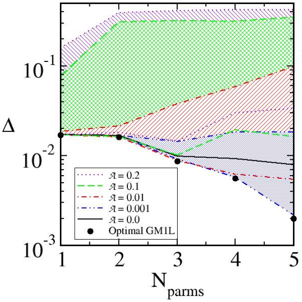
- Solve the SSP^{-1} to determine the best fit spectral parameters λ_k for each of the 1000 noisy $\{M_a, R_a\}$ curves in the collections with noise amplitudes $\mathcal{A} = \{0.001, 0.01, 0.1, 0.2\}$.
- Determine the accuracy of each SSP^{-1} solution by evaluating the equation of state errors Δ determined for the resulting λ_k.



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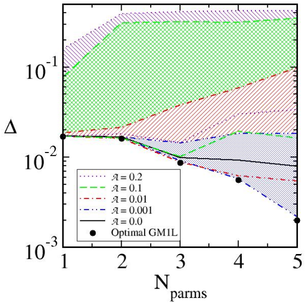
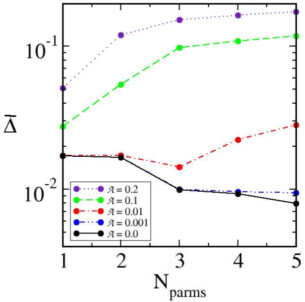
- Determine the accuracy of each SSP^{-1} solution by evaluating the equation of state errors Δ determined for the resulting λ_k.
- Then average the equation of state errors Δ over the 1000 noisy mass-radius curves in the ensemble with noise amplitude $\mathcal{A}$.





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- The neutron-star equation of state can be determined from noisy mass-radius data with an accuracy commensurate with the accuracy of the data.
- There is an optimal number of equation of state parameters to use when solving the inverse stellar structure problem using noisy mass-radius data. Somewhat counter intuitively, using too many parameters can result in less (not more) accurate equation of state models.