Solving Einstein's Equation Numerically on Manifolds with Arbitrary Spatial Topologies

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Mini-Workshop on Recent Advances in Gravitation
Nester Center for Mathematics and Theoretical Physics
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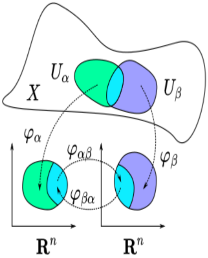
- Develop computational methods for representing and constructing three-dimensional manifolds with arbitrary topologies.
- Develop numerical methods for solving PDEs (e.g. Einstein's equation) on manifolds with topology $R \times \Sigma$, where Σ is a three-dimensional manifold with arbitrary topology.

Differentiable Manifolds

- Manifolds are topological spaces covered by a collection of open sets, each of which is homeomorphic to a subset of $\mathbb{R}^n$. These homeomorphisms are the coordinate charts.
- In a differentiable manifold the maps between coordinate charts must be differentiable in regions where the coordinate patches overlap.



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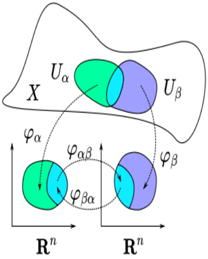

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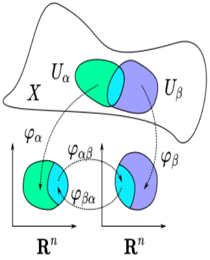

- What is the most convenient and efficient way to represent manifolds in a computer code?
- Is there a general way to organize these representations in a way makes it possible to change from one manifold to another without completely re-writing major parts of the code?
- Where can we find an extensive catalog of three-manifolds that includes all the information needed to use them for computations?

Representations of Arbitrary Three-Manifolds

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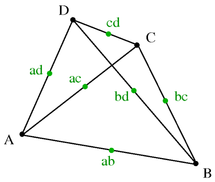

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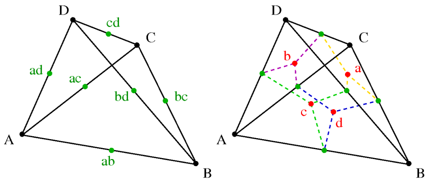

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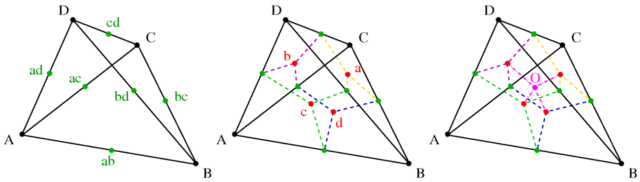

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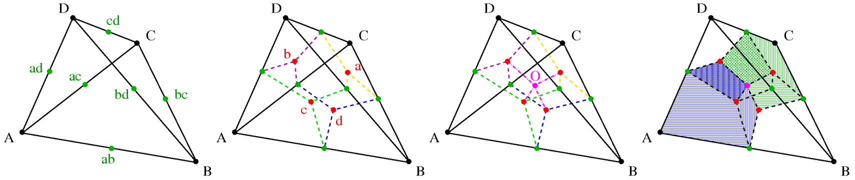
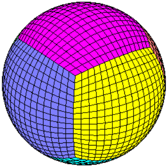
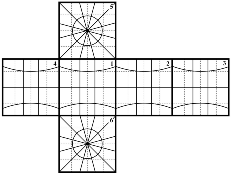


"Multicube" Representations of Manifolds

- Multicube representations of topological manifolds consist of a set of cubic regions, $\mathcal{B}_A$, plus maps that identify the faces of neighboring regions, $\Psi^{A\alpha}_{B\beta}(\partial_\beta \mathcal{B}_B) = \partial_\alpha \mathcal{B}_A$.





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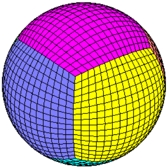
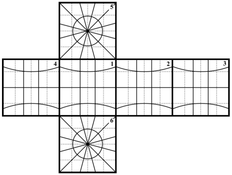
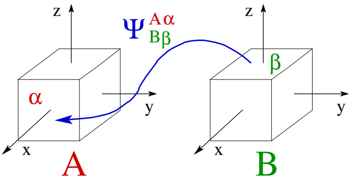


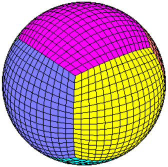


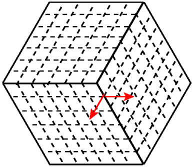
- Choose cubic regions to have uniform size and orientation.
- Choose linear interface identification maps $\Psi^{A\alpha}_{B\beta}$: $x^i_A = c^i_{A\,\alpha} + C^{A\,\alpha\,i}_{B\,\beta\,k}(x^k_B - c^k_{B\,\beta})$, where $C^{A\,\alpha\,i}_{B\,\beta\,k}$ is a rotation-reflection matrix, and $c^i_{A\,\alpha}$ is center of the α face of region A.



Fixing the Differential Structure

- The boundary identification maps, $\Psi^{A\alpha}_{B\beta}$, used to construct multicube topological manifolds are continuous, but typically are not differentiable at the interfaces.





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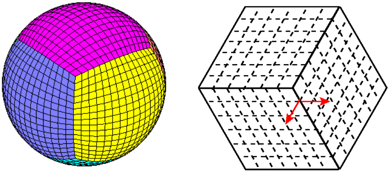

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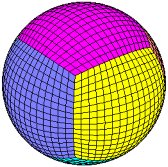

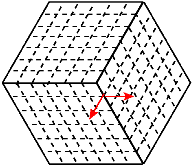

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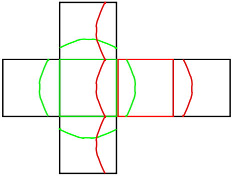
- The differential structure provides the framework in which smooth functions and tensors are defined on a manifold.
- The standard construction assumes the existence of overlapping coordinate domains having smooth transition maps.
- Multicube manifolds need an additional layer of infrastructure: e.g., overlapping domains $\mathcal{D}_A \supset \mathcal{B}_A$ with transition maps that are smooth in the overlap regions.



Fixing the Differential Structure II

- All that is needed to define continuous tensor fields at interface boundaries is the Jacobian $J^{A\alpha i}_{B\beta k}$ and its dual $J^{*B\beta k}_{A\alpha i}$ that transform tensors from one multicube coordinate region to another.
- Define the transformed tensors across interface boundaries:

$$\langle v^i_B\rangle_A = J^{A\alpha i}_{B\beta k} v^k_B, \qquad \langle w_{Bi}\rangle_A = J^{*B\beta k}_{A\alpha i} w_{Bk}.$$

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- If there exists a covariant derivative $\tilde{\nabla}_i$ determined by a smooth connection, then differentiability across interface boundaries can be defined as continuity of the covariant derivatives:

$$\tilde{\nabla}_{Aj} v^i_A = \langle \tilde{\nabla}_{Bj} v^i_B \rangle_A, \qquad \tilde{\nabla}_{Aj} w_{Ai} = \langle \tilde{\nabla}_{Bj} w_{Bi} \rangle_A$$

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- A smooth reference metric $\tilde{g}_{ij}$ determines both the needed Jacobians and the smooth connection.

Fixing the Differential Structure III

- Let $\tilde{g}_{Aij}$ and $\tilde{g}_{Bij}$ be the components of a smooth reference metric expressed in the multicube coordinates of regions $\mathcal{B}_A$ and $\mathcal{B}_B$ that are identified at the faces $\partial_\alpha\mathcal{B}_A \leftrightarrow \partial_\beta\mathcal{B}_B$.
- Use the reference metric to define the outward directed unit normals to the identified faces: $\tilde{n}_{A\alpha i}$, $\tilde{n}^i_{A\alpha}$, $\tilde{n}_{B\beta i}$, and $\tilde{n}^i_{B\beta}$.

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- The needed Jacobians are given by

$$J^{A\alpha i}_{B\beta k} = C^{A\alpha i}_{B\beta\ell}\left(\delta^\ell_k - \tilde{n}^\ell_{B\beta}\tilde{n}_{B\beta k}\right) - \tilde{n}^i_{A\alpha}\tilde{n}_{B\beta k},$$

$$J^{*B\beta k}_{A\alpha i} = \left(\delta^\ell_i - \tilde{n}_{A\alpha i}\tilde{n}^\ell_{A\alpha}\right) C^{B\beta k}_{A\alpha\ell} - \tilde{n}_{A\alpha i}\tilde{n}^k_{B\beta}.$$

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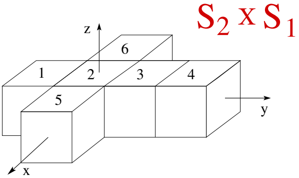
- These Jacobians satisfy:

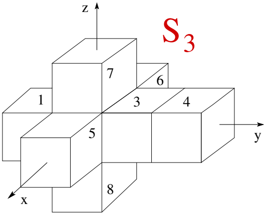
$$\tilde{n}^i_{A\alpha} = -J^{A\alpha i}_{B\beta k}\tilde{n}^k_{B\beta}, \qquad \tilde{n}_{A\alpha i} = -J^{*B\beta k}_{A\alpha i}\tilde{n}_{B\beta k}$$

$$u^i_{A\alpha} = J^{A\alpha i}_{B\beta k}u^k_{B\beta} = C^{A\alpha i}_{B\beta k}u^k_{B\beta}, \qquad \delta^i_k = J^{A\alpha i}_{B\beta\ell}J^{*B\beta\ell}_{A\alpha k},$$

where u^i is any vector tangent to the interface boundary.

Solving PDEs on Multicube Manifolds





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- Use boundary conditions on cube faces to enforce appropriate continuity conditions, thus selecting the correct global solution.

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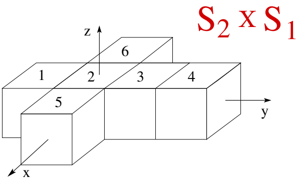

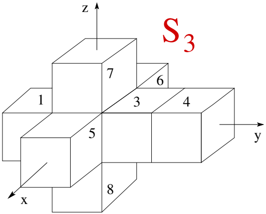

- Solve PDEs within each cubic region using any standard method.
- Use boundary conditions on cube faces to enforce appropriate continuity conditions, thus selecting the correct global solution.
- For first-order symmetric hyperbolic systems whose dynamical fields are tensors: set incoming characteristic fields, $\hat{u}^-$, with outgoing characteristics, $\hat{u}^+$, from neighbor,

$$\hat{u}_A^- = \langle \hat{u}_B^+ \rangle_A \qquad\qquad \hat{u}_B^- = \langle \hat{u}_A^+ \rangle_B.$$

Solving Einstein's Equation on Multi-Cube Manifolds

- Multi-cube methods were designed to solve first-order hyperbolic systems, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\tilde{\nabla}_k u^\beta = F^\alpha(u)$, where the dynamical fields u^α are tensors that can be transformed across interface boundaries using the Jacobians $J^{A\alpha i}_{B\beta k}$, etc.

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- The usual first-order representations of Einstein's equation fail to meet these conditions in two important ways:
  - The usual choice of dynamical fields, $u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c\partial_c\psi_{ab}, \Phi_{iab} = \partial_i\psi_{ab}\}$ are not tensor fields.
  - The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of Π_{ab}, $\Pi_{ab} = -t^c\partial_c\psi_{ab}$, and the one that comes from preserving the constraint $C_{iab} = \Phi_{iab} - \partial_i\psi_{ab}$, $t^c\partial_c C_{iab} = -\gamma_2 C_{iab}$.

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- Our attempts to construct the transformations for non-tensor quantities like $\partial_i\psi_{ab}$ and Φ_{iab} across the non-smooth multi-cube interface boundaries failed to result in stable numerical evolutions.
- A spatially covariant first-order representation of the Einstein evolution system seems to be needed.

Covariant Representations of Einstein's Equation

- Let $\tilde\psi_{ab}$ denote a smooth reference metric on the manifold $R \times \Sigma$. For convenience we choose $ds^2 = \tilde\psi_{ab}dx^a dx^b = -dt^2 + \tilde g_{ij}dx^i dx^j$, where $\tilde g_{ij}$ is the smooth multi-cube reference three-metric on Σ.

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- A fully covariant expression for the Ricci tensor can be obtained using the reference covariant derivative $\tilde\nabla_a$:

$$\begin{aligned} R_{ab} \;&=\; -\tfrac{1}{2}\psi^{cd}\tilde\nabla_c\tilde\nabla_d\psi_{ab} + \nabla_{(a}\Delta_{b)} - \psi^{cd}\tilde R^{e}{}_{cd(a}\psi_{b)e} \\ &\quad + \psi^{cd}\psi^{ef}\left(\tilde\nabla_e\psi_{ca}\tilde\nabla_f\psi_{ab} - \Delta_{ace}\Delta_{bdf}\right), \end{aligned}$$

where $\Delta_{abc} = \psi_{ad}\left(\Gamma^d_{bc} - \tilde\Gamma^d_{bc}\right)$, and $\Delta_a = \psi^{bc}\Delta_{abc}$.

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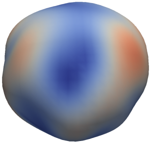
  where $\Delta_{abc}=\psi_{ad}\left(\Gamma^d_{bc}-\tilde{\Gamma}^d_{bc}\right)$, and $\Delta_a=\psi^{bc}\Delta_{abc}$.
- A fully-covariant manifestly hyperbolic representation of the Einstein equations can be obtained by fixing the gauge with a covariant generalized harmonic condition: $\Delta_a=-H_a(\psi_{cd})$.
- The vacuum Einstein equations then become:

$$\begin{aligned} \psi^{cd}\tilde{\nabla}_c\tilde{\nabla}_d\psi_{ab} &= -2\nabla_{(a}H_{b)}+2\psi^{cd}\psi^{ef}\left(\tilde{\nabla}_e\psi_{ca}\tilde{\nabla}_f\psi_{ab}-\Delta_{ace}\Delta_{bdf}\right) \\ &\quad -2\psi^{cd}\tilde{R}^e{}_{cd(a}\psi_{b)e}+\gamma_0\left[2\delta^c_{(a}t_{b)}-\psi_{ab}t^c\right](H_c+\Delta_c). \end{aligned}$$

Solving Einstein's Equation on Multicube Manifolds

- Examine a solution to the non-linear coupled Einstein-Klein-Gordon complex scalar-field equations numerically with perturbations in the ’“tensor” modes of the system (that represent gravitational wave degrees of freedom) away from the static “Einstein Universe” solution.
- Visualize $\sqrt{\delta\psi_{ab}\delta\psi^{ab}}$ on the equatorial $\chi = \pi/2$ two-sphere.



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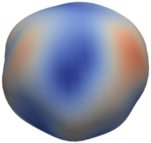
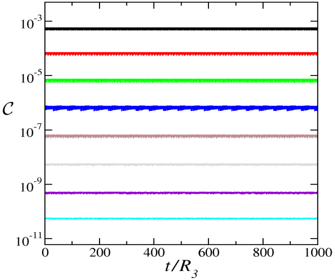


- The constraints $\mathcal{C}$ converge to zero, so the numerical solution converges to a solution of the exact equations.



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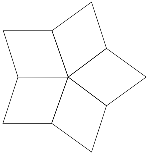

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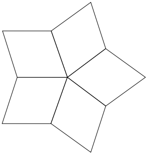


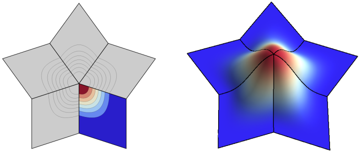
- Next choose the flat metric in this star-shaped domain by setting:

$$ds^2 = \bar{g}^I_{ab} dx^a_A dx^b_A = dx^2_A \pm 2\cos\theta_I \, dx_A \, dy_A + dy^2_A$$

in each square. This metric is smooth across all the internal interface boundaries, and ensures there is no cone singularity.

Choosing a Reference Metric II

- Combine the flat reference metrics defined at each corner of each multicube region using a partition of unity: $\bar{g}_{ab} = \sum_I u_I(\vec{x})\bar{g}^I_{ab}$.
- The weight functions $u_I(\vec{x})$ are chosen to be non-negative $u_I(\vec{x}) \geq 0$, sum to unity at each point $1 = \sum_I u_I(\vec{x})$, and fall to zero at the outer boundaries of the star-shaped domains.



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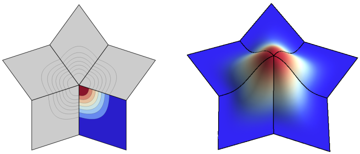

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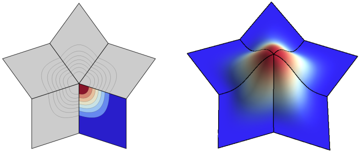

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- Modify these C^0 metrics by adding corrections, $\tilde{g}_{ab} = \bar{g}_{ab} + \delta g_{ab}$, where the δg_{ab} are chosen to make the extrinsic curvature $\tilde{K}_{ab}$ continuous across each interface boundary.

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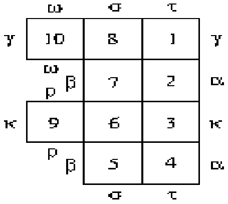

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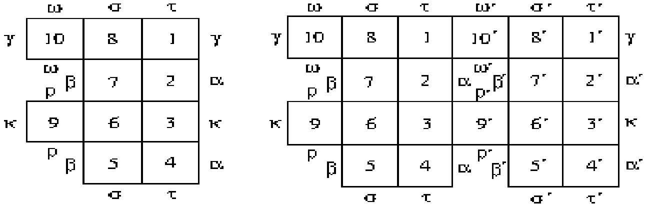

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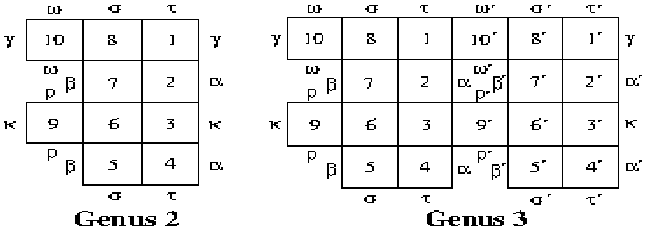

- Reference metrics constructed on these structures make it possible to solve differential equations numerically on any compact orientable two-dimensional manifold.

Smooth Reference Metrics

- As an example, we have solved the Ricci flow equation numerically on these manifolds:

$$\partial_t g_{ab} \;=\; -2R_{ab} + \nabla_a H_b + \nabla_b H_a - \mu \frac{V(t) - V_0}{V(t)}\, g_{ab} + \langle R(t) \rangle\, g_{ab},$$

where $H_a = g_{ab}\, g^{cd}\left(\Gamma^b_{cd} - \tilde{\Gamma}^b_{cd}\right)$ is the DeTurk term that fixes the gauge and makes the equation strongly parabolic, $V(t)$ is the volume, and $\langle R(t) \rangle$ is the volume averaged scalar curvature.

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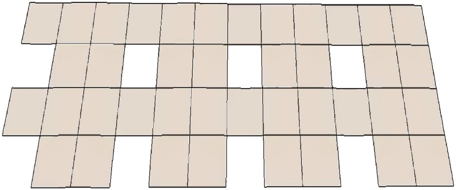

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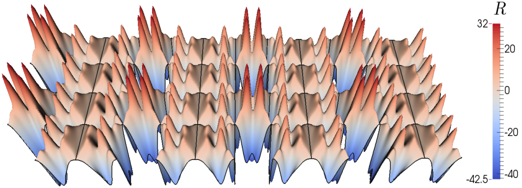

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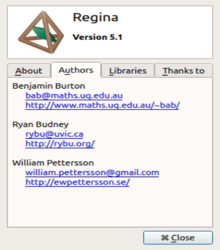
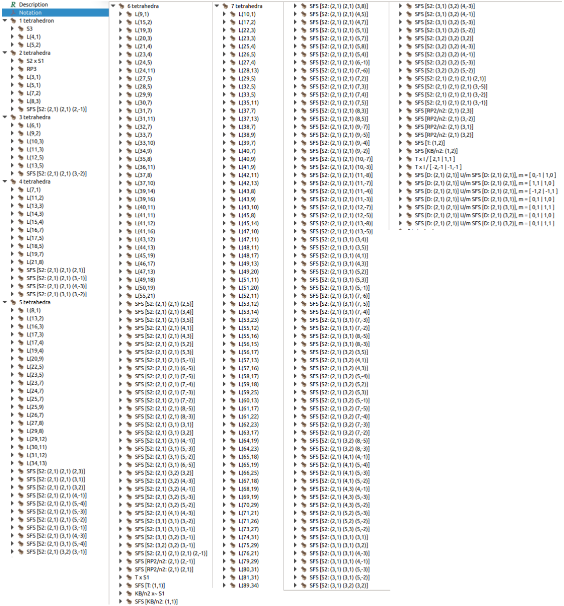
- Regina is a software tool for creating, manipulating, and visualizing triangulations of arbitrary three-manifolds, developed by Benjamin Burton, Rayan Budney and William Pettersson.



Catalogs of Three-Manifolds



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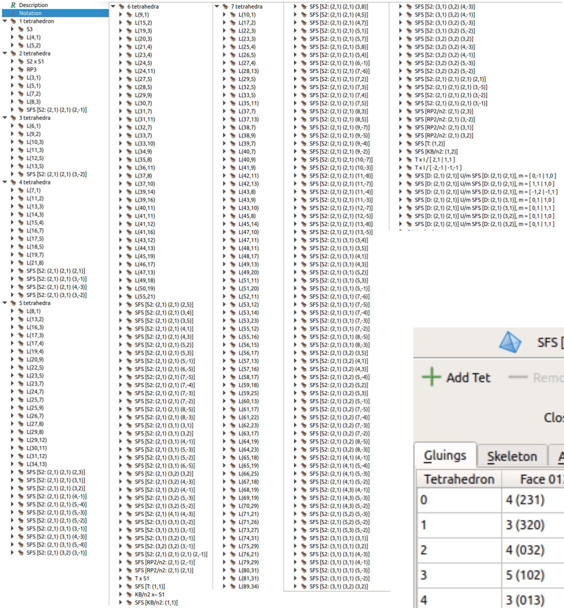

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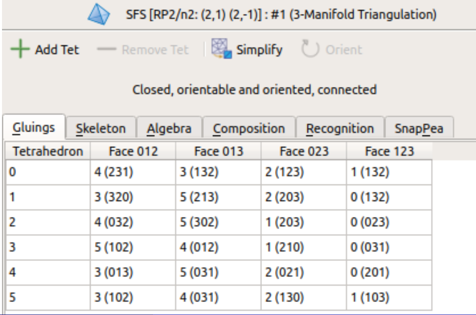

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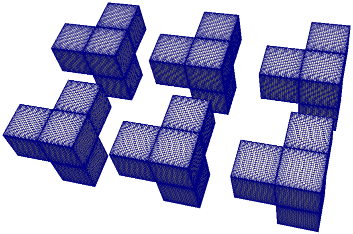

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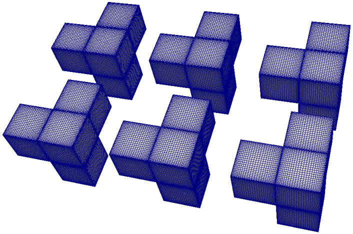

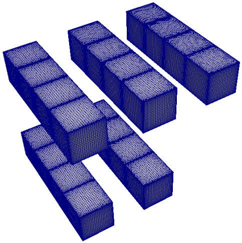

- Multicube structures have also been constructed by hand for some three-manifolds constructed by identifying the faces of polyhedra. Figure on the right shows a multicube structure for Seifert-Weber space.

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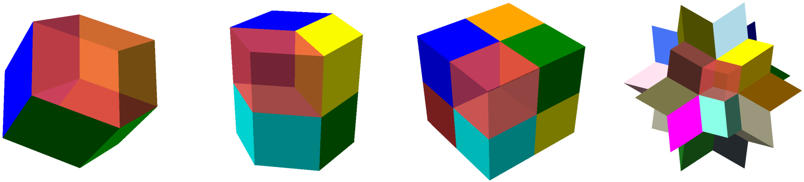

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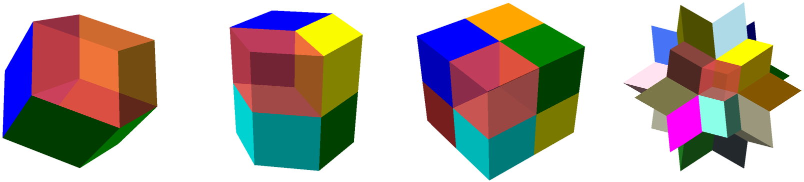

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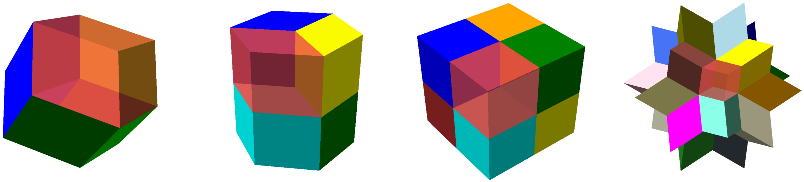

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- In three dimensions it is convenient to parameterize the flat inverse metrics in each cube using the dihedral angles between cube faces $\psi_{A\{xy\}}$, $\psi_{A\{yz\}}$, and $\psi_{A\{xz\}}$:

$$\begin{aligned} ds^{-2} &= \bar{g}^{ab}\,\partial_a\,\partial_b = \partial_x^2 + \partial_y^2 + \partial_z^2 \pm 2\cos\psi_{A\{xy\}}\,\partial_x\,\partial_y \\ &\qquad \pm 2\cos\psi_{A\{yz\}}\,\partial_y\,\partial_z \pm 2\cos\psi_{A\{xz\}}\,\partial_x\,\partial_z. \end{aligned}$$

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- A more complicated method of choosing the dihedral angles allows the construction of reference metrics on 17 additional manifolds.

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to determine the conformal factor ϕ that transforms the reference metric $\tilde{g}_{ab}$ into the physical metric: $g_{ab}=\phi^4\tilde{g}_{ab}$. $\tilde{R}$ is the scalar curvature associated with $\tilde{g}_{ab}$ and $\langle\tilde{R}\,\rangle$ is its spatial average.

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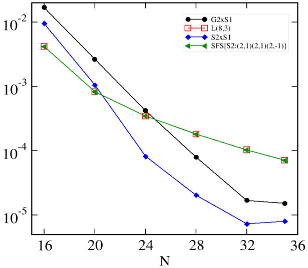

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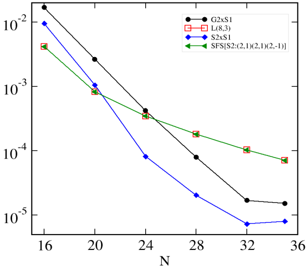

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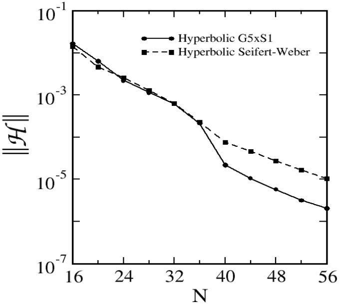
- The method works well on manifolds with negative scalar curvatures: $\langle\tilde R\rangle < 0$ but fails when $\langle\tilde R\rangle > 0$.



Solving the Constraints on Multicube Manifolds II

- The standard numerical methods for solving elliptic equations on these manifolds are very inefficient.
- In some cases a method called hyperbolic relaxation can be used to solve elliptic equations more efficiently.
- The hyperbolic relaxation version of the constant mean curvature equation described in the last slide is given by,

$$-\partial_t^2\phi - \kappa\,\partial_t\phi + \tilde\nabla^a\tilde\nabla_a\phi \;=\; \tfrac{1}{8}\phi\left(\tilde R - \phi^4\langle\tilde R\,\rangle\right).$$

- This method is designed to damp away $\partial_t\phi$ leaving a solution to the elliptic equation at late times.
- The method works well on manifolds with negative scalar curvatures: $\langle\tilde R\,\rangle < 0$ but fails when $\langle\tilde R\,\rangle > 0$.
- The accuracy of the hyperbolic relaxation solutions can be improved using the results as initial guesses for standard elliptic solves.