The Relativistic Inverse Stellar Structure Problem

Lee Lindblom

Department of Physics
University of California at San Diego

Fundamental Theory Group Seminar
National Central University, Taiwan — 13 December 2023

The Relativistic Inverse Stellar Structure Problem

Lee Lindblom

Department of Physics
University of California at San Diego

Fundamental Theory Group Seminar
National Central University, Taiwan — 13 December 2023

- Can the equation of state of the matter in a neutron star be determined from astronomical observations?

The Relativistic Inverse Stellar Structure Problem

Lee Lindblom

Department of Physics
University of California at San Diego

Fundamental Theory Group Seminar
National Central University, Taiwan — 13 December 2023

- Can the equation of state of the matter in a neutron star be determined from astronomical observations?
- This talk describes some mathematical aspects of this question.
  - What is the relativistic inverse stellar structure problem?
  - How can it be solved?
  - How well does the solution work in practice?
  - How can gravitational radiation observations inform this problem?

Relativistic Stellar Structure Problem (SSP)

- Given an equation of state, $\epsilon = \epsilon(p)$, solve Einstein's equations,

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \qquad \frac{dp}{dr} = -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r - 2m)},$$

with boundary conditions at $r = 0$, $m(0) = 0$ and $p(0) = p_c$, to determine the structures of relativistic stars.

Relativistic Stellar Structure Problem (SSP)

- Given an equation of state, $\epsilon = \epsilon(p)$, solve Einstein's equations,

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \qquad \frac{dp}{dr} = -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)},$$

  with boundary conditions at $r = 0$, $m(0) = 0$ and $p(0) = p_c$, to determine the structures of relativistic stars.
- Find the radius $p(R) = 0$ and mass $M = m(R)$ for each star.
- Determine $M(p_c)$ and $R(p_c)$ for all physically relevant values of p_c.

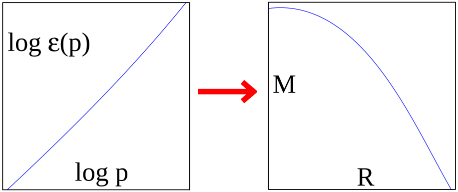
Relativistic Stellar Structure Problem (SSP)

- Given an equation of state, $\epsilon = \epsilon(p)$, solve Einstein's equations,

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \qquad \frac{dp}{dr} = -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)},$$

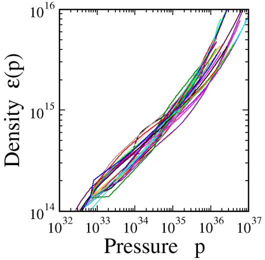
with boundary conditions at $r = 0$, $m(0) = 0$ and $p(0) = p_c$, to determine the structures of relativistic stars.

- Find the radius $p(R) = 0$ and mass $M = m(R)$ for each star.
- Determine $M(p_c)$ and $R(p_c)$ for all physically relevant values of p_c.
- SSP can be thought of as a map from the equation of state $\epsilon = \epsilon(p)$ to the M-R curve $\{R(p_c), M(p_c)\}$.



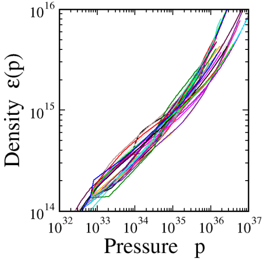
Relativistic Stellar Structure Problem (SSP) II

- Given an equation of state, $\epsilon = \epsilon(p)$, it is straightforward to solve Einstein's equations to determine the structures of neutron stars.
- Unfortunately, the equation of state of neutron-star matter is not well understood. Here are several dozen examples of published neutron-star equations of state.

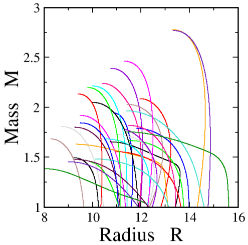


Relativistic Stellar Structure Problem (SSP) II

- Given an equation of state, $\epsilon = \epsilon(p)$, it is straightforward to solve Einstein's equations to determine the structures of neutron stars.
- Unfortunately, the equation of state of neutron-star matter is not well understood. Here are several dozen examples of published neutron-star equations of state.



- These equations of state produce a wide range of neutron-star models by solving the relativistic stellar structure problem.



Relativistic Stellar Structure Problem (SSP) III

- How can the relativistic stellar structure problem be used to interpret observations of neutron stars?
- One simple minded approach would be to use observations of neutron-star masses M and radii R to eliminate particular equation of state models.

Relativistic Stellar Structure Problem (SSP) III

- How can the relativistic stellar structure problem be used to interpret observations of neutron stars?
- One simple minded approach would be to use observations of neutron-star masses M and radii R to eliminate particular equation of state models.
- A more sophisticated approach would be to adjust the parameters of a particular nuclear theory model for the equation of state by fitting the resulting neutron-star models to the observations.
- Versions of this more sophisticated approach have been implemented by James Lattimer and collaborators, and also by Feryal Özel and collaborators.

Relativistic Stellar Structure Problem (SSP) III

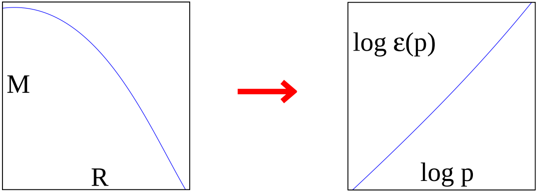
- How can the relativistic stellar structure problem be used to interpret observations of neutron stars?
- One simple minded approach would be to use observations of neutron-star masses M and radii R to eliminate particular equation of state models.
- A more sophisticated approach would be to adjust the parameters of a particular nuclear theory model for the equation of state by fitting the resulting neutron-star models to the observations.
- Versions of this more sophisticated approach have been implemented by James Lattimer and collaborators, and also by Feryal Özel and collaborators.
- Can we can do better?
- Do Einstein's equations determine the neutron-star equation of state directly without assuming any nuclear-theory model?

Relativistic Inverse Stellar Structure Problem (SSP^{-1})

- The inverse stellar structure problem (SSP^{-1}) finds the equation of state $\epsilon = \epsilon(p)$ from the macroscopic observables of the stars, e.g. their masses and radii.

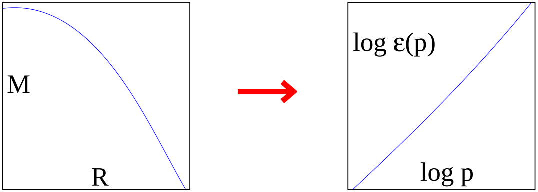
Relativistic Inverse Stellar Structure Problem (SSP^{-1})

- The inverse stellar structure problem (SSP^{-1}) finds the equation of state $\epsilon = \epsilon(p)$ from the macroscopic observables of the stars, e.g. their masses and radii.
- SSP^{-1} can be thought of as the map from the M-R curve $\{R(p_c), M(p_c)\}$ to the equation of state $\epsilon = \epsilon(p)$.



Relativistic Inverse Stellar Structure Problem (SSP^{-1})

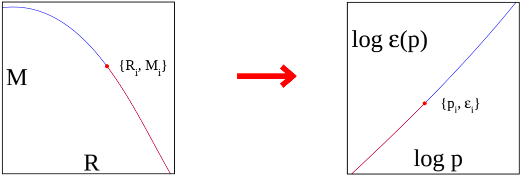
- The inverse stellar structure problem (SSP^{-1}) finds the equation of state $\epsilon = \epsilon(p)$ from the macroscopic observables of the stars, e.g. their masses and radii.
- SSP^{-1} can be thought of as the map from the M-R curve $\{R(p_c), M(p_c)\}$ to the equation of state $\epsilon = \epsilon(p)$.



- The basic mathematical questions then become, "Does this problem have a solution?", "Is the solution unique?", and "How do we solve it?"

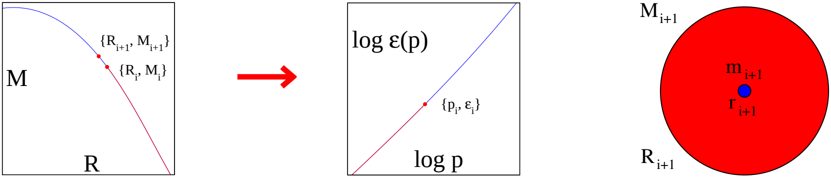
“Formal” Solution to SSP^{-1}

- Assume the complete M-R curve is known, including the point $\{R_i, M_i\} = \{R(p_i), M(p_i)\}$.
- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.



"Formal" Solution to SSP^{-1}

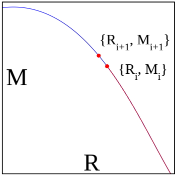
- Assume the complete M-R curve is known, including the point $\{R_i, M_i\} = \{R(p_i), M(p_i)\}$.
- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.

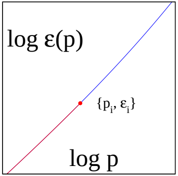


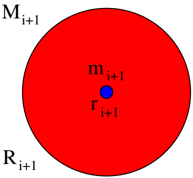
- Choose a new point on the M-R curve, $\{R_{i+1}, M_{i+1}\}$, having slightly larger central density.

"Formal" Solution to SSP^{-1}

- Assume the complete M-R curve is known, including the point $\{R_i, M_i\} = \{R(p_i), M(p_i)\}$.
- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.







- Choose a new point on the M-R curve, $\{R_{i+1}, M_{i+1}\}$, having slightly larger central density.
- Integrate Einstein's equations,

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \qquad \frac{dp}{dr} = -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)},$$

through the outer parts of the star, to determine the mass and radius, $\{r_{i+1}, m_{i+1}\}$, of the small core with large densities $\epsilon \geq \epsilon_i$.

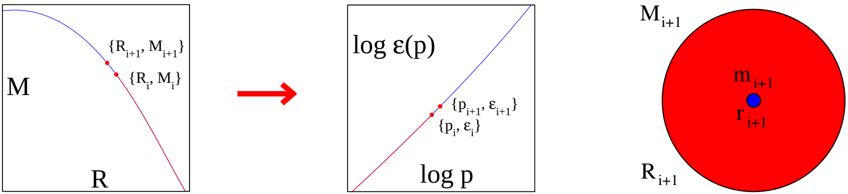
Formal Solution to SSP^{-1} II

- For very small cores, $\{r_{i+1}, m_{i+1}\}$, the solution to the OV equations is described by the power series solution:

$$m_{i+1} = \frac{4\pi}{3}\epsilon_{i+1} r_{i+1}^3 + \mathcal{O}(r_{i+1}^5),$$

$$p_i = p_{i+1} - \frac{2\pi}{3}(\epsilon_{i+1} + p_{i+1})(\epsilon_{i+1} + 3p_{i+1}) r_{i+1}^2 + \mathcal{O}(r_{i+1}^4).$$

- Invert these series to determine the central pressure and density, $\{p_{i+1}, \epsilon_{i+1}\}$, in terms of the known quantities, m_{i+1}, r_{i+1}, p_i, ϵ_i.



Can the Formal Solution to SSP^{-1} be Improved?

- Formal solution to the relativistic SSP^{-1} finds the equation of state, $\epsilon = \epsilon(p)$, represented as a table, $\{p_i, \epsilon_i\}$ for $i = 1, \ldots, N$, and an interpolation formula.
- Formal solution has several weaknesses:
  - Solution converges (slowly) with the number of points, as N^{-q}.
  - Each new equation of state point, $\{p_i, \epsilon_i\}$, requires the knowledge of a separate new M-R curve point, $\{R_i, M_i\}$.
  - Accurate M-R curve points $\{R_i, M_i\}$ for neutron stars are scarce.

Can the Formal Solution to SSP^{-1} be Improved?

- Formal solution to the relativistic SSP^{-1} finds the equation of state, $\epsilon = \epsilon(p)$, represented as a table, $\{p_i, \epsilon_i\}$ for $i = 1, \ldots, N$, and an interpolation formula.

- Formal solution has several weaknesses:
  - Solution converges (slowly) with the number of points, as N^{-q}.
  - Each new equation of state point, $\{p_i, \epsilon_i\}$, requires the knowledge of a separate new M-R curve point, $\{R_i, M_i\}$.
  - Accurate M-R curve points $\{R_i, M_i\}$ for neutron stars are scarce.

- Spectral numerical methods typically converge more rapidly, and represent functions more efficiently than finite difference methods.

Can the Formal Solution to SSP^{-1} be Improved?

- Formal solution to the relativistic SSP^{-1} finds the equation of state, $\epsilon = \epsilon(p)$, represented as a table, $\{p_i, \epsilon_i\}$ for $i = 1, ..., N$, and an interpolation formula.
- Formal solution has several weaknesses:
  - Solution converges (slowly) with the number of points, as N^{-q}.
  - Each new equation of state point, $\{p_i, \epsilon_i\}$, requires the knowledge of a separate new M-R curve point, $\{R_i, M_i\}$.
  - Accurate M-R curve points $\{R_i, M_i\}$ for neutron stars are scarce.
- Spectral numerical methods typically converge more rapidly, and represent functions more efficiently than finite difference methods.
- Can spectral methods provide better (*i.e.* more practical and more accurate) solutions to the SSP^{-1}?
- Can spectral methods provide interesting solutions to SSP^{-1} when only a few (*e.g.* two or three) M-R data points are available?

Outline for Solving SSP^{-1} Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \gamma_k)$, where the γ_k are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \gamma_k) = \sum_k \gamma_k \Phi_k(p)$, where the $\Phi_k(p)$ are spectral basis functions, *e.g.* $\Phi_k(p) = e^{ikp}$, or $\Phi_k(p) = P_k(p)$.

Outline for Solving SSP^{-1} Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \gamma_k)$, where the γ_k are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \gamma_k) = \sum_k \gamma_k \Phi_k(p)$, where the $\Phi_k(p)$ are spectral basis functions, *e.g.* $\Phi_k(p) = e^{ikp}$, or $\Phi_k(p) = P_k(p)$.
- For a given equation of state, *i.e.* a particular choice of γ_k, solve the SSP to obtain a model M-R curve: $\{R(p_c, \gamma_k), M(p_c, \gamma_k)\}$.

Outline for Solving SSP^{-1} Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \gamma_k)$, where the γ_k are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \gamma_k) = \sum_k \gamma_k \Phi_k(p)$, where the $\Phi_k(p)$ are spectral basis functions, *e.g.* $\Phi_k(p) = e^{ikp}$, or $\Phi_k(p) = P_k(p)$.
- For a given equation of state, *i.e.* a particular choice of γ_k, solve the SSP to obtain a model M-R curve: $\{R(p_c, \gamma_k), M(p_c, \gamma_k)\}$.
- Given a set of points from the "real" M-R curve, $\{R_i, M_i\}$, choose the parameters γ_k and p_c^i that minimize the difference measure:

$$\chi^2 = \frac{1}{N_{\rm stars}} \sum_{i=1}^{N_{\rm stars}} \left\{ \left[\log\left(\frac{R(p_c^i, \gamma_k)}{R_i} \right) \right]^2 + \left[\log\left(\frac{M(p_c^i, \gamma_k)}{M_i} \right) \right]^2 \right\}$$

Outline for Solving SSP^{-1} Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \gamma_k)$, where the γ_k are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \gamma_k) = \sum_k \gamma_k \Phi_k(p)$, where the $\Phi_k(p)$ are spectral basis functions, *e.g.* $\Phi_k(p) = e^{ikp}$, or $\Phi_k(p) = P_k(p)$.
- For a given equation of state, *i.e.* a particular choice of γ_k, solve the SSP to obtain a model M-R curve: $\{R(p_c, \gamma_k), M(p_c, \gamma_k)\}$.
- Given a set of points from the "real" M-R curve, $\{R_i, M_i\}$, choose the parameters γ_k and p_c^i that minimize the difference measure:

$$\chi^2 = \frac{1}{N_{\text{stars}}} \sum_{i=1}^{N_{\text{stars}}} \left\{ \left[\log\left(\frac{R(p_c^i, \gamma_k)}{R_i} \right) \right]^2 + \left[\log\left(\frac{M(p_c^i, \gamma_k)}{M_i} \right) \right]^2 \right\}$$

- Resulting γ_k for $k = 1, ..., N_{\gamma_k}$ determines an equation of state, $\epsilon = \epsilon(p, \gamma_k)$, that provides an approximate solution of SSP^{-1}.

Basic Questions

- Do spectral expansions provide an efficient way to represent realistic neutron-star equations of state?
- What choice of spectral basis functions is useful?

Basic Questions

- Do spectral expansions provide an efficient way to represent realistic neutron-star equations of state?
- What choice of spectral basis functions is useful?
- Can the spectral parameters γ_k be determined accurately and robustly by matching model masses and radii $\{R(p_c^i, \gamma_k), M(p_c^i, \gamma_k)\}$ to given $\{R_i, M_i\}$ data?

Basic Questions

- Do spectral expansions provide an efficient way to represent realistic neutron-star equations of state?
- What choice of spectral basis functions is useful?
- Can the spectral parameters γ_k be determined accurately and robustly by matching model masses and radii $\{R(p_c^i, \gamma_k), M(p_c^i, \gamma_k)\}$ to given $\{R_i, M_i\}$ data?
- These questions are best answered using a somewhat different form of the standard stellar structure problem (SSP).
- Digress (briefly) now to describe this alternate formulation that provides a more efficient and more accurate way to solve the SSP.

Alternative Representations of the SSP

- The standard Oppenheimer-Volkoff (OV) representation of the SSP equations determines $m(r)$ and $p(r)$, given an equation of state of the form $\epsilon = \epsilon(p)$.
- The outer boundary of the star is the point where $p(R) = 0$. This is difficult to solve numerically because the pressure goes to zero non-linearly there: $p \propto (R - r)^{\Gamma_0/(\Gamma_0 - 1)} \approx (R - r)^{5/2}$.

Alternative Representations of the SSP

- The standard Oppenheimer-Volkoff (OV) representation of the SSP equations determines $m(r)$ and $p(r)$, given an equation of state of the form $\epsilon = \epsilon(p)$.
- The outer boundary of the star is the point where $p(R) = 0$. This is difficult to solve numerically because the pressure goes to zero non-linearly there: $p \propto (R - r)^{\Gamma_0/(\Gamma_0 - 1)} \approx (R - r)^{5/2}$.
- This problem can be avoided by introducing the relativistic enthalpy $h(p) = \int_0^p dp'/[\epsilon(p') + p']$, and re-writing the OV equations in terms of it:

$$\frac{dh}{dr} = \frac{1}{\epsilon + p}\frac{dp}{dr} = -\frac{m + 4\pi r^3 p(h)}{r(r - 2m)}.$$

Alternative Representations of the SSP

- The standard Oppenheimer-Volkoff (OV) representation of the SSP equations determines $m(r)$ and $p(r)$, given an equation of state of the form $\epsilon = \epsilon(p)$.
- The outer boundary of the star is the point where $p(R) = 0$. This is difficult to solve numerically because the pressure goes to zero non-linearly there: $p \propto (R - r)^{\Gamma_0/(\Gamma_0 - 1)} \approx (R - r)^{5/2}$.
- This problem can be avoided by introducing the relativistic enthalpy $h(p) = \int_0^p dp'/[\epsilon(p') + p']$, and re-writing the OV equations in terms of it:

$$\frac{dh}{dr} = \frac{1}{\epsilon + p}\frac{dp}{dr} = -\frac{m + 4\pi r^3 p(h)}{r(r - 2m)}.$$

- The surface of the star is now the point where $h(R) = 0$. This condition is easier to solve numerically because the enthalpy goes to zero linearly there: $h(r) \propto (R - r)$.

Alternative Representations of the SSP II

- Simplify again by swapping the roles of h and r:

$$\frac{dr}{dh} = -\frac{r(r-2m)}{m+4\pi r^3 p(h)}, \qquad \frac{dm}{dh} = -\frac{4\pi\epsilon(h)r^3(r-2m)}{m+4\pi r^3 p(h)}.$$

Alternative Representations of the SSP II

- Simplify again by swapping the roles of h and r:

$$\frac{dr}{dh} = -\frac{r(r-2m)}{m+4\pi r^3 p(h)}, \qquad \frac{dm}{dh} = -\frac{4\pi\epsilon(h) r^3 (r-2m)}{m+4\pi r^3 p(h)}.$$

- This form of the equations is easier to solve numerically:
  - The domain on which the solution $\{r(h), m(h)\}$ is defined, $h_c \geq h \geq 0$, is known *a priori*.
  - The total mass M and radius R are determined simply by evaluating the solution at $h = 0$, $\{R, M\} = \{r(0), m(0)\}$.

Alternative Representations of the SSP II

- Simplify again by swapping the roles of h and r:

$$\frac{dr}{dh} = -\frac{r(r-2m)}{m+4\pi r^3 p(h)}, \qquad \frac{dm}{dh} = -\frac{4\pi\epsilon(h) r^3 (r-2m)}{m+4\pi r^3 p(h)}.$$

- This form of the equations is easier to solve numerically:
  - The domain on which the solution $\{r(h), m(h)\}$ is defined, $h_c \geq h \geq 0$, is known *a priori*.
  - The total mass M and radius R are determined simply by evaluating the solution at $h=0$, $\{R, M\} = \{r(0), m(0)\}$.
- These alternative OV equations require that the equation of state, $\epsilon = \epsilon(p)$, be re-written as $\epsilon = \epsilon(h)$ and $p = p(h)$:

Alternative Representations of the SSP II

- Simplify again by swapping the roles of h and r:

$$\frac{dr}{dh} = -\frac{r(r-2m)}{m+4\pi r^3 p(h)}, \qquad \frac{dm}{dh} = -\frac{4\pi\epsilon(h) r^3 (r-2m)}{m+4\pi r^3 p(h)}.$$

- This form of the equations is easier to solve numerically:
  - The domain on which the solution $\{r(h), m(h)\}$ is defined, $h_c \geq h \geq 0$, is known *a priori*.
  - The total mass M and radius R are determined simply by evaluating the solution at $h = 0$, $\{R, M\} = \{r(0), m(0)\}$.
- These alternative OV equations require that the equation of state, $\epsilon = \epsilon(p)$, be re-written as $\epsilon = \epsilon(h)$ and $p = p(h)$:
  - Start with the standard, $\epsilon = \epsilon(p)$.
  - Compute, $h(p) = \int_0^p dp'/[\epsilon(p') + p']$.
  - Invert to give $p = p(h)$.
  - Compose $\epsilon = \epsilon(p)$ with $p = p(h)$, to give $\epsilon = \epsilon(h) = \epsilon[p(h)]$.

Faithful Spectral Expansions of the Equation of State

- Physical equations of state, $\epsilon = \epsilon(h)$ and $p = p(h)$, are positive monotonic increasing functions.
- Naive spectral representations, $\epsilon = \epsilon(h, \alpha_k) = \sum_k \alpha_k \Phi_k(h)$ and $p = p(h, \beta_k) = \sum_k \beta_k \Phi_k(h)$, are not faithful.

Faithful Spectral Expansions of the Equation of State

- Physical equations of state, $\epsilon = \epsilon(h)$ and $p = p(h)$, are positive monotonic increasing functions.
- Naive spectral representations, $\epsilon = \epsilon(h, \alpha_k) = \sum_k \alpha_k \Phi_k(h)$ and $p = p(h, \beta_k) = \sum_k \beta_k \Phi_k(h)$, are not faithful.
- Faithful here means:
  - *i)* Every choice of spectral parameters, α_k and β_k, corresponds to a possible physical equation of state.
  - *ii)* Every physical equation of state can be represented by such an expansion.

Faithful Spectral Expansions of the Equation of State

- Physical equations of state, $\epsilon = \epsilon(h)$ and $p = p(h)$, are positive monotonic increasing functions.
- Naive spectral representations, $\epsilon = \epsilon(h, \alpha_k) = \sum_k \alpha_k \Phi_k(h)$ and $p = p(h, \beta_k) = \sum_k \beta_k \Phi_k(h)$, are not faithful.
- Faithful here means:
  - *i)* Every choice of spectral parameters, α_k and β_k, corresponds to a possible physical equation of state.
  - *ii)* Every physical equation of state can be represented by such an expansion.
- Faithful spectral expansions of the adiabatic index Γ do exist:

$$\Gamma(h) = \frac{\epsilon + p}{p}\frac{dp}{d\epsilon} = \exp\left[\sum_k \gamma_k \Phi_k(h)\right].$$

Faithful Spectral Expansions of the Equation of State II

- Every equation of state is determined by the adiabatic index $\Gamma(h)$:

$$\frac{dp}{dh} = \epsilon + p, \qquad \frac{d\epsilon}{dh} = \frac{(\epsilon + p)^2}{p\,\Gamma(h)}.$$

Faithful Spectral Expansions of the Equation of State II

- Every equation of state is determined by the adiabatic index $\Gamma(h)$:

$$\frac{dp}{dh} = \epsilon + p, \qquad \frac{d\epsilon}{dh} = \frac{(\epsilon + p)^2}{p\,\Gamma(h)}.$$

- The solutions to these equations can be reduced to quadratures:

$$\begin{aligned}
\mu(h) &= \frac{p_0 e^{h_0}}{\epsilon_0 + p_0} + \int_{h_0}^{h} \frac{\Gamma(h') - 1}{\Gamma(h')} e^{h'} dh', \\
p(h) &= p_0 \exp\left[\int_{h_0}^{h} \frac{e^{h'} dh'}{\mu(h')}\right], \\
\epsilon(h) &= p(h) \frac{e^h - \mu(h)}{\mu(h)}.
\end{aligned}$$

Faithful Spectral Expansions of the Equation of State II

- Every equation of state is determined by the adiabatic index $\Gamma(h)$:

$$\frac{dp}{dh} = \epsilon + p, \qquad \frac{d\epsilon}{dh} = \frac{(\epsilon+p)^2}{p\,\Gamma(h)}.$$

- The solutions to these equations can be reduced to quadratures:

$$\mu(h) = \frac{p_0 e^{h_0}}{\epsilon_0 + p_0} + \int_{h_0}^{h} \frac{\Gamma(h') - 1}{\Gamma(h')} e^{h'} dh',$$

$$p(h) = p_0 \exp\left[\int_{h_0}^{h} \frac{e^{h'} dh'}{\mu(h')}\right],$$

$$\epsilon(h) = p(h)\frac{e^h - \mu(h)}{\mu(h)}.$$

- Choosing, $\log\Gamma(h) = \sum_k \gamma_k \Phi_k(h)$, for any spectral basis functions, $\Phi_k(h)$, results in a faithful parameterized equation of state of the desired form: $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$.

Causal Spectral Representations

- In a barotropic fluid the speed of sound v is related to the equation of state by: $v^2 = dp/d\epsilon$.

Causal Spectral Representations

- In a barotropic fluid the speed of sound v is related to the equation of state by: $v^2 = dp/d\epsilon$.
- The velocity function

$$\Upsilon(h) = \frac{1 - v^2(h)}{v^2(h)} = \frac{d\epsilon}{dp} - 1$$

is non-negative, $\Upsilon(h) \geq 0$, if and only if the fluid is causal: $0 < v^2 \leq 1$.

Causal Spectral Representations

- In a barotropic fluid the speed of sound v is related to the equation of state by: $v^2 = dp/d\epsilon$.
- The velocity function

$$\Upsilon(h) = \frac{1 - v^2(h)}{v^2(h)} = \frac{d\epsilon}{dp} - 1$$

is non-negative, $\Upsilon(h) \geq 0$, if and only if the fluid is causal: $0 < v^2 \leq 1$.

- Construct a spectral representation of $\Upsilon(h)$ that ensures causality:

$$\Upsilon(h) = \exp\left\{\sum_k \upsilon_k \Phi_k(h)\right\}.$$

Causal Spectral Representations II

- The velocity function $\Upsilon(h)$ determines the equation of state, $\epsilon(h)$ and $p(h)$, through the odes that define Υ and h:

$$\frac{d\epsilon}{dh} = (\epsilon + p)\,[1 + \Upsilon(h)]\,,$$
$$\frac{dp}{dh} = \epsilon + p.$$

Causal Spectral Representations II

- The velocity function $\Upsilon(h)$ determines the equation of state, $\epsilon(h)$ and $p(h)$, through the odes that define Υ and h:

$$\begin{aligned}\frac{d\epsilon}{dh} &= (\epsilon + p)\left[1 + \Upsilon(h)\right],\\ \frac{dp}{dh} &= \epsilon + p.\end{aligned}$$

- Integrate these odes to obtain a causal spectral representation of the equation of state:

$$\begin{aligned}\mu(h) &= \exp\left\{\int_{h_0}^{h}\left[2 + \Upsilon(h')\right]dh'\right\},\\ p(h) &= p_0 + (\epsilon_0 + p_0)\int_{h_0}^{h}\mu(h')dh',\\ \epsilon(h) &= \epsilon_0 - p(h) + (\epsilon_0 + p_0)\mu(h).\end{aligned}$$

Fitting Model Neutron-Star Equations of State

- How accurately and efficiently are realistic neutron-star equations of state represented by $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$, when $\Gamma(h)$ is given by

$$\Gamma(h) = \exp\left\{ \sum_{k=0}^{N_{\gamma_k}-1} \gamma_k \left[\log\left(\frac{h}{h_0}\right)\right]^k \right\} ?$$

Fitting Model Neutron-Star Equations of State

- How accurately and efficiently are realistic neutron-star equations of state represented by $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$, when $\Gamma(h)$ is given by

$$\Gamma(h) = \exp\left\{\sum_{k=0}^{N_{\gamma_k}-1} \gamma_k \left[\log\left(\frac{h}{h_0}\right)\right]^k\right\} ?$$

- Let $\{p_i, \epsilon_i, h_i\}$, for $i = 1, \ldots, N_{\mathrm{EOS}}$ denote one of the standard tabulated realistic neutron-star equations of state.
- Find the spectral parameters γ_k that minimize the fitting error:

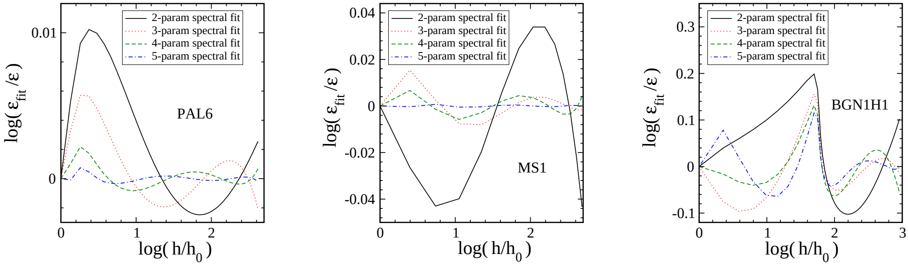
$$\left(\Delta^{EOS}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\mathrm{EOS}}} \sum_{i=1}^{N_{\mathrm{EOS}}} \left(\log\left[\frac{\epsilon(h_i, \gamma_k)}{\epsilon_i}\right]\right)^2 .$$

Fitting Model Neutron-Star Equations of State

- How accurately and efficiently are realistic neutron-star equations of state represented by $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$, when $\Gamma(h)$ is given by
$$\Gamma(h) = \exp\left\{\sum_{k=0}^{N_{\gamma_k}-1} \gamma_k \left[\log\left(\frac{h}{h_0}\right)\right]^k\right\} ?$$
- Let $\{p_i, \epsilon_i, h_i\}$, for $i = 1, \ldots, N_{\rm EOS}$ denote one of the standard tabulated realistic neutron-star equations of state.
- Find the spectral parameters γ_k that minimize the fitting error:
$$\left(\Delta^{EOS}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\rm EOS}} \sum_{i=1}^{N_{\rm EOS}} \left(\log\left[\frac{\epsilon(h_i, \gamma_k)}{\epsilon_i}\right]\right)^2 .$$
- The average values of these fitting errors, $\Delta^{EOS}_{N_{\gamma_k}}$, for 34 realistic neutron-star equations of state are:
$$\Delta^{EOS}_2 = 0.032, \qquad \Delta^{EOS}_3 = 0.017,$$
$$\Delta^{EOS}_4 = 0.012, \qquad \Delta^{EOS}_5 = 0.0089.$$

Spectral Fits of Model Neutron-Star Equations of State

- Relative errors between the spectral equations of state $\epsilon(h, \gamma_k)$, and the exact equation of states $\epsilon(h)$ that they approximate.



- Average errors $\Delta^{EOS}_{N_{\gamma_k}}$ for the best, ‘average’, and worst cases are:

PAL6: $\Delta^{EOS}_2 = 0.0032$, $\Delta^{EOS}_3 = 0.0016$, $\Delta^{EOS}_4 = 0.0005$, $\Delta^{EOS}_5 = 0.0002$,

MS1: $\Delta^{EOS}_2 = 0.0277$, $\Delta^{EOS}_3 = 0.0055$, $\Delta^{EOS}_4 = 0.0035$, $\Delta^{EOS}_5 = 0.0003$,

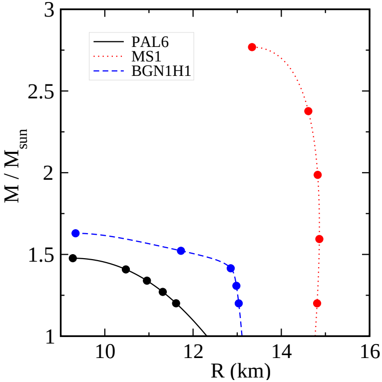
BGN1H1: $\Delta^{EOS}_2 = 0.087$, $\Delta^{EOS}_3 = 0.050$, $\Delta^{EOS}_4 = 0.044$, $\Delta^{EOS}_5 = 0.040$.

Spectral Solution of SSP^{-1}

- Next step: test this spectral approach to solving the SSP^{-1} using mock observational data based on realistic neutron-star models.

Spectral Solution of SSP^{-1}

- Next step: test this spectral approach to solving the SSP^{-1} using mock observational data based on realistic neutron-star models.
- Choose mock data points $\{R_i, M_i\}$ for neutron-star models computed with 34 realistic equations of state.



Spectral Solution of SSP^{-1} II

- Fix the spectral expansion coefficients γ_k by minimizing,

$$\chi^2 = \frac{1}{N_{\rm stars}}\sum_{i=1}^{N_{\rm stars}}\left\{\left[\log\left(\frac{M(h_c^i,\gamma_k)}{M_i}\right)\right]^2+\left[\log\left(\frac{R(h_c^i,\gamma_k)}{R_i}\right)\right]^2\right\}.$$

  with respect to variations in γ_k, and variations in the central values of the enthalpy for each star, h_c^i.

Spectral Solution of SSP^{-1} II

- Fix the spectral expansion coefficients γ_k by minimizing,

$$\chi^2 = \frac{1}{N_{\text{stars}}} \sum_{i=1}^{N_{\text{stars}}} \left\{ \left[\log \left(\frac{M(h_c^i, \gamma_k)}{M_i} \right) \right]^2 + \left[\log \left(\frac{R(h_c^i, \gamma_k)}{R_i} \right) \right]^2 \right\}.$$

  with respect to variations in γ_k, and variations in the central values of the enthalpy for each star, h_c^i.

- These γ_k determine an equation of state, $\epsilon(h, \gamma_k)$ and $p(h, \gamma_k)$, that provides an approximate solution to SSP^{-1}.

Spectral Solution of SSP^{-1} II

- Fix the spectral expansion coefficients γ_k by minimizing,

$$\chi^2 = \frac{1}{N_{\rm stars}}\sum_{i=1}^{N_{\rm stars}}\left\{\left[\log\left(\frac{M(h_c^i,\gamma_k)}{M_i}\right)\right]^2+\left[\log\left(\frac{R(h_c^i,\gamma_k)}{R_i}\right)\right]^2\right\}.$$

  with respect to variations in γ_k, and variations in the central values of the enthalpy for each star, h_c^i.

- These γ_k determine an equation of state, $\epsilon(h,\gamma_k)$ and $p(h,\gamma_k)$, that provides an approximate solution to SSP^{-1}.

- Evaluate the accuracy of this equation of state by measuring the fitting errors, $\Delta^{MR}_{N_{\gamma_k}}$,

$$\left(\Delta^{MR}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\rm EOS}}\sum_{i=1}^{N_{\rm EOS}}\left[\log\left(\frac{\epsilon(h_i,\gamma_k)}{\epsilon_i}\right)\right]^2$$

  to determine how well the spectral expansion $\epsilon=\epsilon(h,\gamma_k)$, matches the exact neutron-star equation of state $\epsilon=\epsilon(h)$.

Spectral Solutions to SSP^{-1} III

- The average values of $\Delta^{MR}_{N_{\gamma_k}}$ (with $N_{\gamma_k} = N_{\mathrm{stars}}$) determined in this way for 34 realistic model equations of state are:

$$\Delta^{MR}_2 = 0.040, \qquad \Delta^{EOS}_2 = 0.032,$$
$$\Delta^{MR}_3 = 0.029, \qquad \Delta^{EOS}_3 = 0.017,$$
$$\Delta^{MR}_4 = 0.028, \qquad \Delta^{EOS}_4 = 0.012,$$
$$\Delta^{MR}_5 = 0.024, \qquad \Delta^{EOS}_5 = 0.0089.$$

Spectral Solutions to SSP^{-1} III

- The average values of $\Delta^{MR}_{N_{\gamma_k}}$ (with $N_{\gamma_k} = N_{\rm stars}$) determined in this way for 34 realistic model equations of state are:

$$\begin{aligned} \Delta^{MR}_2 &= 0.040, & \Delta^{EOS}_2 &= 0.032, \\ \Delta^{MR}_3 &= 0.029, & \Delta^{EOS}_3 &= 0.017, \\ \Delta^{MR}_4 &= 0.028, & \Delta^{EOS}_4 &= 0.012, \\ \Delta^{MR}_5 &= 0.024, & \Delta^{EOS}_5 &= 0.0089. \end{aligned}$$

- The average errors are quite impressive, even using a small number of high accuracy M-R data points.

Spectral Solutions to SSP^{-1} III

- The average values of $\Delta^{MR}_{N_{\gamma_k}}$ (with $N_{\gamma_k} = N_{\rm stars}$) determined in this way for 34 realistic model equations of state are:

$$\begin{aligned} \Delta^{MR}_2 &= 0.040, & \Delta^{EOS}_2 &= 0.032,\\ \Delta^{MR}_3 &= 0.029, & \Delta^{EOS}_3 &= 0.017,\\ \Delta^{MR}_4 &= 0.028, & \Delta^{EOS}_4 &= 0.012,\\ \Delta^{MR}_5 &= 0.024, & \Delta^{EOS}_5 &= 0.0089. \end{aligned}$$

- The average errors are quite impressive, even using a small number of high accuracy M-R data points.

- Average errors $\Delta^{MR}_{N_{\gamma_k}}$ for the best, ‘average’, and worst cases are:

PAL6: $\Delta^{MR}_2 = 0.0034$, $\Delta^{MR}_3 = 0.0018$, $\Delta^{MR}_4 = 0.0007$, $\Delta^{MR}_5 = 0.0003$,

MS1: $\Delta^{MR}_2 = 0.0474$, $\Delta^{MR}_3 = 0.0157$, $\Delta^{MR}_4 = 0.0132$, $\Delta^{MR}_5 = 0.0009$,

BGN1H1: $\Delta^{MR}_2 = 0.135$, $\Delta^{MR}_3 = 0.170$, $\Delta^{MR}_4 = 0.136$, $\Delta^{MR}_5 = 0.138$.

Can SSP^{-1} be Solved with Gravitational Wave Data?

- GW observations of neutron-star binary systems can not determine the radii R of those stars, but can (in principle) determine their masses M and tidal deformabilities Λ.

Can SSP^{-1} be Solved with Gravitational Wave Data?

- GW observations of neutron-star binary systems can not determine the radii R of those stars, but can (in principle) determine their masses M and tidal deformabilities Λ.
- Can we solve the SSP^{-1} by matching model values of $\{M(h_c^i,\gamma_k),\Lambda(h_c^i,\gamma_k)\}$ to gravitational wave observations of $\{M_i,\Lambda_i\}$ by minimizing,

$$\chi^2 = \frac{1}{N_{\rm stars}}\sum_{i=1}^{N_{\rm stars}}\left\{\left[\log\left(\frac{M(h_c^i,\gamma_k)}{M_i}\right)\right]^2+\left[\log\left(\frac{\Lambda(h_c^i,\gamma_k)}{\Lambda_i}\right)\right]^2\right\}?$$

Can SSP^{-1} be Solved with Gravitational Wave Data?

- GW observations of neutron-star binary systems can not determine the radii R of those stars, but can (in principle) determine their masses M and tidal deformabilities Λ.
- Can we solve the SSP^{-1} by matching model values of $\{M(h_c^i,\gamma_k),\Lambda(h_c^i,\gamma_k)\}$ to gravitational wave observations of $\{M_i,\Lambda_i\}$ by minimizing,

$$\chi^2 = \frac{1}{N_{\text{stars}}}\sum_{i=1}^{N_{\text{stars}}}\left\{\left[\log\left(\frac{M(h_c^i,\gamma_k)}{M_i}\right)\right]^2+\left[\log\left(\frac{\Lambda(h_c^i,\gamma_k)}{\Lambda_i}\right)\right]^2\right\}?$$

- Compare average accuracy of the resulting equations of state $\Delta_N^{M\Lambda}$ with those obtained using mass-radius data, Δ_N^{MR}.

$$\Delta_2^{M\Lambda}=0.040,\ \Delta_3^{M\Lambda}=0.030,\ \Delta_4^{M\Lambda}=0.030,\ \Delta_5^{M\Lambda}=0.027,$$

$$\Delta_2^{MR}=0.040,\ \Delta_3^{MR}=0.029,\ \Delta_4^{MR}=0.028,\ \Delta_5^{MR}=0.024.$$

Can SSP^{-1} be Solved with Gravitational Wave Data?

- GW observations of neutron-star binary systems can not determine the radii R of those stars, but can (in principle) determine their masses M and tidal deformabilities Λ.
- Can we solve the SSP^{-1} by matching model values of $\{M(h_c^i,\gamma_k),\Lambda(h_c^i,\gamma_k)\}$ to gravitational wave observations of $\{M_i,\Lambda_i\}$ by minimizing,

$$\chi^2 = \frac{1}{N_{\rm stars}}\sum_{i=1}^{N_{\rm stars}}\left\{\left[\log\left(\frac{M(h_c^i,\gamma_k)}{M_i}\right)\right]^2+\left[\log\left(\frac{\Lambda(h_c^i,\gamma_k)}{\Lambda_i}\right)\right]^2\right\}?$$

- Compare average accuracy of the resulting equations of state $\Delta_N^{M\Lambda}$ with those obtained using mass-radius data, Δ_N^{MR}.

$$\Delta_2^{M\Lambda}=0.040,\ \ \Delta_3^{M\Lambda}=0.030,\ \ \Delta_4^{M\Lambda}=0.030,\ \ \Delta_5^{M\Lambda}=0.027,$$

$$\Delta_2^{MR}=0.040,\ \ \Delta_3^{MR}=0.029,\ \ \Delta_4^{MR}=0.028,\ \ \Delta_5^{MR}=0.024.$$

- The solution to the SSP^{-1} for neutron stars therefore can be done with mass-radius data, $\{M_i,R_i\}$, or with mass-tidal-deformability data, $\{M_i,\Lambda_i\}$, at about the same level of accuracy.

Can SSP^{-1} be Solved with LIGO/VIRGO Data?

- Present GW observations of neutron-star binary systems can determine the masses M_1 and M_2 but not the tidal deformabilities Λ_1 and Λ_2 of the individual stars.

Can SSP^{-1} be Solved with LIGO/VIRGO Data?

- Present GW observations of neutron-star binary systems can determine the masses M_1 and M_2 but not the tidal deformabilities Λ_1 and Λ_2 of the individual stars.
- Measurements in the LIGO/VIRGO frequency band can however measure a composite tidal deformability of the system:

$$\tilde{\Lambda} = \frac{16}{13}\frac{M_1^4(M_1+12M_2)\Lambda_1 + M_2^4(M_2+12M_1)\Lambda_2}{(M_1+M_2)^5}.$$

Can SSP^{-1} be Solved with LIGO/VIRGO Data?

- Present GW observations of neutron-star binary systems can determine the masses M_1 and M_2 but not the tidal deformabilities Λ_1 and Λ_2 of the individual stars.
- Measurements in the LIGO/VIRGO frequency band can however measure a composite tidal deformability of the system:

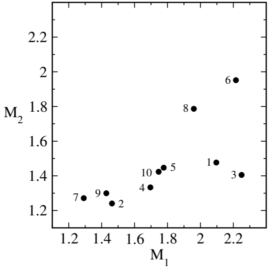
$$\tilde{\Lambda} = \frac{16}{13}\frac{M_1^4(M_1+12M_2)\Lambda_1 + M_2^4(M_2+12M_1)\Lambda_2}{(M_1+M_2)^5}.$$

- Can we solve the SSP^{-1} by matching model values of $\{M_1(h^i_{1c},\epsilon_k), M_2(h^i_{2c}), \tilde{\Lambda}(h^i_{1c}, h^i_{2c},\epsilon_k)\}$ to gravitational wave observations of $\{M_{1i}, M_{2i}, \tilde{\Lambda}_i\}$ by minimizing,

$$\chi^2 = \frac{1}{N_{\rm stars}}\sum_{i=1}^{N_{\rm stars}}\left\{\left[\log\left(\frac{M_1(h^i_{1c},\epsilon_k)}{M_{1i}}\right)\right]^2 + \left[\log\left(\frac{M_2(h^i_{2c},\epsilon_k)}{M_{2i}}\right)\right]^2 + \left[\log\left(\frac{\tilde{\Lambda}(h^i_{1c},h^i_{2c},\epsilon_k)}{\tilde{\Lambda}_i}\right)\right]^2\right\}?$$

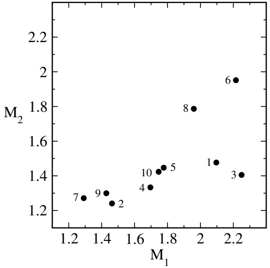
Can SSP^{-1} be Solved with LIGO/VIRGO Data II?

- Test the SSP^{-1} for binaries using high precision mock data $\{M_{1i}, M_{2i}, \tilde{\Lambda}_i\}$ for a randomly chosen collection of binaries based on a simple equation of state: $p = p_0(\epsilon/\epsilon_0)^2$.

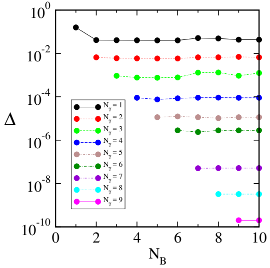


Can SSP^{-1} be Solved with LIGO/VIRGO Data II?

- Test the SSP^{-1} for binaries using high precision mock data $\{M_{1i}, M_{2i}, \tilde{\Lambda}_i\}$ for a randomly chosen collection of binaries based on a simple equation of state: $p = p_0(\epsilon/\epsilon_0)^2$.



- Solve SSP^{-1} for these data by minimizing $\chi^2(h^i_{1c}, h^i_{2c}, \gamma_k)$. Evaluate the accuracy Δ_{N_B} of the resulting approximate equation of state, $\epsilon = \epsilon(h, \gamma_k)$, by comparing to the equation of state used to construct the mock data.



Summary

- Neutron star equations of state can be represented very efficiently with spectral expansions: with (average) accuracies of just a few percent using only 2 or 3 spectral coefficients.

Summary

- Neutron star equations of state can be represented very efficiently with spectral expansions: with (average) accuracies of just a few percent using only 2 or 3 spectral coefficients.
- Matching observed values of neutron star masses M_i and radii R_i to models of $M(h_c^i, \gamma_k)$ and $R(h_c^i, \gamma_k)\}$ can determine the equation of state with (average) accuracies of just a few percent using (high quality) observational data from only 2 or 3 neutron stars.

Summary

- Neutron star equations of state can be represented very efficiently with spectral expansions: with (average) accuracies of just a few percent using only 2 or 3 spectral coefficients.
- Matching observed values of neutron star masses M_i and radii R_i to models of $M(h_c^i, \gamma_k)$ and $R(h_c^i, \gamma_k)\}$ can determine the equation of state with (average) accuracies of just a few percent using (high quality) observational data from only 2 or 3 neutron stars.
- Observations of binary neutron-star masses M_1 and M_2 plus the composite tidal deformability $\tilde{\Lambda}$ can also be used to determine the neutron star equation of state.