What is the relativistic inverse stellar structure problem (SSP$^{-1}$)?

Can spectral methods provide a more effective way to solve it?
Relativistic Stellar Structure Problem (SSP)

- Given an equation of state, \( \epsilon = \epsilon(p) \), solve Einstein’s equations,
  \[
  \frac{dm}{dr} = 4\pi r^2 \epsilon, \\
  \frac{dp}{dr} = -(\epsilon + p) \frac{m + 4\pi r^3 p}{r(r - 2m)},
  \]
to determine the structures of relativistic stars.
- Find the radius \( p(R) = 0 \) and mass \( M = m(R) \) for each star.
- SSP can be thought of as a map from the equation of state \( \epsilon = \epsilon(p) \) to the M-R curve \( \{ R(p_c), M(p_c) \} \).
Relativistic Inverse Stellar Structure Problem (\(\text{SSP}^{-1}\))

- When the equation of state is well understood – as in white dwarf stars – the standard stellar structure problem is useful.
- When the equation of state is poorly known – as in neutron stars – the inverse stellar structure problem may be more interesting.

SSP\(^{-1}\) finds the equation of state \(\epsilon = \epsilon(p)\) from a given mass-radius curve.

SSP\(^{-1}\) can be thought of as a map from the M-R curve \(\{R(p_c), M(p_c)\}\) to the equation of state \(\epsilon = \epsilon(p)\).

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\begin{align*}
\log \epsilon(p) & \rightarrow \\
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- SSP\(^{-1}\) can be thought of as a map from the M-R curve \(\{R(p_c), M(p_c)\}\) to the equation of state \(\epsilon = \epsilon(p)\).
Standard Solution to SSP$^{-1}$

- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.
- Assume the complete M-R curve is known.

\[ \text{M} \quad \text{R} \quad \{R_i, M_i\} \quad \rightarrow \quad \log \epsilon(p) \quad \{p_i, \epsilon_i\} \quad \log p \]
Standard Solution to SSP$^{-1}$

- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.
- Assume the complete M-R curve is known.
- Choose a new point on the M-R curve, $\{R_{i+1}, M_{i+1}\}$, having slightly larger central density.

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- Integrate Einstein’s equations through the outer parts of the star, to determine the mass and radius, \( \{r_{i+1}, m_{i+1}\} \), of the core.
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- Integrate Einstein’s equations through the outer parts of the star, to determine the mass and radius, $\{r_{i+1}, m_{i+1}\}$, of the core.
- Use a power series solution of Einstein’s equations in the core to determine the central pressure and density, $\{p_{i+1}, \epsilon_{i+1}\}$.
Can the Standard Solution to SSP\(^{-1}\) be Improved?

- Standard solution to the relativistic SSP\(^{-1}\) finds the equation of state, \(\epsilon = \epsilon(p)\), represented as a table: \(\{p_i, \epsilon_i\}\) for \(i = 1, \ldots, N\).

- Standard solution has several weaknesses:
  - Solution converges (slowly) with the number of points, as \(N^{-p}\).
  - Each equation of state point found, \(\{p_i, \epsilon_i\}\), requires the knowledge of a separate M-R curve point, \(\{R_i, M_i\}\).
  - Accurate M-R curve points \(\{R_i, M_i\}\) for neutron stars are scarce.

Spectral numerical methods typically converge more rapidly, and represent functions more efficiently than finite difference methods. Can spectral methods provide better solutions to the SSP\(^{-1}\)?

Lee Lindblom (Caltech)
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- Can spectral methods provide better solutions to the SSP\(^{-1}\)?
Assume the equation of state can be written in the form 
\[ \epsilon = \epsilon(p, \lambda_\alpha), \] 
where the \( \lambda_\alpha \) are a set of parameters.

For example, the equation of state could be written as a spectral expansion, 
\[ \epsilon = \epsilon(p, \lambda_\alpha) = \sum_\alpha \lambda_\alpha \Phi_\alpha(p), \] 
where the \( \Phi_\alpha(p) \) are spectral basis functions.
Outline for Solving SSP\(^{-1}\) Using Spectral Methods

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- For a given equation of state, i.e. a particular choice of \( \lambda_\alpha \), solve
  the SSP to obtain the M-R curve: \( \{R(p_c, \lambda_\alpha), M(p_c, \lambda_\alpha)\} \).
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  \[ \{ R(p_c, \lambda_\alpha), M(p_c, \lambda_\alpha) \} . \]

- Given a set of points from the “real” M-R curve, \( \{ R_i, M_i \} \), choose the parameters \( \lambda_\alpha \) and \( p_i \) that minimize the difference measure:

  \[
  \Delta^2_{MR} = \frac{1}{N} \sum_{i=1}^{N} \left \{ \left \{ \log \left ( \frac{R(p_i, \lambda_\alpha)}{R_i} \right ) \right \}^2 + \left \{ \log \left ( \frac{M(p_i, \lambda_\alpha)}{M_i} \right ) \right \}^2 \right \}
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- Resulting \(\lambda_\alpha\) for \(\alpha = 1, \ldots, N\) determine an equation of state, 
  \(\epsilon = \epsilon(p, \lambda_\alpha)\), that provides an approximate solution of SSP\(^{-1}\).
Faithful Spectral Expansions of the Equation of State

- Physical equations of state, \( \epsilon = \epsilon(p) \), are positive monotonic increasing functions. These do not form a vector space.
- The representation, \( \epsilon = \epsilon(p, \lambda_\alpha) = \sum_\alpha \lambda_\alpha \Phi_\alpha(p) \), is not faithful.
- Faithful here means that every choice of \( \lambda_\alpha \) corresponds to a possible physical equation of state, and every equation of state can be represented by such an expansion.
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- Faithful spectral expansions of the adiabatic index do exist:

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\Gamma(p) = \frac{\epsilon c^2 + p}{pc^2} = \exp \left[ \sum_\alpha \gamma_\alpha \Phi_\alpha(p) \right]. 
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- Every equation of state is determined by the adiabatic index $\Gamma(p)$:
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  \mu(p) = \exp \left[ \int_{p_0}^{p} \frac{dp'}{p' \Gamma(p')} \right],
  \]
  \[
  \epsilon(p) = \frac{\epsilon_0}{\mu(p)} + \frac{1}{\mu(p)} \int_{p_0}^{p} \frac{\mu(p')}{c^2 \Gamma(p')} dp'.
  \]
Neutron Star Equations of State

- What choice of spectral basis functions $\Phi_\alpha(p)$ provide efficient representations of realistic neutron star equations of state?

$$
\Gamma(p) = \exp\{\sum_\alpha \gamma_\alpha \left[ \log\left(\frac{p}{p_0}\right)\right]_\alpha\}.
$$

Test its effectiveness by constructing expansions that minimize,

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\Delta^2 \epsilon = \frac{1}{N} \sum_{i=1}^{N} \left[ \log\left(\frac{\epsilon(p_i,\gamma_\alpha)}{\epsilon_i}\right) \right]^2
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Spectral Solution of SSP\(^{-1}\)

- Next step is to test this spectral approach by finding the approximate solution to SSP\(^{-1}\) for realistic neutron star models.

- Choose points \(\{R_i, M_i\}\) from realistic neutron star models, then fix the spectral expansion coefficients \(\gamma_\alpha\) by minimizing,

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\Delta^2_{MR} = \frac{1}{N} \sum_{i=1}^{N} \left\{ \left[ \log \left( \frac{M(p_i, \gamma_\alpha)}{M_i} \right) \right]^2 + \left[ \log \left( \frac{R(p_i, \gamma_\alpha)}{R_i} \right) \right]^2 \right\}.
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Finally evaluate \(\Delta^2\),

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to determine how well the spectral expansion \(\epsilon = \epsilon(p, \gamma_\alpha)\), matches the original realistic neutron star equation of state \(\epsilon = \epsilon(p)\).

Unfortunately, I have run out of time.

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