Gauge conditions are specified in the GH Einstein system by the gauge source function $H^a \equiv \nabla^c \nabla_c x^a$.

How do you choose $H^a$ in a way that provides a reasonable coordinate system and keeps the GH Einstein system hyperbolic?
Gauge Conditions and Hyperbolicity

- The GH Einstein equations may be written (abstractly) as
  \[ \psi^{cd} \partial_c \partial_d \psi_{ab} = \partial_a H_b + \partial_b H_a + Q_{ab}(H, \psi, \partial \psi). \]

- These equations are manifestly hyperbolic when \( H^a \) is specified as a function of \( x^a \) and \( \psi_{ab} \): \( H_a = H_a(x, \psi) \).
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- Elevate \( H_a \) to the status of an independent dynamical field, by choosing an evolution equation for \( H_a \) whose solutions are the desired gauge conditions.
Pretorius proposed evolving $H_a$ using an equation of the form:

\[
\psi^{cd} \partial_c \partial_d H_a = Q_a(x, H, \partial H, \psi, \partial \psi),
\]

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Dynamically very rich, often producing solutions with “interesting” gauge dynamics. This is bad.
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Introduce a new simpler gauge driver:

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Also hyperbolic, but not obviously so.
Choose $Q_a$ so that all solutions $H_a$ evolve toward a target $F_a$:

\[
t^c \partial_c H_a = -\mu (H_a - F_a) + \ldots
\]

New gauge driver has fewer “interesting” solutions.
Damped-Wave Gauge Conditions

- Spatial coordinates satisfying $\nabla^c \nabla_c x^i = 2\mu_S t^c \partial_c x^i$ are called damped-wave coordinates.
- Choose target $F^i = 2\mu_S t^i = -2\mu_S N^{-1} N^i$. 
Damped-Wave Gauge Conditions

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- Choose target $F^i = 2\mu_S t^i = -2\mu_S N^{-1} N^i$.
- The time-component $t^a H_a$ related to spacetime metric by constraints of GH Einstein system:
  \[
  t^a H_a = t^a \partial_a \log \left( \frac{\sqrt{g}}{N} \right) + N^{-1} \partial_k N^k.
  \]
- Choose target $t^a F_a$ to suppress growth in $g = \det g_{ij}$:
  \[
  t^a F_a = -2\mu_L \log \left( \frac{\sqrt{g}}{N} \right).
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- This condition on $t^a H_a = t^a F_a$ is also a damped-wave equation for lapse $N$. 

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Generalized Harmonic Gauge Drivers 
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- This condition on $t^a H_a = t^a F_a$ is also a damped-wave equation for lapse $N$.
- Combined expression for damped-wave target $F_a$:
  \[ F_a = 2\mu_L \log \left( \frac{\sqrt{g}}{N} \right) t_a - 2\mu_S N^{-1} g_{ai} N^i. \]
Testing New Gauge-Driver System:

- **Gauge Driver:** \( \partial_t H_a = -\mu N(H_a - F_a) + \ldots \)
- **Target Gauge:** \( F_a \) representing damped-wave gauge.
- **Initial Data:** Schwarzschild with perturbed lapse and shift.

![Graphs showing the relationship between \( \| H - F \| / \| F \| \) and \( t/M \) for different values of \( \mu \) and \( N_r, L \).]
Summary

- The new gauge driver $\partial_t H_a = -\mu N (H_a - F_a) + \ldots$ allows hyperbolic implementations of a wide variety of gauge conditions.
- Numerical tests show the new gauge driver is effective.

Tests of numerous gauge conditions found the damped-wave gauge very stable and useful for black-hole evolutions. Binary black-hole systems have been evolved successfully through the last orbits plus merger using this new gauge driver and the damped-wave gauge condition.
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