Solving Einstein's Equations the Generalized Harmonic Way

Lee Lindblom
Caltech

Collaborators: Larry Kidder, Keith Matthews, Robert Owen, Harald Pfeiffer, Oliver Rinne, Mark Scheel, Bela Szilagyi, Saul Teukolsky

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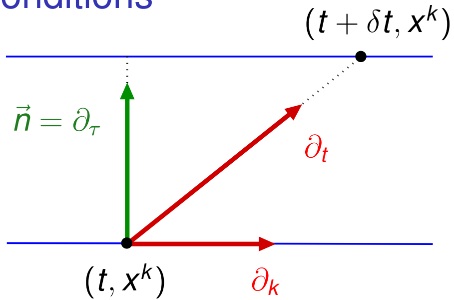
Outline of Talk:

- Methods of Specifying Gauge (Coordinates).
  - Generalized Harmonic (GH) Einstein Equations.
  - Constraint Damping.
- Boundary Conditions.
  - Constraint Preserving.
  - Physical.
- Moving Black Holes in a Spectral Code.
  - Dual Coordinate Frame Evolution.
  - Choosing Coordinates by Feedback Control.
- Gauge Drivers in the GH Einstein System.

Traditional ADM Gauge Conditions

- Construct a foliation of spacetime by spatial slices.
- Choose a time function with t = const. on these slices.
- Choose spatial coordinates, x^k, on each slice.



- Decompose the 4-metric ψ_{ab} into its 3+1 parts: $ds^2 = \psi_{ab}dx^a dx^b = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$.
- The lapse N and shift N^i measure how coordinates are laid out on spacetime:
$$\vec{n} = \partial_\tau = \frac{\partial t}{\partial \tau}\partial_t + \frac{\partial x^k}{\partial \tau}\partial_k, \quad = \frac{1}{N}\partial_t - \frac{N^k}{N}\partial_k.$$
- Spacetime coordinates are determined in the traditional ADM method by specifying the lapse N and shift N^i.

ADM Evolution System

- When the gauge is determined by specifying the lapse N and shift N^k, the Einstein system becomes a set of evolution equations for the spatial metric g_{ij} and extrinsic curvature K_{ij}:

$$\begin{aligned} \partial_t g_{ij} - N^k\partial_k g_{ij} &= -2NK_{ij} + \nabla_i N_j + \nabla_j N_i, \\ \partial_t K_{ij} - N^k\partial_k K_{ij} &= N\left({}^{(3)}R_{ij} - 2K_i{}^k K_{kj} + KK_{ij}\right) \\ &\quad -\nabla_i\nabla_j N + K_{ik}\partial_j N^k + K_{kj}\partial_i N^k. \end{aligned}$$

- The Einstein equations also include constraints:

$$\begin{aligned} 0 &= \mathcal{M}_{\hat{t}} \equiv {}^{(3)}R - K_{ij}K^{ij} + K^2, \\ 0 &= \mathcal{M}_i \equiv \nabla^k K_{ki} - \nabla_i K. \end{aligned}$$

- This traditional form of the Einstein equations is not hyperbolic, and numerical solutions are found to suffer from generic constraint violating instabilities.

Generalized Harmonic Gauge Conditions

- An alternate way to specify the gauge (i.e. coordinates) in the Einstein equations is through the gauge source function H^a:
- Let H^a denote the function obtained by the action of the covariant scalar wave operator on the coordinates x^a:

$$H^a \equiv \nabla^c\nabla_c x^a \;=\; \psi^{bc}(\partial_b\partial_c x^a - \Gamma^e_{bc}\partial_e x^a) = -\Gamma^a,$$

where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

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  where $\Gamma^a = \psi^{bc} \Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

- Specifying coordinates by the *generalized harmonic* (GH) method can be accomplished by choosing a gauge-source function $H^a(x, \psi)$, and requiring that

$$H^a(x, \psi) = -\Gamma^a = \partial_b \left(\sqrt{-\psi}\psi^{ab}\right) / \sqrt{-\psi}.$$

Einstein's Equation with the GH Method

- The spacetime Ricci tensor can be written as:

$$R_{ab} \quad = \quad -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + F_{ab}(\psi,\partial\psi),$$

where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$.

- The Generalized Harmonic Einstein equation is obtained by replacing Γ_a with $-H_a(x,\psi) = -\psi_{ab}H^b(x,\psi)$:

$$R_{ab} - \nabla_{(a}\left[\Gamma_{b)} + H_{b)}\right] = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + F_{ab}(\psi,\partial\psi).$$

- The vacuum GH Einstein equation, $R_{ab} = 0$ with $\Gamma_a + H_a = 0$, is therefore manifestly hyperbolic, having the same principal part as the scalar wave equation:

$$0 = \nabla_a\nabla^a\Phi = \psi^{ab}\partial_a\partial_b\Phi + F(\partial\Phi).$$

Gauge and Constraints in Electromagnetism

- The usual representation of the vacuum Maxwell equations split into evolution equations and constraints:

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0,$$
$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0.$$

  These equations are often written in the more compact 4-dimensional notation: $\nabla^a F_{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$, where F_{ab} has components $\vec{E}$ and $\vec{B}$.

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- Maxwell's equations are often re-expressed in terms of a vector potential $F_{ab} = \nabla_a A_b - \nabla_b A_a$:

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- This form of Maxwell's equations is manifestly hyperbolic as long as the gauge is chosen correctly, e.g., let $\nabla^a A_a = H(x, t)$, giving:

$$\nabla^a \nabla_a A_b \equiv \left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right) A_b = \nabla_b H.$$

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- Modify evolution equations by adding multiples of the constraints:

$$\nabla^a\nabla_a A_b = \nabla_b H + \gamma_0 t_b \mathcal{C} = \nabla_b H + \gamma_0 t_b (\nabla^a A_a - H).$$

- These changes also affect the constraint evolution equation,

$$\nabla^a\nabla_a \mathcal{C} - \gamma_0 t^b \nabla_b \mathcal{C} = 0,$$

so constraint violations are damped when $\gamma_0 > 0$.

Generalized Harmonic Evolution System

- Frans Pretorius wrote a very nice second order finite difference AMR code to solve the generalized harmonic Einstein equations:

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

where $\mathcal{C}_a = H_a + \Gamma_a$. Unfortunately initial code was very unstable.

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  where $\mathcal{C}_a = H_a + \Gamma_a$. Unfortunately initial code was very unstable.
- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by the derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv \left[R_{ab} - \frac{1}{2}\psi_{ab}R\right]n^b = \left[\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\right]n^b.$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 \;=\; R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[n_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\, n^c\, \mathcal{C}_c\right],$$

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  where n^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.
- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[n_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \frac{1}{2}\gamma_0\, n_a\mathcal{C}^c\mathcal{C}_c.$$

  This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

First-Order Einstein Evolution System

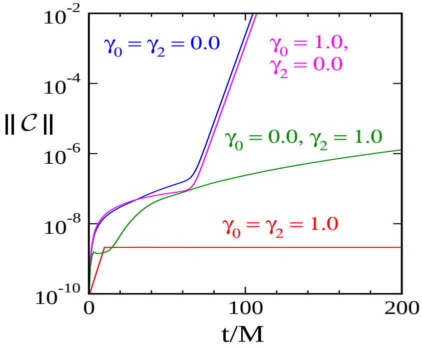
- Introduce new fields Π_{ab} and Φ_{iab} representing the time and space derivatives of the metric ψ_{ab}.
- Our code solves a first-order representation of the GH Einstein evolution system:

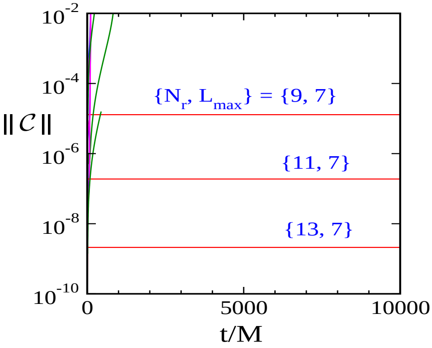
$$\begin{aligned}
&\partial_t\psi_{ab} = -N\Pi_{ab} + N^i\Phi_{iab},\\
&\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} + \gamma_2 N^k\partial_k\psi_{ab} \simeq 0,\\
&\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\Psi_{ab} \simeq 0.
\end{aligned}$$

- Violations of the additional constraint, $\mathcal{C}_{iab} = \Phi_{iab} - \partial_i\psi_{ab}$, are suppressed on the timescale $1/\gamma_2$ by this evolution system.
- This evolution system can be written very abstractly as: $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\partial_k u^\beta = F^\alpha(u)$, where $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{iab}\}$.
- This system is symmetric hyperbolic because there exists a positive definite symmetric $S_{\alpha\beta}$ that symmetrizes the characteristic matrices: $A^k_{\alpha\beta} = A^k_{\beta\alpha} = S_{\alpha\gamma}A^{k\,\gamma}{}_\beta$.

Numerical Tests of the First-Order GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.





- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

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Boundary Conditions

- Boundary conditions are straightforward to formulate for first-order hyperbolic evolutions systems,

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- Find the eigenvectors of the characteristic matrix $s_k A^{k\,\alpha}{}_\beta$ at each boundary point:

$$e^{\hat\alpha}{}_\alpha\; s_k A^{k\,\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta,$$

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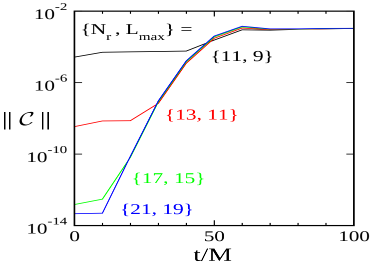
- A boundary condition must be imposed on each incoming characteristic field (*i.e.* every field with $v_{(\hat\alpha)} < 0$), and must not be imposed on any outgoing field (*i.e.* any field with $v_{(\hat\alpha)} > 0$).

Evolutions of a Perturbed Schwarzschild Black Hole

- A black-hole spacetime is perturbed by an incoming gravitational wave that excites quasi-normal oscillations.
- Use boundary conditions that *Freeze* the remaining incoming characteristic fields.
- The resulting outgoing waves interact with the boundary of the computational domain and produce constraint violations.



Lapse Movie   Constraint Movie

Constraint Evolution for the First-Order GH System

- The evolution of the constraints, $c^A = \{\mathcal{C}_a, \mathcal{C}_{kab}, \mathcal{M}_a \approx n^c\partial_c\mathcal{C}_a, \mathcal{C}_{ka} \approx \partial_k\mathcal{C}_a, \mathcal{C}_{klab} = \partial_{[k}\Phi_{l]ab}\}$ are determined by the evolution of the fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$:

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$$\partial_t c^A + A^{k\,A}{}_B(u)\partial_k c^B = F^A{}_B(u, \partial u)\, c^B.$$
- This constraint evolution system is symmetric hyperbolic with principal part:
$$\begin{aligned}
\partial_t\mathcal{C}_a &\simeq 0,\\
\partial_t\mathcal{M}_a - N^k\partial_k\mathcal{M}_a - Ng^{ij}\partial_i\mathcal{C}_{ja} &\simeq 0,\\
\partial_t\mathcal{C}_{ia} - N^k\partial_k\mathcal{C}_{ia} - N\partial_i\mathcal{M}_a &\simeq 0,\\
\partial_t\mathcal{C}_{iab} - (1+\gamma_1)N^k\partial_k\mathcal{C}_{iab} &\simeq 0,\\
\partial_t\mathcal{C}_{ijab} - N^k\partial_k\mathcal{C}_{ijab} &\simeq 0.
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\end{aligned}$$
- An analysis of this system shows that all of the constraints are damped in the WKB limit when $\gamma_0 > 0$ and $\gamma_2 > 0$. So, this system has constraint suppression properties that are similar to those of the Pretorius (and Gundlach, et al.) system.

Constraint Preserving Boundary Conditions

- Construct the characteristic fields, $\hat{c}^{\hat{A}} = e^{\hat{A}}{}_{A} c^{A}$, associated with the constraint evolution system, $\partial_t c^A + A^{k\,A}{}_{B} \partial_k c^B = F^A{}_B c^B$.

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- The incoming characteristic fields mush vanish on the boundaries, $\hat{c}^- = 0$, if the influx of constraint violations is to be prevented.
- The constraints depend on the primary evolution fields (and their derivatives). We find that $\hat{c}^-$ for the GH system can be expressed:

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$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$

- Set boundary conditions on the fields $\hat{u}^-$ by requiring

$$d_\perp \hat{u}^- = -\hat{F}(u, d_\parallel u).$$

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  - This condition is translated into a BC by expressing Ψ_0 in terms of the incoming characteristic fields: $\Psi_0 = d_\perp \hat{u}^- + \hat{F}(u, d_{\|} u)$.

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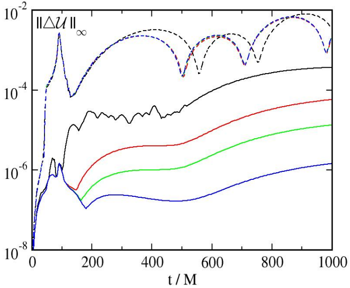
- Constraint preserving and physical boundary conditions involve derivatives of $\hat{u}^\alpha$, and standard well-posedness proofs fail.
- Oliver Rinne (2006) used Fourier-Laplace analysis to show that these BC satisfy the Kreiss (1970) condition which is necessary for well-posedness (but not sufficient for this type of BC).

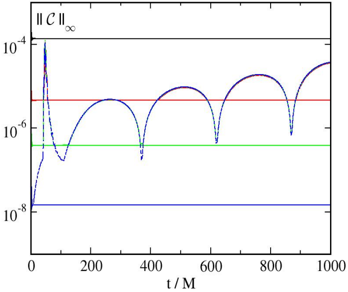
Numerical Tests of Boundary Conditions

- Compare the solution obtained on a “small” computational domain with a reference solution obtained on a “large” domain where the boundary is not in causal contact with the comparison region.

Solution Differences



Constraints



- Solutions using “Freezing” BC (dashed curves) have differences and constraints that do not converge to zero.
- Solutions using constraint preserving and physical BC (solid curves) have much smaller differences and constraints that converge to zero.

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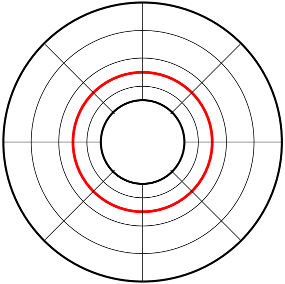
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.



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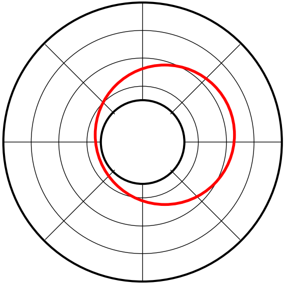

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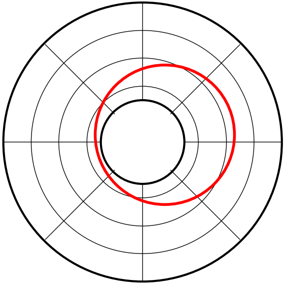

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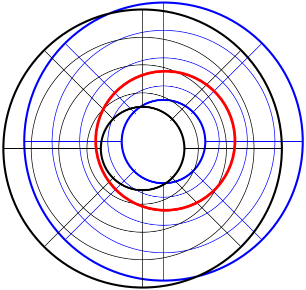

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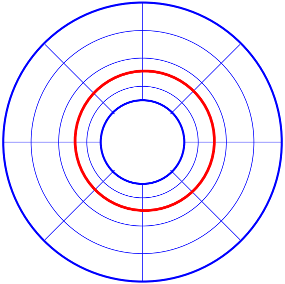

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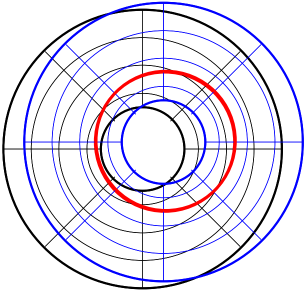

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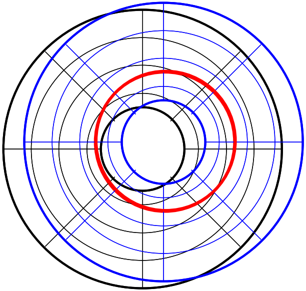

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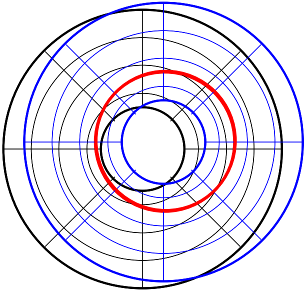

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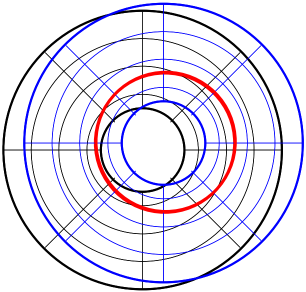

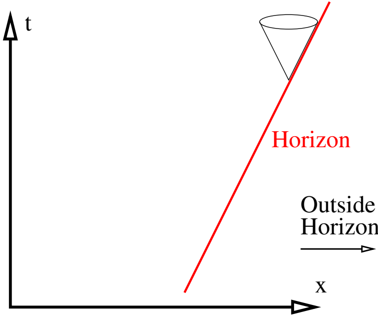

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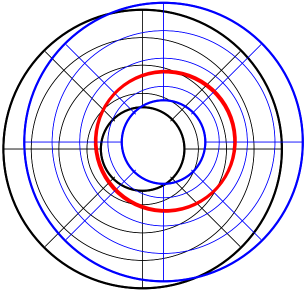

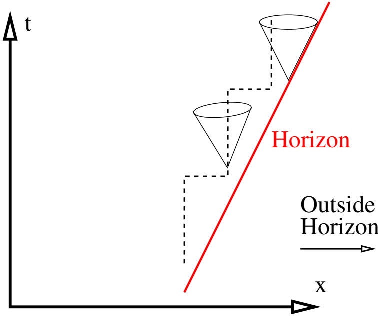

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- Problems:
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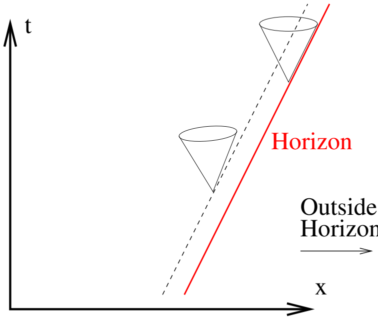
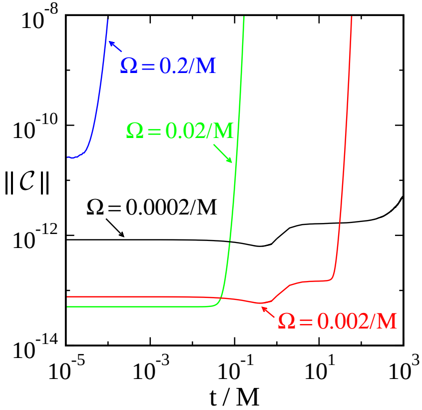
- Solution:
  Choose coordinates that smoothly track the location of the black hole.

  For a black hole binary this means using coordinates that rotate with respect to inertial frames at infinity.



Evolving Black Holes in Rotating Frames

- Coordinates that rotate with respect to the inertial frames at infinity are needed to track the horizons of orbiting black holes.
- Evolutions of Schwarzschild in rotating coordinates are unstable.



- Evolutions shown use a computational domain that extends to $r = 1000M$.
- Angular velocity needed to track the horizons of an equal mass binary at merger is about $\Omega \approx 0.2/M$.
- Problem caused by asymptotic behavior of metric in rotating coordinates: $\psi_{tt} \sim \rho^2\Omega^2$, $\psi_{ti} \sim \rho\Omega$, $\psi_{ij} \sim 1$.

Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar a} = \{\bar t, x^{\bar\imath}\}$, to define field components, $u^{\bar\alpha} = \{\psi_{\bar a\bar b}, \Pi_{\bar a\bar b}, \Phi_{\bar\imath\bar a\bar b}\}$, and the same coordinates to determine these components by solving Einstein's equation for $u^{\bar\alpha} = u^{\bar\alpha}(x^{\bar a})$:

$$\partial_{\bar t} u^{\bar\alpha} + A^{\bar k\,\bar\alpha}{}_{\bar\beta}\partial_{\bar k} u^{\bar\beta} = F^{\bar\alpha}.$$

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- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^{\bar a})$, to represent these components as functions, $u^{\bar\alpha} = u^{\bar\alpha}(x^a)$.

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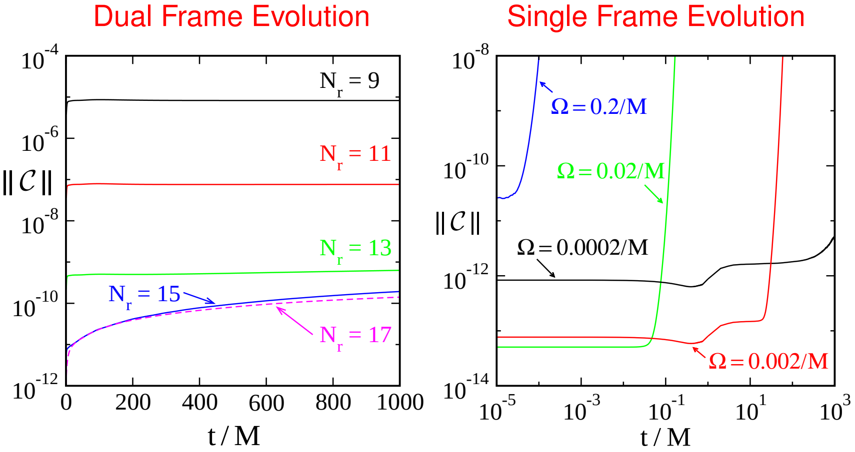
- These functions are determined by solving the transformed Einstein equation:

$$\partial_t u^{\bar\alpha} + \left[\frac{\partial x^i}{\partial \bar t}\delta^{\bar\alpha}{}_{\bar\beta} + \frac{\partial x^i}{\partial x^{\bar k}} A^{\bar k\,\bar\alpha}{}_{\bar\beta}\right]\partial_i u^{\bar\beta} = F^{\bar\alpha}.$$

Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:



- Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/M$ on a domain with outer radius $r = 1000M$.

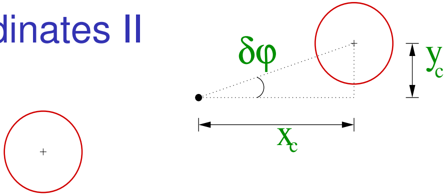
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- A coordinate transformation from inertial coordinates, $(\bar{x}, \bar{y}, \bar{z})$, to co-moving coordinates (x, y, z), consisting of a rotation followed by an expansion,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{a(\bar{t})} \begin{pmatrix} \cos\varphi(\bar{t}) & -\sin\varphi(\bar{t}) & 0 \\ \sin\varphi(\bar{t}) & \cos\varphi(\bar{t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix},$$

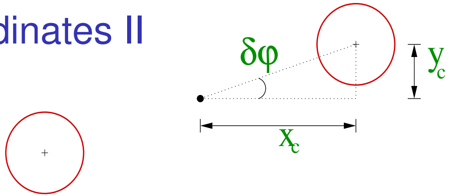
  with $t = \bar{t}$, is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions $a(\bar{t})$ and $\varphi(\bar{t})$.
- Since the motions of the holes are not known *a priori*, the functions $a(\bar{t})$ and $\varphi(\bar{t})$ must be chosen dynamically and adaptively as the system evolves.

Horizon Tracking Coordinates II



- Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.
- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.

Horizon Tracking Coordinates II



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- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.
- Changing the map parameters by the small amounts, δa and $\delta\varphi$, results in associated small changes in δQ^x and δQ^y:

$$\delta Q^x = -\delta a, \qquad \delta Q^y = -\delta\varphi.$$

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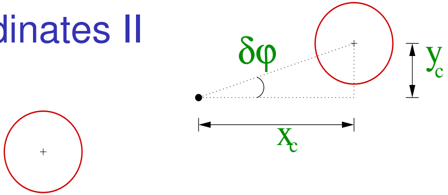

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- Measure the quantities $Q^y(t)$, $dQ^y(t)/dt$, $d^2Q^y(t)/dt^2$, and set

$$\frac{d^3\varphi}{dt^3} = \lambda^3 Q^y + 3\lambda^2\frac{dQ^y}{dt} + 3\lambda\frac{d^2Q^y}{dt^2} = -\frac{d^3Q^y}{dt^3}.$$

  The solutions to this "closed-loop" equation for Q^y have the form $Q^y(t) = (At^2 + Bt + C)e^{-\lambda t}$, so Q^y always decreases as $t \to \infty$.

Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times $t = t_i$.
- In the time interval $t_i < t < t_{i+1}$ we set:

$$\begin{aligned}\varphi(t) &= \varphi_i + (t - t_i)\frac{d\varphi_i}{dt} + \frac{(t-t_i)^2}{2}\frac{d^2\varphi_i}{dt^2} \\ &\quad + \frac{(t-t_i)^3}{2}\left(\lambda\frac{d^2Q_i^y}{dt^2} + \lambda^2\frac{dQ_i^y}{dt} + \lambda^3\frac{Q_i^y}{3}\right),\end{aligned}$$

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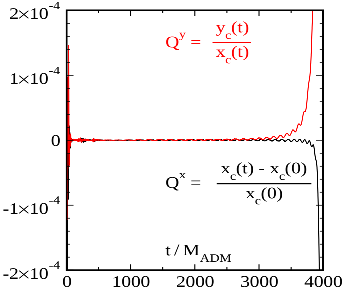
  where Q^x, Q^y, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop equation at $t = t_i$.

- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.



Outline of Talk:

- Methods of Specifying Gauge (Coordinates).
  - Generalized Harmonic (GH) Einstein Equations.
  - Constraint Damping.
- Boundary Conditions.
  - Constraint Preserving.
  - Physical.
- Moving Black Holes in a Spectral Code.
  - Dual Coordinate Frame Evolution.
  - Choosing Coordinates by Feedback Control.
- Gauge Drivers in the GH Einstein System.

Gauge Conditions and Hyperbolicity

- The GH Einstein equations may be written (abstractly) as

$$\psi^{cd}\partial_c\partial_d\psi_{ab} \;=\; \nabla_a H_b + \nabla_b H_a + Q_{ab}(\psi, \partial\psi).$$

- These equations are manifestly hyperbolic when H^a is specified as a function of x^a and ψ_{ab}: $\quad H^a = H^a(x, \psi)$.

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- These equations are manifestly hyperbolic when H^a is specified as a function of x^a and ψ_{ab}: $H^a = H^a(x,\psi)$.
- Unfortunately, most gauge conditions found useful in numerical relativity are conditions on ψ_{ab} and $\partial_c\psi_{ab}$.
- The GH Einstein equations are typically not hyperbolic for gauge conditions of this type: $H^a = H^a(x,\psi,\partial\psi)$.

Solution: Gauge Driver Equations

- Elevate H_a to the status of a dynamical field (Pretorius) and evolve it along with the spacetime metric ψ_{ab}:

$$
\begin{aligned}
&\text{Gauge Driver :} \qquad && \psi^{cd}\partial_c\partial_d H_a &&= Q_a(x, H, \partial H, \psi, \partial\psi),\\
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- Any gauge driver of this form makes the combined Einstein–Gauge system hyperbolic.
- Choose Q_a so that H_a evolves toward the desired gauge target F_a as the system evolves: $H_a \to F_a$.
- We have shown that the simple gauge driver:

$$\psi^{cd}\partial_c\partial_d H_a = Q_a = \mu^2(H_a - F_a) + 2\mu N^{-1}\partial_t H_a + ...$$

drives $H_a \to F_a$ for some of the standard numerical relativity gauges: *Phys. Rev. D* **77** 084001 (2008).

Recent Gauge Driver Improvements

- Replace the wave operator in the gauge driver by a flat-space wave operator:

$$\eta^{cd}\partial_c\partial_d H_a \quad = \quad Q_a(x, H, \partial H, \psi, \partial\psi).$$

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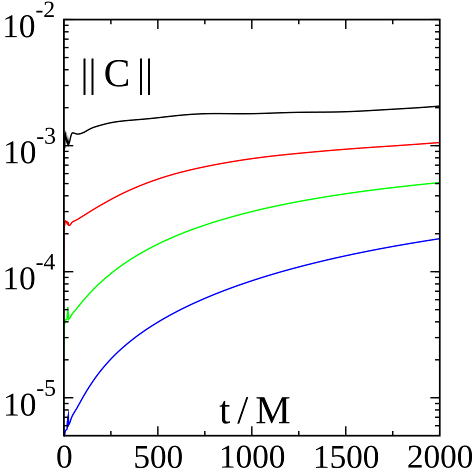
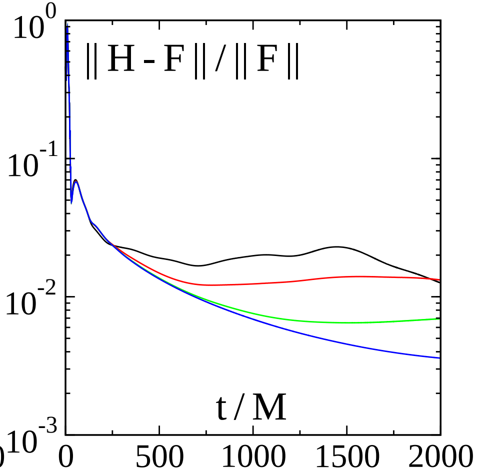
- The gauge field H_a transforms like the trace of a connection. Evolutions done in a co-moving reference frame need an appropriate Hessian term added to F_a:

$$F_a \to F_a - \psi_{ab}\frac{\partial^2 x^b}{\partial x^{\bar{b}}\partial x^{\bar{c}}}\psi^{\bar{b}\bar{c}}.$$

Testing the Improved Gauge Driver System:

- Initial Data: use Schwarzschild with perturbed lapse and shift.
- Gauge Driver: use F_a representing one of the Bona-Masso slicing conditions and one of the Γ-driver shift conditions.





In Progress: Binary Black Hole Evolutions

- Our gauge driver system is now robust enough to perform binary black hole evolutions.
- Which gauge conditions are both stable and effective for performing BBH mergers?
- BBH mergers have been performed using driver versions of the following gauge conditions,

$$\begin{aligned} \partial_t N - N^k \partial_k N &= -\lambda N K, \\ \partial_t N^i &= \nu \left[{}^{(3)}\tilde{\Gamma}^i - \eta \Upsilon^i \right], \\ \partial_t \Upsilon^i + \eta \Upsilon^i &= {}^{(3)}\tilde{\Gamma}^i. \end{aligned}$$

Summary

- Generalized Harmonic method produces manifestly hyperbolic representations of the Einstein equations for any choice of coordinates (when imposed in the appropriate way).
- Constraint damping makes the modified GH equations stable for numerical simulations.
- Constraint preserving and physical boundary conditions ensure that waves propagate through computational boundaries without (much) reflection.
- Dual coordinate frame evolution makes evolutions stable in coordinates that track the black hole motions.
- Feedback control systems can be used to construct co-moving coordinates that accurately track the black hole motions.
- Gauge drivers allow a wide range of useful gauge conditions in the generalized harmonic framework.