Solving Einstein's Equation Numerically on Manifolds with Arbitrary Spatial Topologies

Lee Lindblom

#### Department of Physics University of California at San Diego

Mini-Workshop on Recent Advances in Gravitation Nester Center for Mathematics and Theoretical Physics National Central University, Taiwan — 9 December 2023 Solving Einstein's Equation Numerically on Manifolds with Arbitrary Spatial Topologies

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• Develop computational methods for representing and constructing three-dimensional manifolds with arbitrary topologies.

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• Develop computational methods for representing and constructing three-dimensional manifolds with arbitrary topologies.

• Develop numerical methods for solving PDEs (e.g. Einstein's equation) on manifolds with topology  $R \times \Sigma$ , where  $\Sigma$  is a three-dimensional manifold with arbitrary topology.

# **Differentiable Manifolds**

- Manifolds are topological spaces covered by a collection of open sets, each of which is homeomorphic to a subset of R<sup>n</sup>. These homeomorphisms are the coordinate charts.
- In a differentiable manifold the maps between coordinate charts must be differentiable in regions where the coordinate patches overlap.



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- What is the most convenient and efficient way to represent manifolds in a computer code?
- Is there a general way to organize these representations in a way makes it possible to change from one manifold to another without completely re-writing major parts of the code?
- Where can we find an extensive catalog of three-manifolds that includes all the information needed to use them for computations?

 Keeping track of the overlap regions between coordinate charts is complicated and time consuming. Can we find a way to represent differentiable manifolds using non-overlapping coordinate charts?

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### "Multicube" Representations of Manifolds

 Multicube representations of topological manifolds consist of a set of cubic regions, B<sub>A</sub>, plus maps that identify the faces of neighboring regions, Ψ<sup>Aα</sup><sub>BB</sub>(∂<sub>β</sub>B<sub>B</sub>) = ∂<sub>α</sub>B<sub>A</sub>.





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- Choose cubic regions to have uniform size and orientation.
- Choose linear interface identification maps  $\Psi_{B\beta}^{A\alpha}$ :  $x_A^i = c_{A\alpha}^i + C_{B\beta}^{A\alpha}(x_B^k - c_{B\beta}^k)$ , where  $C_{B\beta}^{A\alpha}(x_B^k - c_{B\beta}^k)$ , reflection matrix, and  $c_{A\alpha}^i$  is center of the  $\alpha$  face of region *A*.



• The boundary identification maps,  $\Psi^{A\alpha}_{B\beta}$ , used to construct multicube topological manifolds are continuous, but typically are not differentiable at the interfaces.



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- The differential structure provides the framework in which smooth functions and tensors are defined on a manifold.
- The standard construction assumes the existence of overlapping coordinate domains having smooth transition maps.

Multicube manifolds need an additional layer of infrastructure:
 e.g., overlapping domains D<sub>A</sub> ⊃ B<sub>A</sub> with transition maps that are smooth in the overlap regions.



- All that is needed to define continuous tensor fields at interface boundaries is the Jacobian  $J_{B\beta k}^{A\alpha i}$  and its dual  $J_{A\alpha i}^{*B\beta k}$  that transform tensors from one multicube coordinate region to another.
- Define the transformed tensors across interface boundaries:

$$\langle v_B^i \rangle_A = J_{B\beta k}^{A\alpha i} v_B^k, \qquad \langle w_{Bi} \rangle_A = J_{A\alpha i}^{*B\beta k} w_{Bk}$$

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 Tensor fields are continuous across interface boundaries if they are equal to their transformed neighbors:

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• A smooth reference metric  $\tilde{g}_{ij}$  determines both the needed Jacobians and the smooth connection.

- Let  $\tilde{g}_{Aij}$  and  $\tilde{g}_{Bij}$  be the components of a smooth reference metric expressed in the multicube coordinates of regions  $\mathcal{B}_A$  and  $\mathcal{B}_B$  that are identified at the faces  $\partial_{\alpha}\mathcal{B}_A \leftrightarrow \partial_{\beta}\mathcal{B}_B$ .
- Use the reference metric to define the outward directed unit normals to the identified faces: ñ<sub>Aαi</sub>, ñ<sup>i</sup><sub>Aα</sub>, ñ<sub>Bβi</sub>, and ñ<sup>i</sup><sub>Bβ</sub>.

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- Use the reference metric to define the outward directed unit normals to the identified faces:  $\tilde{n}_{A\alpha i}$ ,  $\tilde{n}^{i}_{A\alpha}$ ,  $\tilde{n}_{B\beta i}$ , and  $\tilde{n}^{i}_{B\beta}$ .
- The needed Jacobians are given by

$$\begin{aligned} J_{B\beta k}^{A\alpha i} &= C_{B\beta \ell}^{A\alpha i} \left( \delta_k^{\ell} - \tilde{n}_{B\beta}^{\ell} \tilde{n}_{B\beta k} \right) - \tilde{n}_{A\alpha}^{i} \tilde{n}_{B\beta k}, \\ J_{A\alpha i}^{*B\beta k} &= \left( \delta_i^{\ell} - \tilde{n}_{A\alpha i} \tilde{n}_{A\alpha}^{\ell} \right) C_{A\alpha \ell}^{B\beta k} - \tilde{n}_{A\alpha i} \tilde{n}_{B\beta}^{k}. \end{aligned}$$

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• These Jacobians satisfy:

$$\begin{split} \tilde{n}_{A\alpha}^{i} &= -J_{B\beta k}^{A\alpha i} \tilde{n}_{B\beta}^{k}, \qquad \tilde{n}_{A\alpha i} = -J_{A\alpha i}^{*B\beta k} \tilde{n}_{B\beta k} \\ u_{A\alpha}^{i} &= J_{B\beta k}^{A\alpha i} u_{B\beta}^{k} = C_{B\beta k}^{A\alpha i} u_{B\beta}^{k}, \qquad \delta_{k}^{i} = J_{B\beta \ell}^{A\alpha i} J_{A\alpha k}^{*B\beta \ell}, \end{split}$$

where  $u^i$  is any vector tangent to the interface boundary.

# Solving PDEs on Multicube Manifolds



- Solve PDEs within each cubic region using any standard method.
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- Use boundary conditions on cube faces to enforce appropriate continuity conditions, thus selecting the correct global solution.
- For first-order symmetric hyperbolic systems whose dynamical fields are tensors: set incoming characteristic fields, û<sup>-</sup>, with outgoing characteristics, û<sup>+</sup>, from neighbor,

$$\hat{u}_A^- = \langle \hat{u}_B^+ 
angle_A \qquad \qquad \hat{u}_B^- = \langle \hat{u}_A^+ 
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#### Solving Einstein's Equation on Multi-Cube Manifolds

Multi-cube methods were designed to solve first-order hyperbolic systems, ∂<sub>t</sub>u<sup>α</sup> + A<sup>k α</sup><sub>β</sub>(u) ∇̃<sub>k</sub>u<sup>β</sup> = F<sup>α</sup>(u), where the dynamical fields u<sup>α</sup> are tensors that can be transformed across interface boundaries using the Jacobians J<sup>Aαi</sup><sub>Bβk</sub>, etc.

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- The usual first-order representations of Einstein's equation fail to meet these conditions in two important ways:
  - The usual choice of dynamical fields,
    - $u^{\alpha} = \{\psi_{ab}, \Pi_{ab} = -t^{c}\partial_{c}\psi_{ab}, \Phi_{iab} = \partial_{i}\psi_{ab}\}$  are not tensor fields.
  - The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of  $\Pi_{ab}$ ,  $\Pi_{ab} = -t^c \partial_c \psi_{ab}$ , and the one that comes from preserving the constraint  $C_{iab} = \Phi_{iab} \partial_i \psi_{ab}$ ,  $t^c \partial_c C_{iab} = -\gamma_2 C_{iab}$ .

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- Our attempts to construct the transformations for non-tensor quantities like  $\partial_i \psi_{ab}$  and  $\Phi_{iab}$  across the non-smooth multi-cube interface boundaries failed to result in stable numerical evolutions.
- A spatially covariant first-order representation of the Einstein evolution system seems to be needed.

#### Covariant Representations of Einstein's Equation

• Let  $\tilde{\psi}_{ab}$  denote a smooth reference metric on the manifold  $R \times \Sigma$ . For convenience we choose  $ds^2 = \tilde{\psi}_{ab} dx^a dx^b = -dt^2 + \tilde{g}_{ij} dx^i dx^j$ , where  $\tilde{g}_{ij}$  is the smooth multi-cube reference three-metric on  $\Sigma$ .

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 $\begin{aligned} R_{ab} &= -\frac{1}{2}\psi^{cd}\tilde{\nabla}_{c}\tilde{\nabla}_{d}\psi_{ab} + \nabla_{(a}\Delta_{b)} - \psi^{cd}\tilde{R}^{e}{}_{cd(a}\psi_{b)e} \\ &+ \psi^{cd}\psi^{ef}\left(\tilde{\nabla}_{e}\psi_{ca}\tilde{\nabla}_{f}\psi_{ab} - \Delta_{ace}\Delta_{bdf}\right), \end{aligned}$ 

where  $\Delta_{abc} = \psi_{ad} \left( \Gamma^d_{bc} - \tilde{\Gamma}^d_{bc} \right)$ , and  $\Delta_a = \psi^{bc} \Delta_{abc}$ .

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- A fully covariant expression for the Ricci tensor can be obtained using the reference covariant derivative \$\tilde{\nabla}\_a\$:

 $\begin{aligned} R_{ab} &= -\frac{1}{2}\psi^{cd}\tilde{\nabla}_{c}\tilde{\nabla}_{d}\psi_{ab} + \nabla_{(a}\Delta_{b)} - \psi^{cd}\tilde{R}^{e}{}_{cd(a}\psi_{b)e} \\ &+ \psi^{cd}\psi^{ef}\left(\tilde{\nabla}_{e}\psi_{ca}\tilde{\nabla}_{f}\psi_{ab} - \Delta_{ace}\Delta_{bdf}\right), \end{aligned}$ 

where  $\Delta_{abc} = \psi_{ad} \left( \Gamma^d_{bc} - \tilde{\Gamma}^d_{bc} \right)$ , and  $\Delta_a = \psi^{bc} \Delta_{abc}$ .

- A fully-covariant manifestly hyperbolic representation of the Einstein equations can be obtained by fixing the gauge with a covariant generalized harmonic condition: Δ<sub>a</sub> = -H<sub>a</sub>(ψ<sub>cd</sub>).
- The vacuum Einstein equations then become:

$$\begin{split} \psi^{cd} \tilde{\nabla}_{c} \tilde{\nabla}_{d} \psi_{ab} &= -2 \nabla_{(a} H_{b)} + 2 \psi^{cd} \psi^{ef} \left( \tilde{\nabla}_{e} \psi_{ca} \tilde{\nabla}_{f} \psi_{ab} - \Delta_{ace} \Delta_{bdf} \right) \\ &- 2 \psi^{cd} \tilde{R}^{e}{}_{cd(a} \psi_{b)e} + \gamma_{0} \left[ 2 \delta^{c}_{(a} t_{b)} - \psi_{ab} t^{c} \right] \left( H_{c} + \Delta_{c} \right). \end{split}$$

# Solving Einstein's Equation on Multicube Manifolds

- Examine a solution to the non-linear coupled Einstein-Klein-Gordon complex scalar-field equations numerically with perturbations in the "tensor" modes of the system (that represent gravitational wave degrees of freedom) away from the static "Einstein Universe" solution.
- Visualize  $\sqrt{\delta \psi_{ab} \delta \psi^{ab}}$  on the equatorial  $\chi = \pi/2$  two-sphere.



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- Visualize  $\sqrt{\delta \psi_{ab} \delta \psi^{ab}}$  on the equatorial  $\chi = \pi/2$  two-sphere.

• The constraints C converge to zero, so the numerical solution converges to a solution of the exact equations.





# Choosing a Reference Metric

- Finding an appropriate reference metric is the most difficult step in constructing a multicube representation of a manifold.
- For simple familiar manifolds, e.g.  $S^3$ ,  $S^2 \times S^1$ , etc., it is easy to use their standard metrics by transforming them into multicube Cartesian coordinates, but very difficult for arbitrary manifolds.

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- For arbitrary 2D manifolds a step by step method exists for constructing the needed reference metrics:
- First, choose the vertex opening angles θ<sub>i</sub> satisfying 2π = Σ<sub>I</sub><sup>N</sup> θ<sub>I</sub> at each vertex of the multicube structure, e.g. θ<sub>I</sub> = <sup>2π</sup>/<sub>N</sub> where N is the number of squares that intersect at that vertex.


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• Next choose the flat metric in this star-shaped domain by setting:

$$ds^2 = \bar{g}_{ab}^{l} dx_A^a dx_A^b = dx_A^2 \pm 2\cos\theta_l \, dx_A \, dy_A + dy_A^2$$

in each square. This metric is smooth across all the internal interface boundaries, and ensures there is no cone singularity.

# Choosing a Reference Metric II

- Combine the flat reference metrics defined at each corner of each multicube region using a partition of unity:  $\bar{g}_{ab} = \sum_{l} u_{l}(\vec{x})\bar{g}_{ab}^{l}$ .
- The weight functions  $u_l(\vec{x})$  are chosen to be non-negative  $u_l(\vec{x}) \ge 0$ , sum to unity at each point  $1 = \sum_l u_l(\vec{x})$ , and fall to zero at the outer boundaries of the star-shaped domains.



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- Unfortunately, they are not (in general) differentiable across those boundaries.
- Modify these  $C^0$  metrics by adding corrections,  $\tilde{g}_{ab} = \bar{g}_{ab} + \delta g_{ab}$ , where the  $\delta g_{ab}$  are chosen to make the extrinsic curvature  $\tilde{K}_{ab}$  continuous across each interface boundary.

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 Reference metrics constructed on these structures make it possible to solve differential equations numerically on any compact orientable two-dimensional manifold.

#### **Smooth Reference Metrics**

• As an example, we have solved the Ricci flow equation numerically on these manifolds:

$$\partial_t g_{ab} = -2R_{ab} + \nabla_a H_b + \nabla_b H_a - \mu \frac{V(t) - V_0}{V(t)} g_{ab} + \langle R(t) \rangle g_{ab},$$

where  $H_a = g_{ab} g^{cd} \left( \Gamma^b_{cd} - \tilde{\Gamma}^b_{cd} \right)$  is the DeTurk term that fixes the gauge and makes the equation strongly parabolic, V(t) is the volume, and  $\langle R(t) \rangle$  is the volume averaged scalar curvature.

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where  $H_a = g_{ab} g^{cd} \left( \Gamma^b_{cd} - \tilde{\Gamma}^b_{cd} \right)$  is the DeTurk term that fixes the gauge and makes the equation strongly parabolic, V(t) is the volume, and  $\langle R(t) \rangle$  is the volume averaged scalar curvature.

• In this example Ricci flow on the genus number  $N_g = 5$  multicube manifold transforms the  $C^{2-}$  reference metric used as initial data into a smooth constant curvature metric:



#### **Smooth Reference Metrics**

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• In this example Ricci flow on the genus number  $N_g = 5$  multicube manifold transforms the  $C^{2-}$  reference metric used as initial data into a smooth constant curvature metric:

- Three dimensional manifolds are much more complicated:
  - There is no complete catalog of three-dimensional manifolds.

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- While no complete catalog of three-dimensional manifolds exists, there are catalogs containing triangulation based representations of large diverse collections of three-manifolds. One of these is part of the Regina software package.
- Regina is a software tool for creating, manipulating, and visualizing triangulations of arbitrary three-manifolds, developed by Benjamin Burton, Rayan Budney and William Pettersson.



#### Catalogs of Three-Manifolds

> > SESTS2:13.11.03.21.05.401 SFS [52: (3,1) (3,2) (5,2)] SFS [52: (3,2) (3,2) (3,2) \$\*\$ \$P\$ [\$2: (3,2) (3,2) (4,-3)]

575 [52: (3,2) (3,2) (4,-1)] \* \* 575 [52: (3,2) (3,2) (5,-3)] 575 [52: (3,2) (3,2) (5,-2)] SFS [52: (2,1) (2,1) (2,1) (2,1)

STS [S2: (2,1) (2,1) (2,1) (3,-5] SFS [52: (2,1) (2,1) (2,1) (3,-2)] SFS [S2: (2,1) (2,1) (2,1) (0,-1)] SFS [RP2/h2: (2,1) (2,3)]

SFS [RP2/h2: (2,1) (3-2)] SES [RP2/n2: (2,1) (3,1)] SFSTRP2/n2: (2,1) (3,2) SESTE 01.20 SES DKB/h2 (1.2)

Txt/[21]11] + \* Tx1/[-2,1]-1,1]

K De	iscription	P % 61	etrahedra	<b>7 %</b>	7 tetrahedra	- F 1	SES [52: (2,1) (2,1) (3,8)]
N(	otation		L(9,1)		% L(10,1)	- F 1	SFS [52: (2,1) (2,1) (4,5)]
	tetrahedron		L(15,2)		% L(17,2)	- F 1	SFS [52: (2,1) (2,1) (4,7)]
1.2	23		L(19,3)		L(22,3)		SFS [52: (2,1) (2,1) (5,1)]
1.2	L(4,1)		L(20,3)		L(23,3)		SFS [52: (2,1) (2,1) (5,7)]
1.7	10,0		L(21,4)		% L(25,4)		SFS [52: (2,1) (2,1) (5,8)]
	tetranedra		L(23,4)		% L(26,5)		SPS [S2: (2,1) (2,1) (5,4)]
1.7	52 X 51		L(24,5)		‰ L(27,4)		575 [52: (2,1) (2,1) (6,-1)]
1.7	HP/3		L(24,11)		% L(28,13)		SPS [S2: (2,1) (2,1) (7,-6)]
1.7	L(3,1)		L(27,5)		🐌 L(29,5)		srs [s2: (2,1) (2,1) (7,2)]
1.7	L(5,1)		L(28,5)		% L(32,5)		srs [s2: (2,1) (2,1) (7,3)]
2.2	L(7,2)		L(29,9)		😼 L(33,5)		5FS [52: (2,1) (2,1) (7,4)]
2.2	L(8,3)		L(30,7)		😼 L(35,11)	- × 1	5FS [52: (2,1) (2,1) (7,5)]
1.7	SPS [52: (2,1) (2,1) (2,1)]		L01.7)		⁵ ((37,7)		SFS [52: (2,1) (2,1) (8,3)]
	tetranogra		L(31,11)		S L(37,13)		SFS [52: (2,1) (2,1) (8,5)]
12	1(5,1)		L(32,7)		% L(38,7)		SFS [52: (2,1) (2,1) (9,-7)]
12	100,40	1.2.2	L(33,7)	1.1	L(38,9)	1.11	SFS [52: (2,1) (2,1) (9,-5)]
12	100.00	1.2	(133,10)		► L(39,7)		SES [52: (2,1) (2,1) (9,-4)]
12	102.00	1.2	L(34,9)		► L(40,7)		SES [52: (2,1) (2,1) (9,-2)]
12	102.00	1.2	L(35,8)		% L(40,9)	1.11	585 [52: (2,1) (2,1) (10,-7)]
12	CELEBRA CONTRACTOR AND	1.2	L(36,11)		% L(41,9)	1.11	5F5 [52: (2,1) (2,1) (10,-3)]
	set a fact (a, i) (a, i) (a, ii)	1.2	L(37,8)	1.1	% L(42,11)	1.23	5P5 [52: (2,1) (2,1) (11,4)]
1.2	1/7 1)	1.2	1(37,10)		1 L(42,13)		5F5 [52: (2,1) (2,1) (11,-7)]
1.2	101120	1.12	1(39,14)	1.1	10 L(43,0)	1.13	srs (sz. (z, i) (2,1) (11,-4) reg (rs. (s. i) (1,1) (11,-4)
1.5	1(13.3)	1.12	1(19,10)	1.1	1 L(43,9)	1.13	srs (sz. (z, i) (2,1) (11,-3)]
1.5	1(14.3)	1.1.2	1(41.11)	1.1	- L(43,10)	1.13	an a (M2 (2,1) (2,1) (12,-7))
14	1(15.4)	112	1 (41,47)	1.1	The second second	1.13	<ul> <li>an a part (a, i) (2,1) (12,0)</li> <li>and (12,0) (12,0)</li> </ul>
1.5	1016.70	1.2	100.00	1.1	B 1(47.10)	1.1	5 ST 122 (2.1) (2.1) (12.10)
1.5	1(17.5)	1.2	100 000	1.1	B ((47,10)	1.1	5 ST 102 (2.0 (2.0 (1.0 (0)))
1.5	1(18.5)	1.2	100,00	1.1	· ((**,11)	1.1	5 ST 12 0 0 0 0 0 0 0
1.5	L(12.7)	112	1 (45 19)	1.1	b 1649 17)	1.1	SEC [52:72:11 (0.17(0.0)]
1.5	L(21.8)	1.2	106 17)	1.1	b 1649 120	1.1	SEC 152-72-11 (2-11/4-201
1.5	SPS [52: (2.1) (2.1) (2.1)]	112	1047 120	1.1	<ul> <li>L649 200</li> </ul>	1.1	SEC 152-72-11 (2-11/5-201
1.5	SPS [52: (2.1) (2.1) (3-1)]	112	1 (49,10)	1.1	b 1/51 11)	1.1	SES 152-72-10 (0.11/6.20]
1.5	SFS [52: (2,1) (2,1) (4,-3)]	114	1(50.19)	1.	1/51 200	1.1	SES 152 (2:11 (3:11 (5:11)
1.5	SFS [52: (2,1) (3,1) (3,-2)]	1.5	1(55.21)	1.1	1(52.11)	1.1	SES 152:12 11 (3.1) (76)
\$ 51	tetrahedra	1.5	9502-01000	1.1	1(53.12)	1.1	SES 52-12 11 (3.1) (7.5)
1.5	L(8,1)	1.5	SES [52: (2.1) (2.1) (3.4)]	1.1	L(53,14)		SES [52: (2.1) (3.1) (740]
	L(13,2)	1.5	SFS [52: (2.1) (2.1) (3.5]]		L(53,23)	1.1	SFS [52: (2.1) (3.1) (730]
	L(16,3)	1.5	SPS [52: (2.1) (2.1) (4.1)]		L(55.12)	1.1	SFS [52: (2.1) (3.1) (7-2)]
1.5	L(17,3)	1.5	SPS [S2: (2,1) (2,1) (4,3)]		L(55,16)	1.1	5PS [52: (2,1) (3,1) (0,-5)]
- P - %	L(17,4)		SPS [52: (2,1) (2,1) (5,2)]		L(56,15)	1 1	575 [52: (2,1) (3,1) (0,-3)]
- P - %	L(19,4)		SPS [52: (2,1) (2,1) (5,3)]		L(56,17)	1 1	575 [52: (2,1) (3,2) (3,5)]
- P - %	L(20,9)		SPS [52: (2,1) (2,1) (5,-1)]		L(57,13)	1 1	5FS [52: (2,1) (3,2) (4,1)]
	L(22,5)		SFS [52: (2,1) (2,1) (6,-5)]		% L(57,16)	- 11	5FS [52: (2,1) (3,2) (4,3)]
	L(23,5)		SFS [52: (2,1) (2,1) (7,-5)]		L(58,17)	- 11	SFS [52: (2,1) (3,2) (5,-4)]
2.2	L(23,7)		SFS [S2: (2,1) (2,1) (7,-4)]		L(59,18)	- 11	SFS [S2: (2,1) (3,2) (5,2)]
2.2	L(24,7)		SFS [S2: (2,1) (2,1) (7,-3)]		L(59,25)	- F 1	SFS [S2: (2,1) (3,2) (5,3)]
2.2	L(25,7)		SFS [S2: (2,1) (2,1) (7,-2)]		L(60,13)	- F 1	SFS [S2: (2,1) (3,2) (5,-1)]
2.2	L(25,9)		SFS [52: (2,1) (2,1) (8,-5)]		L(61,17)	- 11	SES [52: (2,1) (3,2) (7,-5)]
12	L(26,7)		SFS [52: (2,1) (2,1) (8,-3)]		L(61,22)		SFS [52: (2,1) (3,2) (7,-4)]
12	100.00	1.2	SPS [S2: (2,1) (3,1) (3,1)]		L(62,25)		585 [52: (2,1) (3,2) (7,3)]
12	100.00	1.2	SPS [52: (2,1) (3,1) (3,2)]		L(63,17)	1.11	585 [52: (2,1) (3,2) (7,-2)]
12	1/20 11)	1.2	595 [52: (2,1) (3,1) (4,-1)]	1.1	L(64,19)	1.23	5P5 [52: (2,1) (3,2) (8,-5)]
1.2	1(31.12)	1.2	5P5 [52: (2,1) (3,1) (5,3)]		1 L(04,23)		575 [52: (2,1) (3,2) (0,-3)]
1.2	1/34 120	1.2	575 [52: (2,1) (3,1) (5,2)]		1 L(05,10)		5F5 [52: (2,1) [4,1) (4,1)]
1.2	SEC 152-12 11 12 11 12 201	1.2	575 [52: (2,1) (3,1) (6,-3)]		1 L(05,19)	1.1.1	5F5 [52: (2,1) [4,1) (5,-4)]
1.5	SES 152: (2 11 (2 1) (3 11]	12	SPS [S2: (2,1) (3,2) (3,2)]	1.1	1 (00,25)		SFS [52: (2,1) [4,1) (5,-3)]
1.5	SES [52: (2 1) (2 1) (3 2)]	1.2	ar a [ac. (a, i) (a, a) (i(-a))	1.	b 1(cs.10)	1.1	
1.5	SF5 [52: (2.1) (2.1) (4.1)]	112	SEC [52-12 11 (1 2) (5.3 <sup>1</sup> )	1.1	L(co.cr)		ar a pactor, (a, () (4,3) (4,1))
1.5	SFS [52: (2.1) (2.1) (540]	112	SPS [52-72-11 (1-2) (5-2)]	1.	1(70.29)	1.14	SES (52-72-11 (4-3) (5-20)
1.5	SFS [52: (2,1) (2,1) (5,-3)]	1.5	SES (\$2: (2.1) (4.1) (4.1)]	1.	h (71.21)	1.14	SFS [52: (2.1) (5.2) (5-30]
1.5	SFS [52: (2,1) (2,1) (5,-2)]	115	SES [52: (3 1) (3 1) (3 -2)]	1.1	1(7126)	1.1	SES [52: (2 1) (5 2) (5 -2)
1.5	SPS [S2: (2,1) (3,1) (3,-1)]	1.5	SFS[52: (3.1) (3.1) (3.10]	1.1	L(73,27)	1.1	SES 152: (2,1) (5,3) (5-24
1.5	SPS [S2: (2,1) (3,1) (4,-3)]	1.5	SES [52: (3.1) (3.2) (3.1)]	- i - i	L(74,31)	1.1	SES 152: (3.1) (3.1) (3.1)
1.5	SPS [S2: (2,1) (3,1) (5,-4)]	1.5	SES [52: (3.2) (3.2) (3.1)]	- i - i	L(75,29)	1.1	SES 152: (3.1) (3.1) (3.2)]
+ %	SFS [S2: (2,1) (3,2) (3,-1)]	1.5	SFS [S2: (2,1) (2,1) (2,1) (2,-1)]	- i - i	L(76,21)	1.1	SFS [52: (3,1) (3,1) (4-30]
		1 1 5	SFS [RP2/n2: (2,1) (2,-1)]	1.1	L(79,29)	1.1	5F5 [52: (3,1) (3,1) (4,-1)]
		1.5	SFS [RP2/h2: (2,1) (2,1)]		% L(80,31)	- F 1	5F5 [52: (3,1) (3,1) (5,-3)]
		1.5	Tx51		% L(81,31)	1 1 1	5F5 [52: (3,1) (3,1) (5,-2)]
		1.5	SFS [T: (1,1)]	- F 1	L(89,34)	- F 1	5F5 [52: (3,1) (3,2) (3,2)]
		1.2	K8/n2 x- 51				
		- F %	SPS (K6/n2: (1.1))				

The Regina software package includes a complete catalog of all compact orientable three-manifolds that SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [0,1] 1.0 SPS [D: (2,1) (2,1)] U/m SPS [D: (2,1) (2,1)], m = [1,1 | 1,0] can be represented by SPS [D: (2,1) (2,1)] U/m SPS [D: (2,1) (2,1)], m = [-1,2]-1,1 SPS [D: (2,1) (2,1)] U/m SPS [D: (2,1) (3,1)], m = [ 0,1 ] 1,0 STS [D: (2,1) (2,1)] U/m STS [D: (2,1) (3,1)], m = [ 0,1 ] 1,1 SFS (D: (2,1) (2,1)) U/m SFS (D: (2,1) (3,2)), m = [ 0,1 [ 1,0 ] SFS (D: (2,1) (2,1)] U/m SFS (D: (2,1) (3,2)], m = [0,1 | 1,1] triangulations consisting of up to eleven tetrahedra.

#### Catalogs of Three-Manifolds

R Description	🖛 🐂 6 tetrahedra	<ul> <li>% 7 tetrahedra</li> </ul>	SFS [52: (2,1) (2,1) (3,8)]
Notation	> % L(9,1)	h % L(10,1)	SFS [52: (2,1) (2,1) (4,5)]
1 tetrahedron	b % L(15,2)	h % L(17,2)	SFS [52: (2,1) (2,1) (4,7)]
• % S3	b % L(19,3)	h % L(22,3)	SFS [52: (2,1) (2,1) (5,1)]
P 19 ((4,1)	h % L(20,3)	h % L(23,3)	SFS [52: (2,1) (2,1) (5,7)]
• • (D,0	h % L(21,4)	h % L(25,4)	SFS [S2: (2,1) (2,1) (5,8)]
1 2 cecranedra	h % L(23,4)	h % L(26,5)	SPS [S2: (2,1) (2,1) (5,4)]
• • 52X51	h % L(24,5)	h % L(27,4)	SPS [S2: (2,1) (2,1) (6,-1)]
F 19 HP3	h % L(24,11)	h % L(28,13)	SPS [S2: (2,1) (2,1) (7,-6)]
• • L(3,1)	h % L(27,5)	h % L(29,5)	SPS [S2: (2,1) (2,1) (7,2)]
• • L(5,1)	IL(28,5)	IL(32,5)	SFS [S2: (2,1) (2,1) (7,3)]
• • · · (/,2)	• S L(29,9)	<ul> <li>% L(33,5)</li> </ul>	SFS [S2: (2,1) (2,1) (7,4)]
<ul> <li>W L(0,3)</li> <li>A COLUMN COLUMN</li></ul>	• % L(30,7)	L(35,11)	SFS [S2: (2,1) (2,1) (7,5)]
Statistics (2,1) (2,1) (2,1)	• % U01.7)	<ul> <li>% L(37,7)</li> </ul>	SFS [S2: (2,1) (2,1) (8,3)]
Stetranedra	U01,11)	<ul> <li>% L(37,13)</li> </ul>	SFS [52: (2,1) (2,1) (8,5)]
<ul> <li>b (0,1)</li> <li>b (0,1)</li> </ul>	• • u)z,n	U(38,7)	SES [52: (2,1) (2,1) (9,-7)]
<ul> <li>b b 1(10.1)</li> </ul>	• • (133,7)	EL(38,9)	SES [52: (2,1) (2,1) (9,-5)]
	U(33,10)	L(39,7)	SES [52: (2,1) (2,1) (9,-4)]
<ul> <li>b 1/12 5)</li> </ul>	U34,9)	• • L(40,7)	SFS [52: (2,1) (2,1) (9,-2)]
h th 1/13 53	U35,8)	• • L(40,9)	SFS [S2: (2,1) (2,1) (10,-7)]
b % SES [52-72 10 /2 10 /2 -20]	<ul> <li>U(36,11)</li> </ul>	<ul> <li>E(41,9)</li> </ul>	F % SFS [S2: (2,1) (2,1) (10,3)]
S distributes	<ul> <li>C(37,0)</li> </ul>	<ul> <li>C(42,11)</li> </ul>	srs [sz: (z,1) (z,1) (11,4)]
107.10	<ul> <li>A 100000</li> </ul>	<ul> <li>m (c(d,10))</li> <li>h (h) ((d,0))</li> </ul>	<ul> <li></li></ul>
10120	<ul> <li>A 100000</li> </ul>	<ul> <li></li></ul>	<ul> <li></li></ul>
1(13.3)	<ul> <li>A 1000 100</li> </ul>	<ul> <li></li></ul>	<ul> <li></li></ul>
• > L(14.3)	b 5 ((41.11))	<ul> <li>b 1645 m)</li> </ul>	<ul> <li>an a part (a, r) (a, r) (12,-7)</li> <li>an a part (a, r) (a, r) (12,-7)</li> </ul>
I 10 1015.40	b b ((41.17)	b 1645 140	E SES (52-72 1) (2 1) (13,40)
h % L(16.7)	b > 1(41.16)	h h 1647 100	E SES [52-72 1] (2 1) (13,5)]
h % L(17.5)	b 1(43.12)	h h 1667 11)	E SES [52-72 1] (1 1) (1 40]
h % L(18.5)	h the 1044 170	h h 1649 11)	b SEC [52-72 1] (1 1) (2 5)]
b % L(19,7)	1(45.19)	b % 1/48 17)	SESTS2: (2.1) (3.1) (4.1)]
b % L(21,0)	b % ((46.17))	b % ((49.13))	SESTS2: (2.1) (3.1) (4.3)]
\$ \$75 [52: (2,1) (2,1) (2,1)]	b % 1(47.13)	h 1629 200	SESTS2:02.11(3.1)(5.2)]
\$ \$ \$P\$ [\$2: (2,1) (2,1) (3,-1)]	> % L(49,18)	b % L(51,11)	SES [52: (2,1) (3,1) (5,3)]
\$75 [52: (2,1) (2,1) (4,-3)]	b % US0.19)	L(51,20)	SES [52: (2.1) (3.1) (5.1)]
SFS [S2: (2,1) (3,1) (3,-2)]	b % US5.21)	L(52,11)	SES [52: (2.1) (3.1) (7.6)]
🐞 Stetrahedra	SFS [52: (2,1) (2,1) (2,5]]	h % L(53,12)	SFS [52: (2,1) (3,1) (7,-5)]
IL(8,1)	SFS [52: (2,1) (2,1) (3,4]]	h % L(53,14)	SFS [52: (2,1) (3,1) (7,-4)]
b % L(13,2)	SFS [S2: (2,1) (2,1) (3,5]]	h % L(53,23)	SFS [S2: (2,1) (3,1) (7,-3)]
b % L(16,3)	52'5 [52: (2,1) (2,1) (4,1)]	h % L(55,12)	SPS [S2: (2,1) (3,1) (7,-2)]
L(17,3)	52'5 [52: (2,1) (2,1) (4,3)]	b % L(55,16)	SPS [S2: (2,1) (3,1) (0,-5)]
• • L(17,4)	SPS [52: (2,1) (2,1) (5,2)]	IL(56,15)	SPS [S2: (2,1) (3,1) (0,-3)]
* % L(19,4)	SFS [52: (2,1) (2,1) (5,3)]	IL(56,17)	SPS [S2: (2,1) (3,2) (3,5)]
L(20,9)	SFS [S2: (2,1) (2,1) (5,-1)]	k % L(57,13)	SFS [S2: (2,1) (1,2) (4,1)]
• • • • • • • • • • • • • • • • • • •	SFS [52: (2,1) (2,1) (6,-5)]	<ul> <li>S L(57,16)</li> </ul>	SFS [S2: (2,1) (1,2) (4,3)]
<ul> <li>U(2),5)</li> <li>U(2),70</li> </ul>	SFS [S2: (2,1) (2,1) (7,-5)]	5 L(58,17)	SFS [S2: (2,1) (3,2) (5,-4)]
b the 1/14 million	SES [S21(2,1)(2,1)(7,4)]	<ul> <li>U(59,18)</li> </ul>	SES [52 (2,1) (3,2) (5,2)]
<ul> <li>b 1(25.7)</li> </ul>	SFS[S21(2,1) (2,1) (7,3)]	• • ((59,25)	• • • • • • • • • • • • • • • • • • •
1/25.90	SPS(52, (2,1) (2,1) (7,2))	C(90,13)	• • • • • • • • • • • • • • • • • • •
1/26.7)	SPS [52: (2,1) (2,1) (0,5)]	L(s1,17)	F = 5F5 [52:(2,1) [5;2) (1;5)]
1/27.80	SS(52, (2, 1) (2, 1) (3, 3)]	b to 1/63 220	SSS(2,0,0,2,0,4)
b % L(29.8)	SSSI2:010.0000	b % (63.17)	SSI2:010200-20
I(29,12)	SES 02:72 11 (3.1) (4.10)	b % 1654 190	> > SECTO 10 (12) (8-5)
b % L(30,11)	SF5[52; (2,1) (3,1) (5,3)]	h t(54,23)	SFS [52: (2,1) (3,2) (8-3)]
I(31,12)	SFS [52; (2,1) (3,1) (5,2)]	k % L(55,18)	SFS [52: (2,1) (4,1) (4,1)]
b % L(34,13)	SPS [52; (2,1) (3,1) (6,-5)]	k % L(65,19)	* * 575 [52: (2,1) (4,1) (5,-4)]
SFS [S2: (2,1) (2,1) (2,3)]	\$75 [52: (2,1) (3,2) (3,2)]	h % L(66,25)	575 [52: (2,1) (4,1) (5,-3)]
SFS [S2: (2,1) (2,1) (3,1)]	\$75 [52: (2,1) (3,2) (4,-3)]	k % L(67,10)	F % SFS [S2: (2,1) (4,1) (5,-2)]
SFS [52: (2,1) (2,1) (3,2)]	SFS [S2: (2,1) (3,2) (4,-1)]	b % L(68,19)	SFS [S2: (2,1) (4,3) (4,-1)]
SFS [52: (2,1) (2,1) (4,-1)]	SFS [52: (2,1) (3,2) (5,-3)]	b % L(69,19)	SFS [S2: (2,1) (4,3) (5,-3)]
585 [52: (2,1) (2,1) (5,-4)]	SFS [S2: (2,1) (3,2) (5,-2)]	I (70,29)	SFS [S2: (2,1) (4,3) (5,-2)]
<ul> <li>= 5r5 [52: (2,1) (2,1) (5,-3)]</li> </ul>	SFS [S2: (2,1) (4,1) (4,3)]	1(71,21)	SFS [S2: (2,1) (5,2) (5,-3)]
<ul> <li>srs p.c. (z, () (2,1) (5,-2))</li> </ul>	SFS [S2: (3,1) (3,1) (3,-2)]	L(71,26)	SFS [52: (2,1) (5,2) (5,-2)]
<ul> <li></li></ul>	<ul> <li>SFS [S2: (3,1) (3,1) (3,-1)]</li> </ul>	<ul> <li>% L(73,27)</li> </ul>	<ul> <li>SFS [52: (2,1) (5,3) (5,-2)]</li> </ul>
<ul> <li>an a part (a, (1(3, 1)(4, 3)))</li> <li>an a part (a, (1(3, 1)(4, 3)))</li> </ul>	SES[S2: (3,1) (3,2) (3,-1)]	<ul> <li>To L(74,31)</li> </ul>	• • StS p2: (3,1) (3,1) (3,1)
<ul> <li>SPS [52-72 1] (12) (14,11]</li> </ul>	<ul> <li>SS(S2(3,2)(3,2)(3,1)]</li> <li>CONTRACT OF CONTRACT OF CONTRACT.</li> </ul>	<ul> <li>To L(75,29)</li> </ul>	<ul> <li>SES [32: (3,1) (3,1) (3,2)]</li> </ul>
	<ul> <li>sestemption (2,1) (2,1) (2,1) (2,1)</li> <li>sestemption (2,1) (2,1) (2,1) (2,1)</li> </ul>	<ul> <li>m L(r6,21)</li> <li>h h L(28,20)</li> </ul>	<ul> <li></li></ul>
	<ul> <li>S 500 c/lt2 (2,1) (2,1)</li> <li>S 555 (0023/22 (2.1) (2.1))</li> </ul>	<ul> <li></li></ul>	<ul> <li>South Company (Science)</li> <li>South Company (Science)</li> </ul>
	<ul> <li>TyS1</li> </ul>	k % (81 31)	<ul> <li>Sector (3,1) (3,1) (3,3)</li> <li>Sector (3,1) (3,1) (3,1) (3,3)</li> </ul>
	> > 9507-0.00	b % (89.34)	• • • • • • • • • • • • • • • • • • •
	KB/n2 x- 51		· · · · · · · · · · · · · · · · · · ·
	b % 525 005/v2 (1.10)		

The Regina software package includes a complete catalog of all compact orientable three-manifolds that SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [0,1] 1.0 can be represented by SPS [D: (2,1) (2,1)] U/m SPS [D: (2,1) (2,1)], m = [-1,2]-1,1 SFS (D: (2,1) (2,1)) U/m SFS (D: (2,1) (3,2)), m = [ 0,1 [ 1,0 ] SFS (D: (2,1) (2,1)] U/m SFS (D: (2,1) (3,2)], m = [0,1 | 1,1] triangulations consisting of up to eleven tetrahedra.

SFS [RP2/n2: (2,1) (2,-1)] : #1 (3-Manifold Triangulation)



SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [1,1 | 1,0]

SPS [D: (2,1) (2,1)] U/m SPS [D: (2,1) (3,1)], m = [ 0,1 ] 1,0 STS [D: (2,1) (2,1)] U/m STS [D: (2,1) (3,1)], m = [ 0,1 ] 1,1

> > SESTO-13.11(3.2)(5.30) SFS [52: (3,1) (3,2) (5,2)] SFS [52: (3,2) (3,2) (3,2) \$\*\$ \$P\$ [\$2: (3,2) (3,2) (4,-3)]

575 [52: (3,2) (3,2) (4,-1)] \* \* 575 [52: (3,2) (3,2) (5,-3)] > % 575 [52: (3,2) (3,2) (5,2)] SFS [52: (2,1) (2,1) (2,1) (2,1)

SFS [RP2/h2: (2,1) (3-2)] SES [RP2/n2: (2,1) (3,1)] SFSTRP2/n2: (2,1) (3,2) SESTE 0.20 SES DKB/h2 (1.2)

Txt/[21]11] + \* Tx1/[-2,1]-1,1]

SFS [S2: (2,1) (2,1) (2,1) (3,5] SFS [52: (2,1) (2,1) (2,1) (3,-2)] SFS [S2: (2,1) (2,1) (2,1) (0,-1)] SFS [RP2/n2: (2,1) (2,3)]

Simplify

( ) Orient

#### Closed, orientable and oriented, connected

<u>G</u> luings <u>S</u> k		eleton	Alge	Algebra <u>C</u> ompo		ition <u>R</u> ecogr		nition	Snap <u>P</u>	ea
Tetrahedron		Face 012		Face 013		Face 023		Face 123		
0		4 (231)		3 (132)		2 (123)		1 (132)		
1		3 (320)		5 (213)		2 (203)		0 (132)		
2		4 (032)		5 (302)		1 (203)		0 (023)		
3		5 (102)	5 (102)		4 (012)		1 (210)		0 (031)	
4		3 (013)		5 (031)		2 (021)		0 (201)		
5		3 (102)		4 (0	31)	2 (13	0) 1 (103		)	

Lee Lindblom (Physics Dept.: UCSD)

MiniWorkshop : NCU - 9/12/2023

## **Creating Multicube Representations**

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# **Creating Multicube Representations**

- Oliver Rinne has developed a python code that automatically converts the triangulation gluing structure from a Regina output file into a multicube structure.
- This figure shows the multicube structure for the manifold SFS[RP2/n2:(2,1)(2,-1)] from the Regina catalog.



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- This figure shows the multicube structure for the manifold SFS[RP2/n2:(2,1)(2,-1)] from the Regina catalog.



 Multicube structures have also been constructed by hand for some three-manifolds constructed by identifying the faces of polyhedra. Figure on the right shows a multicube structure for Seifert-Weber space.

- In three dimensions, building *C*<sup>0</sup> reference metrics follows the same basic approach used in two dimensions:
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- Combine the flat metrics in each multicube region using a partition of unity that is continuous across the cube interfaces.
- In three dimensions it is convenient to parameterize the flat inverse metrics in each cube using the dihedral angles between cube faces ψ<sub>A{xy}</sub>, ψ<sub>A{yz}</sub>, and ψ<sub>A{xz}</sub>:

$$ds^{-2} = \bar{g}^{ab} \partial_a \partial_b = \partial_x^2 + \partial_y^2 + \partial_z^2 \pm 2\cos\psi_{A\{xy\}} \partial_x \partial_y \\ \pm 2\cos\psi_{A\{yz\}} \partial_y \partial_z \pm 2\cos\psi_{A\{xz\}} \partial_x \partial_z.$$

The uniform dihedral angle assumption, ψ<sub>A{αβ</sub>} = <sup>2π</sup>/<sub>N<sub>A{αβ</sub></sub> (analogous to the method used in two dimensions) produces flat metrics with minimal coordinate distortion.

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- This assumption has been used to construct reference metrics successfully on each of the three-dimensional multicube structures constructed by hand.
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- In total, reference metrics were constructed successfully using this method on 40 different compact orientable manifolds.
- A more complicated method of choosing the dihedral angles allows the construction of reference metrics on 17 additional manifolds.

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- We use a simple constant mean curvature constraint equation,

$$\tilde{\nabla}^{a}\tilde{\nabla}_{a}\phi = \frac{1}{8}\phi\left(\tilde{R}-\phi^{4}\langle\tilde{R}\rangle
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to determine the conformal factor  $\phi$  that transforms the reference metric  $\tilde{g}_{ab}$  into the physical metric:  $g_{ab} = \phi^4 \tilde{g}_{ab}$ .  $\tilde{R}$  is the scalar curvature associated with  $\tilde{g}_{ab}$  and  $\langle \tilde{R} \rangle$  is its spatial average.

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- The solution to this constraint equation is also a solution to the Yamabe problem.



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- The accuracy of the hyperbolic relaxation solutions can be improved using the results as initial guesses for standard elliptic solves.

