Mathematrical Structure of Einstein's Equation in the Generalized Harmonic Formalism I

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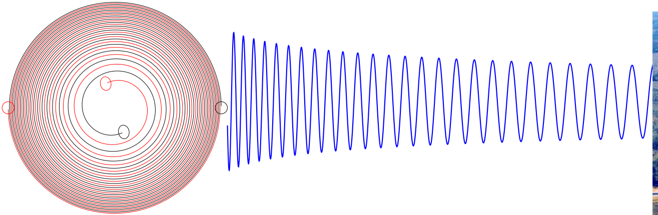

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- Strongest waves (and therefore the most easily detectable waves) are emitted as the two black holes merge into a single hole.
- Full non-linear numerical relativity is needed to construct accurate model waveforms for these spacetimes.

Why Is Numerical Relativity So Difficult?

- Very big computational problem:
  - Must evolve ~ 50 dynamical fields (spacetime metric plus all first derivatives).
  - Must accurately resolve features on many scales from black hole horizons $r \sim M$ to emitted waves $r \sim 100M$.
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  - Evolutions must be stable and accurate for very long times $t \sim 10^5 M$.
- Dynamics of the binary black hole problem is driven by delicate adjustments to orbit due to emission of gravitational waves. Very high accuracy is needed to represent these effects correctly.
- Many representations of the Einstein equations have mathematically ill-posed initial value and/or initial-boundary value problems.
- Constraint violating instabilities destroy stable numerical solutions in many well-posed forms of the equations.

General Relativity Theory

- The spacetime metric ψ_{ab} is determined by Einstein's equation:
  $$R_{ab} - \tfrac{1}{2} R \psi_{ab} = 8\pi T_{ab},$$
  where R_{ab} is the Ricci curvature tensor associated with ψ_{ab}, $R = \psi^{ab} R_{ab}$ is the scalar curvature, and T_{ab} is the stress-energy tensor of the matter present in spacetime.

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- For "vacuum" spacetimes (like binary black hole systems) $T_{ab} = 0$, so Einstein's equations can be reduced to $R_{ab} = 0$.
- The Ricci curvature R_{ab} is determined by derivatives of the metric:
  $$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{bc} + \Gamma^c{}_{cd}\Gamma^d{}_{ab} - \Gamma^c{}_{ad}\Gamma^d{}_{bc},$$
  where $\Gamma^c{}_{ab} = \frac{1}{2}\psi^{cd}(\partial_a \psi_{db} + \partial_b \psi_{da} - \partial_d \psi_{ab})$.
- The Ricci tensor therefore depends on the spacetime metric and its first and second derivatives:
  $$R_{ab} = R_{ab}(\partial\partial\psi, \partial\psi, \psi).$$

General Relativity Theory II

- Einstein's equations are second-order PDEs that (should, hopefully) determine the spacetime metric, e.g. in vacuum

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  - gauge freedom,
  - constraints.

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- The important fundamental ideas needed to understand these questions are:
  - gauge freedom,
  - constraints.
- Maxwell's equations are a simpler system in which these same fundamental issues play analogous roles.

Gauge and Hyperbolicity in Electromagnetism

- The usual representation of the vacuum Maxwell equations split into evolution equations and constraints:

  $$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \quad \partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \quad \vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0.$$

  These equations are often written in the more compact 4-dimensional form $\nabla^a F_{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$, where the antisymmetric F_{ab} has components $\vec{E}$ and $\vec{B}$.

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- Maxwell's equations can be solved in part by introducing a vector potential $F_{ab} = \nabla_a A_b - \nabla_b A_a$. This reduces the system to the single equation: $\nabla^a \nabla_a A_b - \nabla_b \nabla^a A_a = 0.$

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- The choice of Λ effects $\nabla_a A^a$, so fix Λ by setting $\nabla_a A^a = H$, where $H(A, x, t)$ can be chosen arbitrarily.
- The resulting Maxwell equations are manifestly hyperbolic for all choices of $H(A, x, t)$:

  $$\nabla^a\nabla_a A_b = \nabla_b H.$$

Gauge and Hyperbolicity in General Relativity

- The spacetime Ricci curvature tensor can be written as:

$$R_{ab} \;=\; -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi, \partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi_{ad}\psi^{bc}\Gamma^d{}_{bc}$.

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- One way to impose the needed gauge conditions is to specify H^a, the source term for a wave equation for each coordinate x^a:

$$H^a = \nabla^c\nabla_c x^a = \psi^{bc}(\partial_b\partial_c x^a - \Gamma^e{}_{bc}\partial_e x^a) = -\Gamma^a,$$

where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

Gauge and Hyperbolicity in General Relativity II

- Specifying coordinates by the *generalized harmonic* (GH) method is accomplished by choosing a gauge-source function $H^a(x, \psi)$, e.g. $H^a = \psi^{ab}H_b(x)$, and requiring that

$$H^a(x, \psi) = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b\psi_{dc} + \partial_c\psi_{db} - \partial_d\psi_{bc}).$$

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$$R_{ab} = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi,\partial\psi).$$

- The Generalized Harmonic Einstein equation is obtained by replacing $\Gamma_a = \psi_{ab}\Gamma^b$ with $-H_a(x,\psi) = -\psi_{ab}H^b(x,\psi)$:

$$R_{ab} - \nabla_{(a}\left[\Gamma_{b)} + H_{b)}\right] = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(\psi,\partial\psi).$$

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- The vacuum GH Einstein equation, $R_{ab} = 0$ with $\Gamma_a + H_a = 0$, is therefore manifestly hyperbolic, in the sense that it has the same principal part as the scalar wave equation:

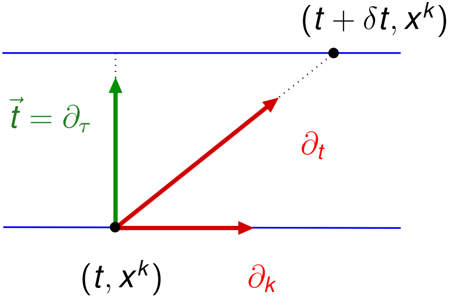
$$0 = \nabla_a\nabla^a\Phi = \psi^{ab}\partial_a\partial_b\Phi + Q(\partial\Phi).$$

ADM 3+1 Approach to Fixing Coordinates

- Decompose the 4-metric ψ_{ab} into its 3+1 parts:

$$ds^2 = \psi_{ab}dx^a dx^b = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

- The unit vector t^a normal to the $t =$constant slices depends only on the lapse N and shift N^i: $\vec{t} = \partial_\tau = \frac{\partial x^a}{\partial \tau}\partial_a = \frac{1}{N}\partial_t - \frac{N^k}{N}\partial_k.$



ADM Approach to the Einstein Evolution System

- Decompose the Einstein equations $R_{ab} = 0$ using the ADM 3+1 coordinate splitting. The resulting system includes evolution equations for the spatial metric g_{ij} and extrinsic curvature K_{ij}:

$$\begin{aligned}
\partial_t g_{ij} - N^k \partial_k g_{ij} &= -2NK_{ij} + g_{jk}\partial_i N^k + g_{ik}\partial_j N^k, \\
\partial_t K_{ij} - N^k \partial_k K_{ij} &= NR^{(3)}_{ij} + K_{jk}\partial_i N^k + K_{ik}\partial_j N^k \\
&\quad -\nabla_i \nabla_j N - 2NK_{ik}K^k{}_j + NK^k{}_k K_{ij}.
\end{aligned}$$

- The resulting system also includes constraints:

$$\begin{aligned}
0 &= R^{(3)} - K_{ij}K^{ij} + (K^k{}_k)^2, \\
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- System includes no evolution equations for lapse N or shift N^i. These quanties can be specified freely to fix the gauge.
- Resolving the issues of hyperbolicity (i.e. well posedness of the initial value problem) and constraint stability are much more complicated in this approach. The most successful version is the BSSN evolution system used by many (most) codes.

Dynamical GH Gauge Conditions

- The spacetime coordinates x^b are fixed in the generalized harmonic Einstein equations by specifying H^b:

$$\nabla^a\nabla_a x^b \equiv H^b.$$

- The generalized harmonic Einstein equations remain hyperbolic as long as the gauge source functions H^b are taken to be functions of the coordinates x^b and the spacetime metric ψ_{ab}.

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- The simplest choice $H^b = 0$ (harmonic gauge) fails for very dynamical spacetimes, like binary black-hole mergers.
- This failure seems to occur because the coordinates themselves become very dynamical solutions of the wave equation $\nabla^a \nabla_a x^b = 0$ in these situations.

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- This failure seems to occur because the coordinates themselves become very dynamical solutions of the wave equation $\nabla^a \nabla_a x^b = 0$ in these situations.
- Another simple choice – keeping H^b fixed in the co-moving frame of the black holes – works well during the long inspiral phase, but fails when the black holes begin to merge.

Dynamical GH Gauge Conditions II

- Some of the extraneous gauge dynamics could be removed by adding a damping term to the harmonic gauge condition:

$$\nabla^a\nabla_a x^b = H^b = \mu t^a \partial_a x^b = \mu t^b.$$

- This works well for the spatial coordinates x^i, driving them toward solutions of the spatial Laplace equation on the timescale $1/\mu$.

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- For the time coordinate t, this damped wave condition drives t to a time independent constant, which is not a good coordinate.
- A better choice sets $t^a H_a = -\mu \log\sqrt{g/N}$. The gauge condition in this case becomes

$$t^a\partial_a \log\sqrt{g/N} = -\mu\log\sqrt{g/N} + N^{-1}\partial_k N^k$$

  This coordinate condition keeps g/N close to unity, even during binary black hole mergers (where it became of order 100 using simpler gauge conditions).

The Constraint Problem

- Fixing the gauge in an appropriate way makes the Einstein equations hyperbolic, so the initial value problem becomes well-posed mathematically.
- In a well-posed representation, the constraints, $\mathcal{C} = 0$, remain satisfied for all time if they are satisfied initially.

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- There is no guarantee, however, that constraints that are “small” initially will remain “small”.
- Constraint violating instabilities were one of the major problems that made progress on the binary black hole problem so slow.
- Special representations of the Einstein equations are needed that control the growth of any constraint violations.

Constraint Damping in Electromagnetism

- Electromagnetism is described by the hyperbolic evolution equation $\nabla^a\nabla_a A_b = \nabla_b H$. Are there any constraints? Where have the usual $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ constraints gone?

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- Gauge condition becomes a constraint: $0 = \mathcal{C} \equiv \nabla^b A_b - H$.
- Maxwell's equations imply that this constraint is preserved:

$$0 = \nabla^a\nabla_a(\nabla^b A_b - H) = \nabla^a\nabla_a\mathcal{C}.$$

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- Modify evolution equations by adding multiples of the constraints:

$$\nabla^a\nabla_a A_b = \nabla_b H + \gamma_0 t_b\,\mathcal{C} = \nabla_b H + \gamma_0 t_b\,(\nabla^a A_a - H).$$

- These changes effect the constraint evolution equation,

$$\nabla^a\nabla_a\mathcal{C} - \gamma_0 t^b\nabla_b\mathcal{C} = 0,$$

so constraint violations are damped when $\gamma_0 > 0$.

Constraints in the GH Evolution System

- The GH evolution system has the form,

$$\begin{aligned} 0 &= R_{ab} - \nabla_{(a}\Gamma_{b)} - \nabla_{(a}H_{b)}, \\ &= R_{ab} - \nabla_{(a}\mathcal{C}_{b)}, \end{aligned}$$

  where $\mathcal{C}_a = H_a + \Gamma_a$ plays the role of a constraint. Without constraint damping, these equations are very unstable to constraint violating instabilities.
- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv \Big[R_{ab} - \tfrac{1}{2}\psi_{ab}R\Big]t^b = \Big[\nabla_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\Big]t^b.$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 \;=\; R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[t_{(a}\mathcal{C}_{b)} - \tfrac{1}{2}\psi_{ab}\, t^c\,\mathcal{C}_c\right],$$

  where t^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

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- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[t_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \tfrac{1}{2}\gamma_0\, t_a\mathcal{C}^c\mathcal{C}_c.$$

  This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

Summary of the GH Einstein System

- Choose coordinates by fixing a gauge-source function $H^a(x,\psi)$, e.g. $H^a = \psi^{ab}H_b(x)$, and requiring that

$$H^a(x,\psi) = \nabla^c\nabla_c x^a = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b\psi_{dc} + \partial_c\psi_{db} - \partial_d\psi_{bc}).$$

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- Principal part of evolution system becomes manifestly hyperbolic:

$$R_{ab} - \nabla_{(a}\mathcal{C}_{b)} = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(\psi,\partial\psi).$$

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$$R_{ab}-\nabla_{(a}\mathcal{C}_{b)}=-\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab}-\nabla_{(a}H_{b)}+Q_{ab}(\psi,\partial\psi).$$

- Add constraint damping terms for stability:

$$0 \;=\; R_{ab}-\nabla_{(a}\mathcal{C}_{b)}+\gamma_0\left[t_{(a}\mathcal{C}_{b)}-\tfrac{1}{2}\psi_{ab}\,t^c\,\mathcal{C}_c\right],$$

where t^a is a unit timelike vector field. Since $\mathcal{C}_a=H_a+\Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

What Do We Mean By Hyperbolic?

- Einstein's equation is "manifestly hyperbolic" in the sense that its principal part is the same as the scalar wave equation.
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- Using equations with well-posed initial-boundary value problems is critical. We want to solve problems with physical boundaries (e.g. boundaries near spatial infinity or just inside black-hole horizons).
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- Some numerical methods (e.g. spectral) cut space into many small computational domains. Exchanging dynamical information across these domain boundaries without loss of accuracy is essential.
- Symmetric hyperbolic systems are one class of equations for which suitable well-posedness theorems exist, and which are general enough to include Einstein's equation together with most of the other dynamical field equations used by physicists.

Symmetric Hyperbolic Systems

- Evolution equations of the form,

$$\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u,x,t)\partial_k u^\beta = F^\alpha(u,x,t),$$

for a collection of dynamical fields u^α, are called symmetric hyperbolic if there exists a positive definite $S_{\alpha\beta}$ having the property that $S_{\alpha\gamma}A^{k\,\gamma}{}_\beta \equiv A^k_{\alpha\beta} = A^k_{\beta\alpha}$.

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- Consider the scalar wave equation in space with arbitrary spatial metric g_{ij}:
$$0 = -\partial_t^2\psi + \nabla^k\nabla_k\psi = -\partial_t^2\psi + g^{k\ell}(\partial_k\partial_\ell\psi - \Gamma^n_{k\ell}\partial_n\psi).$$

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- Define the first-order dynamical fields, $u^\alpha = \{\psi, \Pi, \Phi_k\}$, which satisfy the following evolution equations:
$$\partial_t\psi = -\Pi, \qquad \partial_t\Pi + \nabla^k\Phi_k = 0, \qquad \partial_t\Phi_k + \nabla_k\Pi = 0.$$

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$$\partial_t\psi = -\Pi, \qquad \partial_t\Pi + \nabla^k\Phi_k = 0, \qquad \partial_t\Phi_k + \nabla_k\Pi = 0.$$
- Solutions to this first order system are the same as the solutions to the original second-order wave equation as long as the constraints are satisfied: $\mathcal{C}_i \equiv \partial_i\psi - \Phi_i = 0$, and $\mathcal{C}_{ij} \equiv \partial_{[i}\Phi_{j]} = 0$.

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- The first-order scalar field system is symmetric hyperbolic with the symmetrizer
$$dS^2 = S_{\alpha\beta}du^\alpha du^\beta = \Lambda^2 d\psi^2 + d\Pi^2 + g^{ij}d\Phi_i d\Phi_j.$$

First Order Generalized Harmonic Evolution System

- GH Einstein equations can be written as a symmetric-hyperbolic first-order system (Fischer and Marsden 1972, Alvi 2002). It is straightforward to write it as a first-order evolution system:

$$
\begin{aligned}
\partial_t\psi_{ab} - N^k\partial_k\psi_{ab} &= -N\,\Pi_{ab},\\
\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} &\simeq 0,\\
\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} &\simeq 0,
\end{aligned}
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  where $\simeq$ means equality up to terms depending on the fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$ but not their derivatives.

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  where $\simeq$ means equality up to terms depending on the fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$ but not their derivatives.

- This system is symmetric hyperbolic because there exists a symmetrizer, which can written in the form:

$$\begin{aligned} dS^2 &= S_{\alpha\beta}du^\alpha du^\beta \\ &= m^{ab}m^{cd}\left(\Lambda^2 d\psi_{ac}d\psi_{bd} + d\Pi_{ac}d\Pi_{bd} + g^{ij}d\Phi_{iac}d\Phi_{jbd}\right), \end{aligned}$$

  where m^{ab} is any positive definite metric (e.g. $m^{ab} = \delta^{ab}$).

Constraints in the First-Order Einstein System

- The first-order symmetric hyperbolic evolution system is equivalent to the original second-order Einstein equation so long as the following constraints are satisfied:

$$\begin{aligned}
\mathcal{C}_a &= H_a + \Gamma_a = 0, \\
\mathcal{F}_a &= \partial_t \mathcal{C}_b = 0, \\
\mathcal{C}_{ia} &= \partial_i \mathcal{C}_b = 0, \\
\mathcal{C}_{iab} &= \partial_i \psi_{ab} - \Phi_{iab} = 0, \\
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- This first-order system has (at least) two potential problems:
  - The new constraints, e.g. in particular $\mathcal{C}_{kab} = \partial_k \psi_{ab} - \Phi_{kab}$, tend to grow exponentially during numerical evolutions.
  - This system is not linearly degenerate, so it is possible (likely?) that shocks will develop (e.g. the components that determine shift evolution have the form $\partial_t N^i - N^k \partial_k N^i \simeq 0$).

A 'New' Generalized Harmonic Evolution System

- We can correct these problems by adding additional multiples of the constraints to the evolution system:

$$\partial_t\psi_{ab} - (1+\gamma_1)N^k\partial_k\psi_{ab} = -N\Pi_{ab} - \gamma_1 N^k\Phi_{kab},$$
$$\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} - \gamma_1\gamma_2 N^k\partial_k\psi_{ab} \simeq -\gamma_1\gamma_2 N^k\Phi_{kab},$$
$$\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} \simeq -\gamma_2 N\Phi_{iab}.$$

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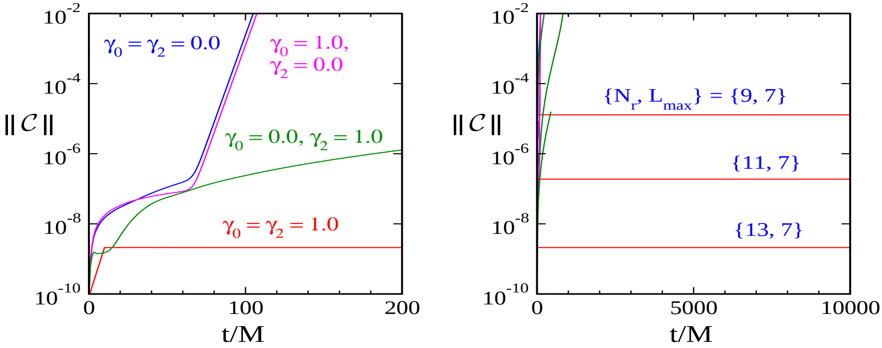
$$\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} \simeq -\gamma_2 N\Phi_{iab}.$$

- This ‘new’ generalized-harmonic evolution system has several nice properties:
  - This system is linearly degenerate for $\gamma_1 = -1$ (and so shocks should not form from smooth initial data).
  - The Φ_{iab} evolution equation can be written in the form, $\partial_t\mathcal{C}_{iab} - N^k\partial_k\mathcal{C}_{iab} \simeq -\gamma_2 N\mathcal{C}_{iab}$, so the new constraints are damped when $\gamma_2 > 0$.
  - This system is symmetric hyperbolic for all values of γ_1 and γ_2.

Numerical Tests of the New GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of our GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.



- The boundary conditions used for this simple test problem fix the boundary data to be the exact analytical values.