# Solving Einstein's Equation Numerically on Manifolds with Arbitrary Spatial Topologies 

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Department of Physics<br>University of California at San Diego

Mini-Workshop on Recent Advances in Gravitation Nester Center for Mathematics and Theoretical Physics National Central University, Taiwan - 9 December 2023

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- Develop computational methods for representing and constructing three-dimensional manifolds with arbitrary topologies.


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- Develop computational methods for representing and constructing three-dimensional manifolds with arbitrary topologies.
- Develop numerical methods for solving PDEs (e.g. Einstein's equation) on manifolds with topology $R \times \Sigma$, where $\Sigma$ is a three-dimensional manifold with arbitrary topology.


## Differentiable Manifolds

- Manifolds are topological spaces covered by a collection of open sets, each of which is homeomorphic to a subset of $\mathbb{R}^{n}$. These homeomorphisms are the coordinate charts.
- In a differentiable manifold the maps between coordinate charts must be differentiable in regions where the coordinate patches overlap.



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- What is the most convenient and efficient way to represent manifolds in a computer code?
- Is there a general way to organize these representations in a way makes it possible to change from one manifold to another without completely re-writing major parts of the code?
- Where can we find an extensive catalog of three-manifolds that includes all the information needed to use them for computations?


## Representations of Arbitrary Three-Manifolds

- Keeping track of the overlap regions between coordinate charts is complicated and time consuming. Can we find a way to represent differentiable manifolds using non-overlapping coordinate charts?


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## "Multicube" Representations of Manifolds

- Multicube representations of topological manifolds consist of a set of cubic regions, $\mathcal{B}_{A}$, plus maps that identify the faces of neighboring regions, $\Psi_{B \beta}^{A \alpha}\left(\partial_{\beta} \mathcal{B}_{B}\right)=\partial_{\alpha} \mathcal{B}_{A}$.



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- Choose cubic regions to have uniform size and orientation.
- Choose linear interface identification maps $\Psi_{B \beta}^{A \alpha}$ : $x_{A}^{i}=C_{A \alpha}^{i}+C_{B \beta k}^{A \alpha i}\left(x_{B}^{k}-C_{B \beta}^{k}\right)$, where $C_{B \beta k}^{A \alpha i}$ is a rotationreflection matrix, and $c_{A \alpha}^{i}$ is center of the $\alpha$ face of
 region $A$.


## Fixing the Differential Structure

- The boundary identification maps, $\Psi_{B \beta}^{A \alpha}$, used to construct multicube topological manifolds are continuous, but typically are not differentiable at the interfaces.

- Smooth tensor fields expressed in multicube Cartesian coordinates are not (in general) even continuous at the interfaces.


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- The differential structure provides the framework in which smooth functions and tensors are defined on a manifold.
- The standard construction assumes the existence of overlapping coordinate domains having smooth transition maps.
- Multicube manifolds need an additional layer of infrastructure: e.g., overlapping domains $\mathcal{D}_{A} \supset \mathcal{B}_{A}$ with transition maps that are smooth in the overlap regions.



## Fixing the Differential Structure II

- All that is needed to define continuous tensor fields at interface boundaries is the Jacobian $J_{B \beta k}^{A \alpha i}$ and its dual $J_{A \alpha i}^{* B \beta k}$ that transform tensors from one multicube coordinate region to another.
- Define the transformed tensors across interface boundaries:

$$
\left\langle v_{B}^{i}\right\rangle_{A}=J_{B \beta k}^{A \alpha i} v_{B}^{k}, \quad\left\langle w_{B i}\right\rangle_{A}=J_{A \alpha i}^{* B \beta k} w_{B k} .
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- If there exists a covariant derivative $\tilde{\nabla}_{i}$ determined by a smooth connection, then differentiability across interface boundaries can be defined as continuity of the covariant derivatives:

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\tilde{\nabla}_{A j} v_{A}^{i}=\left\langle\tilde{\nabla}_{B j} v_{B}^{i}\right\rangle_{A}, \quad \tilde{\nabla}_{A j} w_{A i}=\left\langle\tilde{\nabla}_{B j} w_{B i}\right\rangle_{A}
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$$

- A smooth reference metric $\tilde{g}_{i j}$ determines both the needed Jacobians and the smooth connection.


## Fixing the Differential Structure III

- Let $\tilde{g}_{A i j}$ and $\tilde{g}_{B i j}$ be the components of a smooth reference metric expressed in the multicube coordinates of regions $\mathcal{B}_{A}$ and $\mathcal{B}_{B}$ that are identified at the faces $\partial_{\alpha} \mathcal{B}_{A} \leftrightarrow \partial_{\beta} \mathcal{B}_{B}$.
- Use the reference metric to define the outward directed unit normals to the identified faces: $\tilde{n}_{A \alpha i}, \tilde{n}_{A \alpha}^{i}, \tilde{n}_{B \beta i}$, and $\tilde{n}_{B \beta}^{i}$.


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- The needed Jacobians are given by
$J_{B \beta k}^{A \alpha i}=C_{B \beta \ell}^{A \alpha i}\left(\delta_{k}^{\ell}-\tilde{n}_{B \beta}^{\ell} \tilde{n}_{B \beta k}\right)-\tilde{n}_{A \alpha}^{i} \tilde{n}_{B \beta k}$,
$J_{A \alpha i}^{* B \beta k}=\left(\delta_{i}^{\ell}-\tilde{n}_{A \alpha i} \tilde{n}_{A \alpha}^{\ell}\right) C_{A \alpha \ell}^{B \beta k}-\tilde{n}_{A \alpha i} \tilde{n}_{B \beta}^{k}$.


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$J_{A \alpha i}^{* B \beta k}=\left(\delta_{i}^{\ell}-\tilde{n}_{A \alpha i} \tilde{n}_{A \alpha}^{\ell}\right) C_{A \alpha l}^{B \beta k}-\tilde{n}_{A \alpha i} \tilde{n}_{B \beta}^{k}$.
- These Jacobians satisfy:

$$
\begin{array}{ll}
\tilde{n}_{A \alpha}^{i}=-J_{B \beta k}^{A \alpha i} \tilde{n}_{B \beta}^{k}, & \tilde{n}_{A \alpha i}=-J_{A \alpha i}^{* B \beta k} \tilde{n}_{B \beta k} \\
u_{A \alpha}^{i}=J_{B \beta k}^{A \alpha i} u_{B \beta}^{k}=C_{B \beta k}^{A \alpha i} u_{B \beta}^{k}, & \\
\delta_{k}^{i}=J_{B \beta \ell}^{A \alpha i} J_{A \alpha k}^{* B \beta \ell},
\end{array}
$$

where $u^{i}$ is any vector tangent to the interface boundary.

## Solving PDEs on Multicube Manifolds



- Solve PDEs within each cubic region using any standard method.
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- Use boundary conditions on cube faces to enforce appropriate continuity conditions, thus selecting the correct global solution.
- For first-order symmetric hyperbolic systems whose dynamical fields are tensors: set incoming characteristic fields, $\hat{u}^{-}$, with outgoing characteristics, $\hat{u}^{+}$, from neighbor,

$$
\hat{u}_{A}^{-}=\left\langle\hat{u}_{B}^{+}\right\rangle_{A} \quad \hat{u}_{B}^{-}=\left\langle\hat{u}_{A}^{+}\right\rangle_{B} .
$$

## Solving Einstein's Equation on Multi-Cube Manifolds

- Multi-cube methods were designed to solve first-order hyperbolic systems, $\partial_{t} u^{\alpha}+A^{k \alpha}{ }_{\beta}(u) \tilde{\nabla}_{k} u^{\beta}=F^{\alpha}(u)$, where the dynamical fields $U^{\alpha}$ are tensors that can be transformed across interface boundaries using the Jacobians $J_{B \beta k}^{A \alpha i}$, etc.


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- The usual first-order representations of Einstein's equation fail to meet these conditions in two important ways:
- The usual choice of dynamical fields, $u^{\alpha}=\left\{\psi_{a b}, \Pi_{a b}=-t^{c} \partial_{c} \psi_{a b}, \Phi_{i a b}=\partial_{i} \psi_{a b}\right\}$ are not tensor fields.
- The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of $\Pi_{a b}, \Pi_{a b}=-t^{c} \partial_{c} \psi_{a b}$, and the one that comes from preserving the constraint $C_{i a b}=\Phi_{i a b}-\partial_{i} \psi_{a b}$, $t^{c} \partial_{c} C_{i a b}=-\gamma_{2} C_{i a b}$.


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- The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of $\Pi_{a b}, \Pi_{a b}=-t^{c} \partial_{c} \psi_{a b}$, and the one that comes from preserving the constraint $C_{i a b}=\Phi_{i a b}-\partial_{i} \psi_{a b}$, $t^{c} \partial_{c} C_{i a b}=-\gamma_{2} C_{i a b}$.
- Our attempts to construct the transformations for non-tensor quantities like $\partial_{i} \psi_{a b}$ and $\Phi_{i a b}$ across the non-smooth multi-cube interface boundaries failed to result in stable numerical evolutions.
- A spatially covariant first-order representation of the Einstein evolution system seems to be needed.


## Covariant Representations of Einstein's Equation

- Let $\tilde{\psi}_{a b}$ denote a smooth reference metric on the manifold $R \times \Sigma$. For convenience we choose $d s^{2}=\tilde{\psi}_{a b} d x^{a} d x^{b}=-d t^{2}+\tilde{g}_{i j} d x^{i} d x^{j}$, where $\tilde{g}_{i j}$ is the smooth multi-cube reference three-metric on $\Sigma$.


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- A fully covariant expression for the Ricci tensor can be obtained using the reference covariant derivative $\tilde{\nabla}_{a}$ :

$$
\begin{aligned}
& R_{a b}=-\frac{1}{2} \psi^{c d} \tilde{\nabla}_{c} \tilde{\nabla}_{d} \psi_{a b}+\nabla_{(a} \Delta_{b)}-\psi^{c d} \tilde{R}_{c d}^{e}\left(a \psi_{b) e}\right. \\
&+\psi^{c d} \psi^{e f}\left(\tilde{\nabla}_{e} \psi_{c a} \tilde{\nabla}_{f} \psi_{a b}-\Delta_{a c e} \Delta_{b d f}\right) \\
& \text { where } \Delta_{a b c}=\psi_{a d}\left(\Gamma_{b c}^{d}-\tilde{\Gamma}_{b c}^{d}\right), \text { and } \Delta_{a}=\psi^{b c} \Delta_{a b c} .
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& +\psi^{c d} \psi^{e f}\left(\tilde{\nabla}_{e} \psi_{c a} \tilde{\nabla}_{f} \psi_{a b}-\Delta_{a c e} \Delta_{b d f}\right),
\end{aligned}
$$

where $\Delta_{a b c}=\psi_{a d}\left(\Gamma_{b c}^{d}-\tilde{\Gamma}_{b c}^{d}\right)$, and $\Delta_{a}=\psi^{b c} \Delta_{a b c}$.

- A fully-covariant manifestly hyperbolic representation of the Einstein equations can be obtained by fixing the gauge with a covariant generalized harmonic condition: $\Delta_{a}=-H_{a}\left(\psi_{c d}\right)$.
- The vacuum Einstein equations then become:

$$
\begin{aligned}
\psi^{c d} \tilde{\nabla}_{c} \tilde{\nabla}_{d} \psi_{a b}= & -2 \nabla_{(a} H_{b)}+2 \psi^{c d} \psi^{e f}\left(\tilde{\nabla}_{e} \psi_{c a} \tilde{\nabla}_{f} \psi_{a b}-\Delta_{a c e} \Delta_{\text {bdf }}\right) \\
& -2 \psi^{c d} \tilde{R}_{c d(a}^{e} \psi_{b) e}+\gamma_{0}\left[2 \delta_{(a}^{c} t_{b)}-\psi_{a b} t^{c}\right]\left(H_{c}+\Delta_{c}\right) .
\end{aligned}
$$

## Solving Einstein's Equation on Multicube Manifolds

- Examine a solution to the non-linear coupled Einstein-Klein-Gordon complex scalar-field equations numerically with perturbations in the "tensor" modes of the system (that represent gravitational wave degrees of freedom) away from the static "Einstein Universe" solution.
- Visualize $\sqrt{\delta \psi_{a b} \delta \psi^{a b}}$ on the equatorial $\chi=\pi / 2$ two-sphere.



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- Visualize $\sqrt{\delta \psi_{a b} \delta \psi^{a b}}$ on the equatorial $\chi=\pi / 2$ two-sphere.
- The constraints $\mathcal{C}$ converge to zero, so the numerical solution converges to a solution of the exact equations.



## Choosing a Reference Metric

- Finding an appropriate reference metric is the most difficult step in constructing a multicube representation of a manifold.
- For simple familiar manifolds, e.g. $S^{3}, S^{2} \times S^{1}$, etc., it is easy to use their standard metrics by transforming them into multicube Cartesian coordinates, but very difficult for arbitrary manifolds.


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- For arbitrary 2D manifolds a step by step method exists for constructing the needed reference metrics:
- First, choose the vertex opening angles $\theta_{i}$ satisfying $2 \pi=\sum_{l}^{N} \theta_{l}$ at each vertex of the multicube structure, e.g. $\theta_{l}=\frac{2 \pi}{N}$ where $N$ is the number of squares that intersect at that vertex.



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- Next choose the flat metric in this star-shaped domain by setting:

$$
d s^{2}=\bar{g}_{a b}^{\prime} d x_{A}^{a} d x_{A}^{b}=d x_{A}^{2} \pm 2 \cos \theta_{l} d x_{A} d y_{A}+d y_{A}^{2}
$$

in each square. This metric is smooth across all the internal interface boundaries, and ensures there is no cone singularity.

## Choosing a Reference Metric II

- Combine the flat reference metrics defined at each corner of each multicube region using a partition of unity: $\bar{g}_{a b}=\sum_{l} u_{l}(\vec{x}) \bar{g}_{a b}^{\prime}$.
- The weight functions $u_{l}(\vec{x})$ are chosen to be non-negative $u_{l}(\vec{x}) \geq 0$, sum to unity at each point $1=\sum_{l} u_{l}(\vec{x})$, and fall to zero at the outer boundaries of
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- Reference metrics produced by averaging flat metrics in this way have no conical singularities, and are continuous across all the multicube interface boundaries.
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- Unfortunately, they are not (in general) differentiable across those boundaries.
- Modify these $C^{0}$ metrics by adding corrections, $\tilde{g}_{a b}=\bar{g}_{a b}+\delta g_{a b}$, where the $\delta g_{a b}$ are chosen to make the extrinsic curvature $\tilde{K}_{a b}$ continuous across each interface boundary.


## Multicube Structures for Two-Manifolds

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Genus 2

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Genus 2

|  | Ls | $\square$ | $\tau$ | $\omega^{-}$ | $\square^{\circ}$ | $\tau$ | $\gamma$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | 10 | 8 | ] | $10^{\circ}$ | 8 | $]^{*}$ |  |
|  | ${ }_{p}^{\omega} \beta$ | 7 | 2 | ${ }^{2}{ }_{p}^{\omega p^{\prime}} \sigma^{\prime}$ | $3^{*}$ | 2 | D $x^{*}$ |
| $\kappa$ | 9 | 6 | 3 | 9 | 6 | $3{ }^{\circ}$ | K |
|  | $P_{\beta}$ | 5 | 4 | ${ }_{\text {c }}{ }^{-} \beta^{*}$ | $5{ }^{*}$ | 4 | D ${ }^{\text {c }}$ |
|  |  | $\sigma$ | $\tau$ |  | $\sigma^{\circ}$ | $\tau$ |  |

Genus 3

- Reference metrics constructed on these structures make it possible to solve differential equations numerically on any compact orientable two-dimensional manifold.


## Smooth Reference Metrics

- As an example, we have solved the Ricci flow equation numerically on these manifolds:
$\partial_{t} g_{a b}=-2 R_{a b}+\nabla_{a} H_{b}+\nabla_{b} H_{a}-\mu \frac{V(t)-V_{0}}{V(t)} g_{a b}+\langle R(t)\rangle g_{a b}$, where $H_{a}=g_{a b} g^{c d}\left(\Gamma_{c d}^{b}-\tilde{\Gamma}_{c d}^{b}\right)$ is the DeTurk term that fixes the gauge and makes the equation strongly parabolic, $V(t)$ is the volume, and $\langle R(t)\rangle$ is the volume averaged scalar curvature.


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## Representations of Arbitrary Three-Manifolds

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- Regina is a software tool for creating, manipulating, and visualizing triangulations of arbitrary three-manifolds, developed by Benjamin Burton, Rayan Budney and William Pettersson.



## Cataloas of Three-Manifolds

|  | $R$ Description |
| :---: | :---: |
|  | - \$1 tetrahedron |
|  | , \$3s |
|  | , \$ 44,17 |
|  | ** L5, 2) |
|  | - 2 tetrabedra |
|  | * \# $52 \times 51$ |
|  | - \$ RP3 |
|  | , \$ 43,1$)$ |
|  |  |
|  |  |
|  | *) $4(8,3)$ |
|  | $\geqslant \operatorname{ses} \operatorname{se}(2,1)(2,1)(2,-1)]$ * 3 tetrahedra |
|  | , \$ $L 6,1$ ) |
|  | * \$ $4(9,2)$ |
|  | * L 10.3$)$ |
|  | - * $[11,3)$ |
|  | * \$ L 12.5 ) |
|  | - \% L 13,5 ) |
|  |  |
|  | - 4 tetrahedra |
|  | 1 \$ 47,17 |
|  | : $*$ L 41,2$)$ |
|  | - \$ $L^{(13,3)}$ |
|  | - \$ 414.3$)$ |
|  | - \$ L 15.4 ) |
|  | * \$ 416,7$)$ |
|  | - \$ L 17.5 ) |
|  | * 3 L18,5) |
|  | - \$ $L$ (19,7) |
|  | \# \$ 421.88 |
|  |  |
|  | $\cdots \operatorname{sFS}[152[2,1)(2,1)(3,1)]$ |
|  | $\cdots \operatorname{s5s}(\underline{52}(2,1)(2,1)(4,3)]$ |
|  | $\cdots \operatorname{sfs}[52-(2,1)(3,1)(3,-2)]$ |
|  | - 5 tetusheda |
|  | * $\$ 48,17$ |
|  | : * L 13,2$)$ |
|  | : $*$ L16,3) |
|  | - * L 417,3$)$ |
|  | , \$ L17,4) |
|  | - \% L 419,4 ) |
|  | * ${ }^{\text {L }}$ (20,9) |
|  | - * $4(22,5)$ |
|  | ** L23,5) |
|  | - * 423,7$)$ |
|  | - \$ 424.7$)^{\text {a }}$ |
|  | - \$ 4225,7$)$ |
|  | * L25,9) |
|  | * \$ $426.7{ }^{\text {a }}$ |
|  | - \$ 427.8$)$ |
|  | * \$ L29,8) |
|  | - \$ L29,12] |
|  | * \$ L30, 111 |
|  | - \$ L31,12] |
|  | - ${ }^{\text {c }}$ L34,131 |
|  | $\cdots \operatorname{SFS}[52(2,1)(2,1)(2,3)]$ |
|  | \% $\operatorname{sFS}[152[(2,1)(2,1)(3,1)]$ |
|  | * $\left.\operatorname{sFS}^{[52} \mathbf{( 2 , 1 )}(2,1)(3,2)\right]$ |
|  | * $\operatorname{SFS}^{\text {S }}$ S2 $\left.2(2,1)(2,1)(4,-1)\right]$ |
|  | - \$ $\operatorname{sFs}[\operatorname{sz} 2(2,1)(2,1)(5,4)]$ |
|  | *) SFS [S2 [2, 21$)(2,1)(5,3)]$ |
|  | - $\operatorname{sFs}(\operatorname{szz}(2,1)(2,1)(5,-2)]$ |
|  | * SFS [SE2 [2, 1) (3, 1) (3,-1]] |
|  | * $\operatorname{sFS}[\operatorname{sz} 2(2,1)(3,1)(4,3)]$ |
|  | *) $\operatorname{SFS}(\operatorname{S2} 2(2,1)(3,1)(5,4)]$ |
|  | © $\operatorname{SFS}[52 \mathrm{z}(2,1)(3,2)(3,-1)]$ |



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|  | R Description Notrecon |
| :---: | :---: |
| - | - 1 tetrahedron |
|  | *3s |
|  | - 44,1$)$ |
| * | \$ L5, 2) |
| - | - 2 tetrabedra |
|  | * $52 \times 51$ |
| , | * RP3 |
| , | * L3,1) |
| , | * 45,1$)$ |
|  | * LT, 2) |
| , | * 418,3$)$ |
|  | S SFS [S2 $(2,1)(2,1)(2,-1]]$ <br> 3 tetrahedra |
|  | * L6,1) |
|  | * $4(9,2)$ |
|  | * 410,3$)$ |
| , | * 411,3$)$ |
|  | * 412,5$)$ |
| , | * 413,5$)$ |
| $\stackrel{ }{*}$ | \$ SFS [SE $(2,1)(2,1)(3,2)]$ |
| - | $4 \text { tetrahedra }$ |
|  | - $4(11,2)$ |
| - | * 413,3$)$ |
|  | * $[14.3)$ |
|  | * L(15,4) |
|  | \$ 416,77 |
|  | * 417.5$)$ |
|  | * L18,5) |
|  | * $L(19,7)$ |
|  | * 421.8$)$ |
|  | * SF5 [ [82 $(2,1)(2,1)(2,1)]$ |
|  | $\geqslant \operatorname{sFS}[152(2,1)(2,1)(3,-1)]$ |
|  | * SFS [ $32 \times(2,1)(2,1)(4,3)]$ |
|  | \$ sfs [152(2,1) (3,1)(3,-2)] |
| - | \$ 5 tetrathedra |
|  | * L(8,1) |
|  | * 413,2$)$ |
|  | * 416,37 |
|  | * 417,3$)$ |
|  | \$ 417.4$)^{\text {l }}$ |
|  | * 419,4$)$ |
|  | * 420.9$)$ |
|  | * 422,5$)$ |
|  | * 423,5$)$ |
|  | * 423,7$)$ |
|  | * 424.77 |
|  | * 4225,7$)$ |
| , | * 425,9$)$ |
|  | * $426.7{ }^{\text {a }}$ |
|  | * 427.8$)$ |
|  | * L29,8) |
|  | * 429,12$]$ |
| , | * Liso,11] |
|  | - L L31,12] |
|  | * L34,13\% |
| , | + SFS [ [32 $(2,1)(2,1)(2,3)]$ |
|  | * SFS [\$2 $2(2,1)(2,1)(3,1)]$ |
| , | * $\operatorname{sFs}[\operatorname{sz} 2(2,1)(2,1)(3,2)]$ |
|  | $3 \operatorname{SFS}[152(2,1)(2,1)(4,-1)]$ |
|  | * $\operatorname{sFs}[$ [ $32(2,1)(2,1)(5,-4)]$ |
| , | - $\operatorname{SFS}[152-(2,1)(2,1)(5,3] 1$ |
|  | - $\operatorname{ses}(\underline{5 z} 2(2,1)(2,1)(5,-2)]$ |
| , | * SFS [SE2 [2, 1) (3, 1) (3-1]] |
|  | * $\operatorname{sFs}[5 z=(2,1)(3,1)(4,3)]$ |
|  | \# SFS [122 [2, 1) (3,1) (5,-4] |
|  | * SFS $(\operatorname{sz}(2,1)(3,2)(3,1)]$ |


| 6 tetrahedra | - \$ 7 tetrabedra | * \$ SFs [52 (2, 1) (2, 1) (3,8)] |
| :---: | :---: | :---: |
| * 4 (9,1) | $1 \rightarrow L(0,1)$ | $\cdots \operatorname{sis}[5 z(2,1)(2,1)(4,5)]$ |
| * $[15,2]$ | - $L_{\text {[17,2] }}$ | - $\left.{ }^{\text {SFS }}[52 \mathrm{~L}, 2,1)(2,1)(4,7)\right]$ |
| * 419,3$]$ | - 3 L22,3] |  |
| * 420,3$]$ | , \# 423,3$]$ | - $*$ SFS [ $[52(2,1)(2,1)(5,7)]$ |
| * L[21,4] | - \$ 425,9$]$ | * \#SFS [52 [2, [2) (2,1) (5,8]] |
| \$ $123,4{ }^{\text {a }}$ | - \$ L26.5) | $\rightarrow$ SFs $[52(2,1)(2,1)(5,4)]$ |
| * 424,5$]$ | - \$ 427.4$]$ | - \#SFS SSE [ 2,1 ) (2,1)(6,-1]] |
| * 424,11$]$ | - 4228,13 | $\cdots$ - SFS [5z $(2,1)(2,1)(7,6,6]$ |
| * 427,5$]$ | - \$ 429.5$]$ |  |
| * 428,5$]$ | - \$ 432,5 |  |
| * 429.97 | - * 183,5 | - \# SFS $[52=[2,1)(2,1)[0,4]$ |
| * L30,7) | - 4353,11$)$ |  |
| * L31, ] |  |  |
| * 4313,11 | - $4(37,13)$ |  |
| * L02,7] | - * [138,7] | - \#SFS [5E $(2,1)(2,1)(9,-7]$ |
| * 433,7$)$ | - * 438,9$)$ | - \# SFs $[52(2,2)(2,1)(9,-5]]$ |
| - 403,10 | 1 3 L39,7] | - SFSE[5z: $(2,1)(2,1)$ ) $9,-4]$ |
| * $4(34,9)$ | * L40,7) | $\cdots$ - SFS $[52(2,3)(2,1)(9,2) 2]$ |
| * 435,8$]$ | * L40,9] |  |
| * 436,11 | - $4(41,9]$ |  |
| \$ $[37,8]$ | - \$ $L(42,11\}$ | - $*$ SFS $[52(2,1)(2,1)(11,88]$ |
| $\geqslant 437,10\rangle$ | - $\left.L^{4} 42,13\right\rangle$ |  |
| * $L(39,14)$ |  | - \# SFs $[52,(2,1)(2,1)(11,-4)]$ |
| * L(39,16) | * $4(43,9]$ | - ${ }^{\text {SFS }}$ S5z $(2,2)(2,1)(11,-31]$ |
| * L40,11) | - \$ 4 (43,10) | - ${ }^{\text {SFS }}$ S $[52:(2,1)(2,1)(12,7)]$ |
| * L(41, 11) | - \$ L 45,8 , | * *sFs $[52:(2,1)(2,1)(12,51]$ |
| * $4(41,12\rangle$ | - L(45,14) | - \# SFS [5z: 2,1 ) (2,1)(13,88)] |
| * $4(41,10)$ | - \$ L 47,10 ) | - \# srs $[52(2,2)(2,1)(13,-5]$ |
| * $L(43,12)$ | * L 47,11$\}$ | - ${ }^{\text {SFS }} \operatorname{sF}[52(2,1)(3,1)(3,4)]$ |
| * $L(44,13)$ | - \$ L 488,11$\}$ | * \#SFS [52 [2, 2,1 (3,1) 3,5$]$ |
| * 445.19$)$ |  | * $\%$ SFS [52:(2.1) (3,1) (4,1)] |
| * L46,17] | - $L^{(49,13)}$ |  |
| * 4477,13$\rangle$ | - $L^{40,20]}$ | - ${ }^{\text {SFS }}[52[(2,1)(3,1)(5,2)]$ |
| * L 449,18$)$ | - \$ L51,11) | * \#SFS [52: $2(2,1)(3,1)(5,3]]$ |
| * 450,191 | - $\geqslant 451,20\rangle$ | $\cdots \operatorname{sFs}[52(2,1)(3,1)(5,-1)]$ |
| * L55, 21) | \$ L52,11) | - \% SFS [SE $(2,1)(3,1)[(7,6)]$ |
| * SF5 [52: 2,11 ) $(2,1)(2,5)]$ | - * 453,12$\}$ |  |
| \# SFS [22: 2,11$](2,1)(3,4]]$ | * $4(53,14)$ |  |
| * $\operatorname{sFs}(5 z:(2,1)(2,1)(3,5)]$ | - * 453,23$)$ | - ${ }^{\text {SFS }}$ S5 $\left.52(2,2,1)(3,1)(7 ; 3)\right]$ |
| \# SFS [22: 2,11$](2,1)(4,1)]$ | * L(55,12) | * SFS [52-(2, 1) (3, 1) [(-2)] |
| * SF5 [52: $(2,1)(2,1)(4,3,3)]$ | - 455,16 ) |  |
| - SFS [52:(2, 1 ( 2,12$)(5,2)]$ | - * 456,15 | - SFS [5E [2, 2 )(3,1)(8,-3]] |
| * SFS $[5 z=(2,1)(2,1)(5,3)]$ | - * 456,17$)$ | - + SFs $[52(2,1)(3,2)(3,5])$ |
| \#SFS [22: $2,1,1,2,1)(5,-1]]$ | * 457,131 | * \#SFS [5E-(2, ) (3,2) (4, 17] |
| \# SFS [5z $2(2,1)(2,3)(6,5]$ ] | - * 457,16 ) | * \#SFs $[52(2,2)(3,2)(4,3)]$ |
| $\geqslant \operatorname{SFs}[52:(2,1)(2,1)(7,-5]]$ | * $4(58,17)$ | - \#SFS [52 (2, 1) (3,2) (5,4]] |
| $\cdots \operatorname{sFs}[52:(2,1)(2,1)(1,-4])$ |  | - \$sFS $552(2,1)(3,2)(5,2)]$ |
| \# SFS [52: 2,1 ) $(2,1)(7,3]]$ | - $4(59,25)$ | - Sts5 [52 (2, 1) (3,2) (5,3)] |
|  | * L(60,13) |  |
| $\geqslant \operatorname{SFS}[52:(2,1)(2,1)(8,-5]]$ | - $L^{(61,17)}$ |  |
| - $\operatorname{sFS}[52:(2,1)(2,1)(8,3] 1]$ | \$ $4(61,2 z)$ | - \$ SFs [5z $(2,2)(3,2)[7,48)]$ |
| - $\operatorname{sFs}[52:(2,1)(3,1)(3,1)]$ | - \$ $\mathrm{L}(62,23)$ | - + SFs $[52:(2,1)(3,2)(7,-3)]$ |
| - $\operatorname{srs}[22:(2,1)(3,1)(3,2)]$ | - ${ }^{\text {L }}$ (0, 3,17$)$ | - tosrs $[5 z=(2,1)(3,2)(T,-z)]$ |
| \# $\operatorname{SFS}[52: 12,1)(3,1)(4,-1)]$ | * 464,19$)$ | \# \$ SFS [52: (2, 1) (3,2) (8,5]] |
| \# SFS [52: $(2,1)(3,1)(5,-3] 1]$ | - \$ L 64,231 |  |
| * SFS $[52:(2,1)(3,1)(5,-2]]$ | * L(65,18) | - * SFS [52: 2,1$)(4,1)(4,-1)]$ |
| * SFS [52: 22,11$](3,1)(6,5]$ | * 4 [65,19 | - \$ SFS [SE [ 2,1 ) (4,1) (5,-4]] |
| $\geqslant \operatorname{SFS}[52:(2,1)(3,2)(32)]$ | - $\mathrm{L}^{\text {(66,25 }}$ | - \$ SF5 [5z (2, 1) (4, 1) (5,-3)] |
| \# SFS [52: $2,1,1(3,2)(4,-3] 1$ | - \$ 467,18$)$ | - \# SFS [ $[2=(2,1)(4,1)(5,-2]]$ |
| $\geqslant \operatorname{sFs}[52 \cdot[2,1)(3,2)(4,-1)]$ | - ${ }^{\text {L }}$ (68,19 | * * $\operatorname{sFs}[52,(2,1)(4,3)(4,-1)]$ |
| \# SFS [52:[2,1] (3,2) ( $5,-3] 1$ | * $L^{(69,19)}$ | - \$ SFS [5E:(2, ) (4,3) (5,-3]] |
| * $\operatorname{sfs}(5 z=2,1)(3,2)(5,-2)]$ | * ¢ $^{(70,29)}$ |  |
|  | - \$ L(71,21) | - \#) SFS [5E:(2, 1) (5,2) (5,-3]] |
| \# $\operatorname{sFs}(532:(3,1)(3,1)(3,-2]]$ | - \$ $4(71,26)$ | * \#sFs $[5 z-(2,1)(5,2)(5,-2)]$ |
| \# $\operatorname{sFS}[52:[3,1)(3,1)(3,-1]]$ | - 4 [73,27) | - \% sFS [SE: $(2,1)(5,3)(5,-27]$ |
| * $\operatorname{sFs}[52:(3,1)(3,2)(3,-1)]$ | - 4 L 74,317 | - \$3F5 [5z $(3,1)(3,1)(3,1)]$ |
| - SFS [52: 3,22$)(3,2)(3,-1]]$ | - 4 (75,29) |  |
| * SF5 [5z $(2,1)(2,1)(2,0)(2,-1)]$ | - * 476,21$)$ | - \# SFs [5z $(3,1)(3,1)(4,-3)]$ |
| * SFS [PP2/nz $2,11(2,-1]$ | - 479,29$)$ | - \$ SFS [5z: (3, 1) (3, 1) (4,-7] |
|  | - \% 480,31$)$ | - \#SFs $[5 z(3,1)(3,1)(5,-37]$ |
| - Txst | $\rightarrow$ ( 481,31 | - ${ }^{\text {S }}$ SFS $[5 E[(3,1)(3,1)(5,-2)]$ |
| ¢55 [f: $(1,1)]$ | - * L 499,341 | ${ }^{*} \operatorname{sFs}(52 E(3,1)(3,2)(3,2)]$ |
| kb/n 2 x |  |  |


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## The Regina software package includes a complete catalog of all compact orientable three-manifolds that can be represented by triangulations consisting of up to eleven tetrahedra.

## SFS [RP2/n2: $(2,1)(2,-1)]$ : \#1 (3-Manifold Triangulation)

```
* Add Te
```



Simplify

Closed, orientable and oriented, connected

| Gluings Skeleton | Algebra | Composition | Recognition | SnapPea |
| :--- | :--- | :--- | :--- | :--- | :--- |


| Tetrahedron | Face 012 | Face 013 | Face 023 | Face 123 |
| :--- | :--- | :--- | :--- | :--- |
| 0 | $4(231)$ | $3(132)$ | $2(123)$ | $1(132)$ |
| 1 | $3(320)$ | $5(213)$ | $2(203)$ | $0(132)$ |
| 2 | $4(032)$ | $5(302)$ | $1(203)$ | $0(023)$ |
| 3 | $5(102)$ | $4(012)$ | $1(210)$ | $0(031)$ |
| 4 | $3(013)$ | $5(031)$ | $2(021)$ | $0(201)$ |
| 5 | $3(102)$ | $4(031)$ | $2(130)$ | $1(103)$ |

## Creating Multicube Representations

- Oliver Rinne has developed a python code that automatically converts the triangulation gluing structure from a Regina output file into a multicube structure.


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- Oliver Rinne has developed a python code that automatically converts the triangulation gluing structure from a Regina output file into a multicube structure.
- This figure shows the multicube structure for the manifold SFS[RP2/n2:(2,1)(2,-1)] from the Regina catalog.



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- Oliver Rinne has developed a python code that automatically converts the triangulation gluing structure from a Regina output file into a multicube structure.
- This figure shows the multicube structure for the manifold SFS[RP2/n2:(2,1)(2,-1)] from the Regina catalog.

- Multicube structures have also been constructed by hand for some three-manifolds constructed by identifying the faces of polyhedra. Figure on the right shows a multicube structure for Seifert-Weber space.


## Building Three-Dimensional Reference Metrics

- In three dimensions, building $C^{0}$ reference metrics follows the same basic approach used in two dimensions:
- Construct flat metrics in each star shaped domain surrounding each vertex in the multicube structure.



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## Building Three-Dimensional Reference Metrics

- In three dimensions, building $C^{0}$ reference metrics follows the same basic approach used in two dimensions:
(1) Construct flat metrics in each star shaped domain surrounding each vertex in the multicube structure.

(2) Combine the flat metrics in each multicube region using a partition of unity that is continuous across the cube interfaces.
- In three dimensions it is convenient to parameterize the flat inverse metrics in each cube using the dihedral angles between cube faces $\psi_{A\{x y\}}, \psi_{A\{y z\}}$, and $\psi_{A\{x z\}}$ :

$$
\begin{aligned}
d s^{-2}=\bar{g}^{a b} \partial_{a} \partial_{b} & =\partial_{x}^{2}+\partial_{y}^{2}+\partial_{z}^{2} \pm 2 \cos \psi_{A\{x y\}} \partial_{x} \partial_{y} \\
& \pm 2 \cos \psi_{A\{y z\}} \partial_{y} \partial_{z} \pm 2 \cos \psi_{A\{x z\}} \partial_{x} \partial_{z}
\end{aligned}
$$

## Building Three-Dimensional Reference Metrics II

- The uniform dihedral angle assumption, $\psi_{A\{\alpha \beta\}}=\frac{2 \pi}{N_{A\{\alpha \beta\}}}$ (analogous to the method used in two dimensions) produces flat metrics with minimal coordinate distortion.


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- This assumption has been used to construct reference metrics successfully on each of the three-dimensional multicube structures constructed by hand.


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- This assumption has been used to construct reference metrics successfully on each of the three-dimensional multicube structures constructed by hand.
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- A more complicated method of choosing the dihedral angles allows the construction of reference metrics on 17 additional manifolds.


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to determine the conformal factor $\phi$ that transforms the reference metric $\tilde{g}_{a b}$ into the physical metric: $g_{a b}=\phi^{4} \tilde{g}_{a b} . \tilde{R}$ is the scalar curvature associated with $\tilde{g}_{a b}$ and $\langle\tilde{R}\rangle$ is its spatial average.

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- The accuracy of the hyperbolic relaxation solutions can be improved using the results as initial guesses for standard elliptic solves.


