Solving Einstein's Equation Numerically on Manifolds with Arbitrary Spatial Topologies

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Center for Gravitation and Cosmology Seminar
University of Wisconsin at Milwaukee — 24 May 2013

- Multi-cube representations of arbitrary three-manifolds.
- Boundary conditions for elliptic, parabolic and hyperbolic PDEs.
- Numerical tests for solutions of simple PDEs.
- Covariant first-order representation of Einstein's equation.
- Simple numerical Einstein evolutions.

Representations of Arbitrary Three-Manifolds

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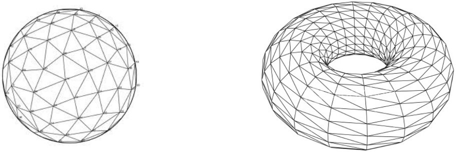

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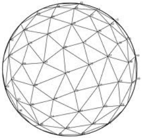

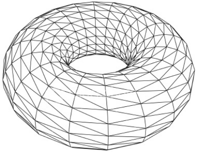

- Cubes make more convenient computational domains for finite difference and spectral numerical methods.
- Can arbitrary two- and three-manifolds be "cubed", i.e. represented as a set of squares or cubes plus a list of rules for gluing their edges or faces together?

“Multi-Cube” Representations of Three-Manifolds

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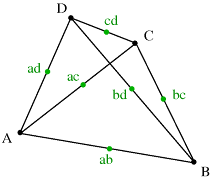

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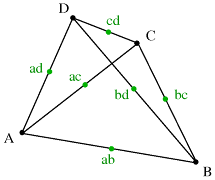

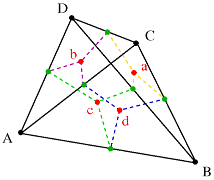

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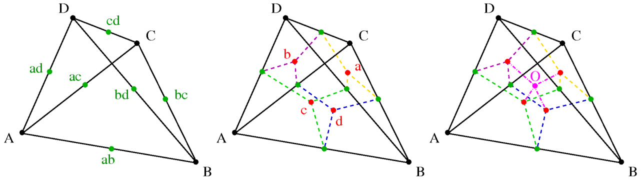

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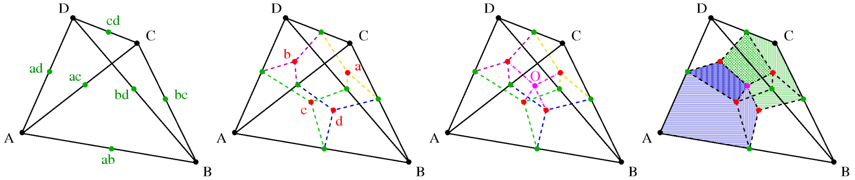

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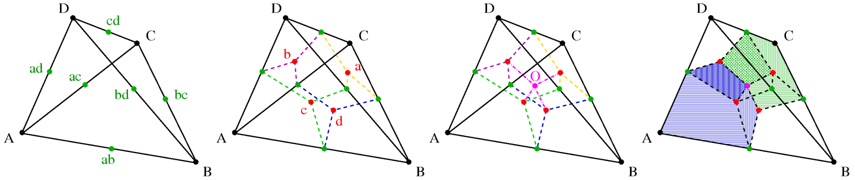
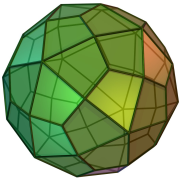
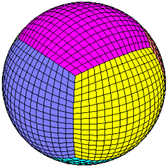
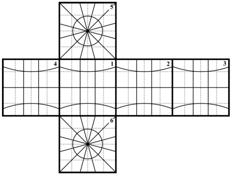


- Every two- or three-manifold can be represented as a set of squares or cubes, plus maps that identify their edges or faces.







Boundary Maps: Fixing the Topology

- Multi-cube representations of topological manifolds consist of a set of cubic regions, $\mathcal{B}_A$, plus maps that identify the faces of neighboring regions, $\Psi^{A\alpha}_{B\beta}(\partial_\beta \mathcal{B}_B) = \partial_\alpha \mathcal{B}_A$.

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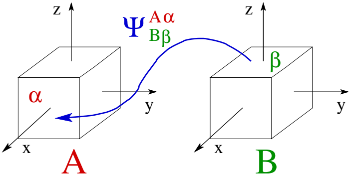
- Choose cubic regions to have uniform size and orientation.
- Choose linear interface identification maps $\Psi^{A\alpha}_{B\beta}$: $x^i_A = c^i_{A\,\alpha} + C^{A\,\alpha\,i}_{B\,\beta\,k}(x^k_B - c^k_{B\,\beta})$, where $C^{A\,\alpha\,i}_{B\,\beta\,k}$ is a rotation-reflection matrix, and $c^i_{A\,\alpha}$ is center of α face of region A.



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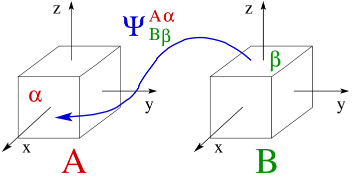
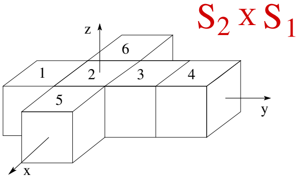
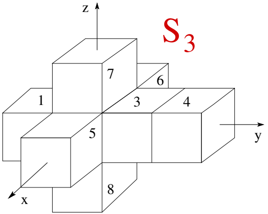


- Examples:





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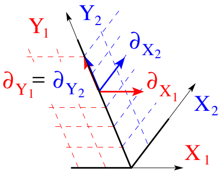

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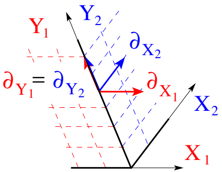

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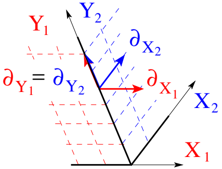

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- Differential structure provides the framework in which smooth functions and tensors are defined on a manifold.
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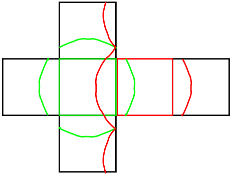
- Multi-cube manifolds need an additional layer of infrastructure: e.g., overlapping domains $\mathcal{D}_A \supset \mathcal{B}_A$ with transition maps that are smooth in the overlap regions.



Fixing the Differential Structure II

- All that is needed to define continuous tensor fields at interface boundaries is the Jacobian $J^{A\alpha i}_{B\beta k}$ and its dual $J^{*B\beta k}_{A\alpha i}$ that transform tensors from one multi-cube coordinate region to another: for example, $v^i_A = J^{A\alpha i}_{B\beta k} v^k_B$ and $w_{Ai} = J^{*B\beta k}_{A\alpha i} w_{Bk}$.

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- A smooth reference metric $\tilde{g}_{ij}$ determines the needed Jacobians.
- Let $\tilde{g}_{Aij}$ and $\tilde{g}_{Bij}$ be the components of a smooth reference metric in the multi-cube coordinates of regions $\mathcal{B}_A$ and $\mathcal{B}_B$ that are identified at the faces $\partial_\alpha \mathcal{B}_A \leftrightarrow \partial_\beta \mathcal{B}_B$.

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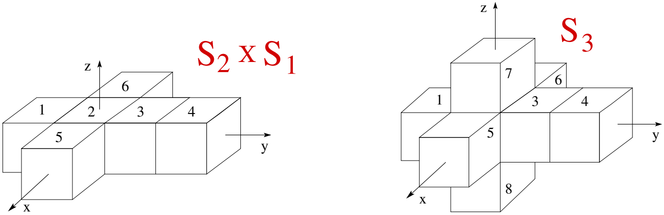
Solving PDEs on Multi-Cube Manifolds



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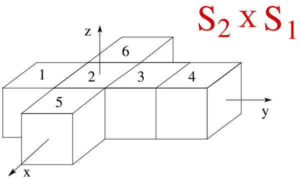

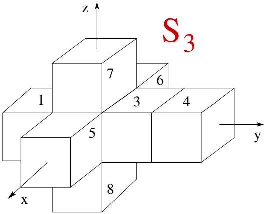

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- For second-order strongly-elliptic systems: enforce continuity on one face and continuity of normal derivatives on neighboring face,

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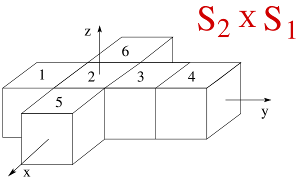

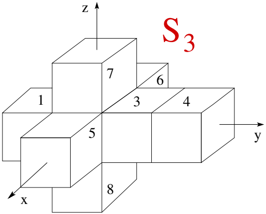

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- For first-order symmetric hyperbolic systems whose dynamical fields are tensors: set incoming characteristic fields with outgoing characteristics from neighbor,

$$\hat{u}_A^- \simeq \hat{u}_B^+ \qquad\qquad \hat{u}_B^- \simeq \hat{u}_A^+.$$

Numerical Methods

- Represent each component of each tensor function as a (finite) sum of spectral basis functions, $\mathbf{u} = \sum_{pqr} \mathbf{u}_{pqr}\, T_p(x)\, T_q(y)\, T_r(z)$, in each cubic region.

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- Evaluate the PDEs and BCs on a set of collocation points, $\{x_i, y_j, z_k\}$, chosen so that $\mathbf{u}(x_i, y_j, z_k)$ can be mapped efficiently onto the spectral coefficients $\mathbf{u}_{pqr}$. Derivatives become linear combinations of the fields: $\partial_x \mathbf{u}(x_i, y_j, z_k) = \sum_\ell D_i{}^\ell\, \mathbf{u}(x_\ell, y_j, z_k)$.

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- For elliptic systems, these pseudo-spectral equations become a system of algebraic equations for $\mathbf{u}(x_i, y_j, z_k)$. Solve these algebraic equations using standard numerical methods.
- For hyperbolic systems these equations become a system of ordinary differential equations for $\mathbf{u}(x_i, y_j, z_k, t)$. Solve these equations by the method of lines using standard ode integrators.

Testing the Elliptic PDE Solver

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- Use the co-variant derivative ∇_i for the round metric on $S^2 \times S^1$:

$$\begin{aligned} ds^2 &= R_1^2 d\chi^2 + R_2^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \\ &= \left(\frac{2\pi R_1}{L}\right)^2 dz_A^2 + \left(\frac{\pi R_2}{2L}\right)^2 \frac{(1+X_A^2)(1+Y_A^2)}{(1+X_A^2+Y_A^2)^2} \\ &\quad \times\left[(1+X_A^2)\,dx_A^2 - 2X_A Y_A\, dx_A\, dy_A + (1+Y_A^2)\,dy_A^2\right]. \end{aligned}$$

where $X_A = \tan\left[\pi(x_A - c_A^x)/2L\right]$ and $Y_A = \tan\left[\pi(y_A - c_A^y)/2L\right]$ are "local" Cartesian coordinates in each cubic region.

Testing the Elliptic PDE Solver

- Solve the elliptic PDE, $\nabla^i\nabla_i\psi - c^2\psi = f$ where c^2 is a constant, and f is a given function.
- Use the co-variant derivative ∇_i for the round metric on $S^2\times S^1$:

$$
\begin{aligned}
ds^2 &= R_1^2 d\chi^2 + R_2^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),\\
&= \left(\frac{2\pi R_1}{L}\right)^2 dz_A^2 + \left(\frac{\pi R_2}{2L}\right)^2 \frac{(1+X_A^2)(1+Y_A^2)}{(1+X_A^2+Y_A^2)^2}\\
&\quad\times\left[(1+X_A^2)\,dx_A^2 - 2X_AY_A\,dx_A\,dy_A + (1+Y_A^2)\,dy_A^2\right].
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- Let $f = -(\omega^2 + c^2)\psi_E$, where $\psi_E = \Re\left[e^{ik\chi}Y_{\ell m}(\theta,\varphi)\right]$. The angles χ, θ and φ are functions of the coordinates x, y and z.

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- The unique, exact, analytical solution to this problem is $\psi = \psi_E$, when $\omega^2 = \ell(\ell+1)/R_2^2 + k^2/R_1^2$.

Testing the Elliptic PDE Solver II

- Measure the accuracy of the numerical solution ψ_N as a function of numerical resolution N (grid points per dimension) in two ways:
  - First, with the residual $R_N \equiv \nabla^i\nabla_i\psi_N - c^2\psi_N - f$, and its norm:
$$\mathcal{E}_R = \sqrt{\frac{\int R_N^2\sqrt{g}d^3x}{\int f^2\sqrt{g}d^3x}}.$$
  - Second, with the solution error, $\Delta\psi = \psi_N - \psi_E$, and its norm:
$$\mathcal{E}_\psi = \sqrt{\frac{\int \Delta\psi^2\sqrt{g}d^3x}{\int \psi_E^2\sqrt{g}d^3x}}.$$

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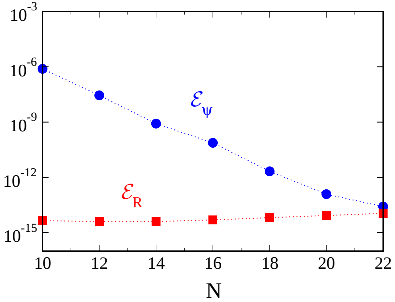

- All these numerical tests were performed by implementing the ideas described here into the Spectral Einstein Code (SpEC) developed originally by the Caltech/Cornell numerical relativity collaboration.

Testing the Hyperbolic PDE Solver

- Solve the equation $\partial_t^2\psi = \nabla_i\nabla^i\psi$ with given initial data.
- Convert the second-order equation into an equivalent first-order system: $\partial_t\psi = -\Pi$, $\partial_t\Pi = -\nabla^i\Phi_i$ and $\partial_t\Phi_i = -\nabla_i\Pi$ with constraint $\mathcal{C}_i = \nabla_i\psi - \Phi_i$.

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- Use the co-variant derivative ∇_i for the round metric on S^3:

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\begin{aligned}
ds^2 &= R_3^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \\
&= \left(\frac{\pi R_3}{2L}\right)^2 \frac{(1+X_A^2)(1+Y_A^2)(1+Z_A^2)}{(1+X_A^2+Y_A^2+Z_A^2)^2}\left[\frac{(1+X_A^2)(1+Y_A^2+Z_A^2)}{(1+Y_A^2)(1+Z_A^2)}dx^2 + \frac{(1+Y_A^2)(1+X_A^2+Z_A^2)}{(1+X_A^2)(1+Z_A^2)}dy^2\right. \\
&\qquad \left. + \frac{(1+Z_A^2)(1+X_A^2+Y_A^2)}{(1+X_A^2)(1+Y_A^2)}dz^2 - \frac{2X_AY_A}{1+Z_A^2}dx\,dy - \frac{2X_AZ_A}{1+Y_A^2}dx\,dz - \frac{2Y_AZ_A}{1+X_A^2}dy\,dz\right].
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\end{aligned}
$$

- Choose initial data with $\psi_{t=0} = \Re[Y_{k\ell m}(\chi,\theta,\varphi)]$, $\Pi_{t=0} = -\Re[i\omega Y_{k\ell m}(\chi,\theta,\varphi)]$ and $\Phi_{i\,t=0} = \Re[\nabla_i Y_{k\ell m}(\chi,\theta,\varphi)]$ where $\omega^2 = k(k+2)/R_3^2$.

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- Solve the equation $\partial_t^2\psi = \nabla_i\nabla^i\psi$ with given initial data.
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- The unique, exact, analytical solution to this problem is $\psi = \psi_E = \Re[e^{i\omega t}Y_{k\ell m}(\chi,\theta,\varphi)]$, $\Pi = -\partial_t\psi_E$, and $\Phi_i = \nabla_i\psi_E$.

Testing the Hyperbolic PDE Solver II

- Measure the accuracy of the numerical solution ψ_N as a function of numerical resolution N (grid points per dimension) in two ways:
  - First, with the solution error, $\Delta\psi = \psi_N - \psi_E$, and its norm:
  $$\mathcal{E}_\psi = \sqrt{\frac{\int \Delta\psi^2 \sqrt{g} d^3x}{\int \psi^2 \sqrt{g} d^3x}}.$$
  - Second, with the constraint error, $\mathcal{C}_i = \Phi_i - \nabla_i\psi$, and its norm:
  $$\mathcal{E}_\mathcal{C} = \sqrt{\frac{\int g^{ij}\mathcal{C}_i\mathcal{C}_j \sqrt{g} d^3x}{\int g^{ij}(\Phi_i\Phi_j + \nabla_i\psi\nabla_j\psi)\sqrt{g} d^3x}}.$$

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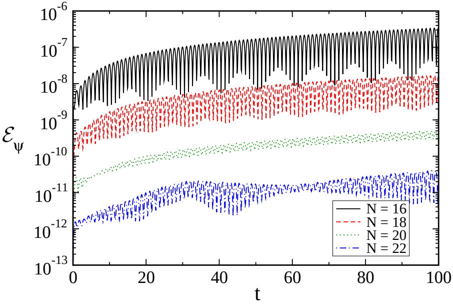

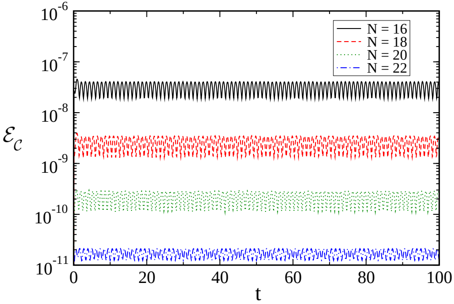

Solving Einstein's Equation on Multi-Cube Manifolds

- Multi-cube methods were designed to solve first-order hyperbolic systems, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\tilde\nabla_k u^\beta = F^\alpha(u)$, where the dynamical fields u^α are tensors that can be transformed across interface boundaries using the Jacobians $J^{A\alpha i}_{B\beta k}$, etc.

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- The usual first-order representations of Einstein's equation fail to meet these conditions in two important ways:
  - The usual choice of dynamical fields, $u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c\partial_c\psi_{ab}, \Phi_{iab} = \partial_i\psi_{ab}\}$ are not tensor fields.
  - The usual first-order evolution equations are not covariant: i.e., the one that comes from the definition of Π_{ab}, $\Pi_{ab} = -t^c\partial_c\psi_{ab}$, and the one that comes from preserving the constraint $C_{iab} = \Phi_{iab} - \partial_i\psi_{ab}$, $t^c\partial_c C_{iab} = -\gamma_2 C_{iab}$.

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- Our attempts to construct the transformations for non-tensor quantities like $\partial_i\psi_{ab}$ and Φ_{iab} across the non-smooth multi-cube interface boundaries failed to result in stable numerical evolutions.
- A spatially covariant first-order representation of the Einstein evolution system seems to be needed.

Covariant Representations of Einstein's Equation

- Let $\tilde{\psi}_{ab}$ denote a smooth reference metric on the manifold $R \times \Sigma$. For convenience we choose $ds^2 = \tilde{\psi}_{ab} dx^a dx^b = -dt^2 + \tilde{g}_{ij} dx^i dx^j$, where $\tilde{g}_{ij}$ is the smooth multi-cube reference three-metric on Σ.

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- A fully covariant expression for the Ricci tensor can be obtained using the reference covariant derivative $\tilde{\nabla}_a$:

$$\begin{aligned} R_{ab} &= -\tfrac{1}{2}\psi^{cd}\tilde{\nabla}_c\tilde{\nabla}_d\psi_{ab} + \nabla_{(a}\Delta_{b)} - \psi^{cd}\tilde{R}^e{}_{cd(a}\psi_{b)e} \\ &\quad +\psi^{cd}\psi^{ef}\left(\tilde{\nabla}_e\psi_{ca}\tilde{\nabla}_f\psi_{ab} - \Delta_{ace}\Delta_{bdf}\right), \end{aligned}$$

where $\Delta_{abc} = \psi_{ad}\left(\Gamma^d_{bc} - \tilde{\Gamma}^d_{bc}\right)$, and $\Delta_a = \psi^{bc}\Delta_{abc}$.

Covariant Representations of Einstein's Equation

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  where $\Delta_{abc}=\psi_{ad}\left(\Gamma^d_{bc}-\tilde\Gamma^d_{bc}\right)$, and $\Delta_a=\psi^{bc}\Delta_{abc}$.
- A fully-covariant manifestly hyperbolic representation of the Einstein equations can be obtained by fixing the gauge with a covariant generalized harmonic condition: $\Delta_a=-H_a(\psi_{cd})$.
- The vacuum Einstein equations then become:

$$\begin{aligned} \psi^{cd}\tilde\nabla_c\tilde\nabla_d\psi_{ab} &= -2\nabla_{(a}H_{b)}+2\psi^{cd}\psi^{ef}\left(\tilde\nabla_e\psi_{ca}\tilde\nabla_f\psi_{ab}-\Delta_{ace}\Delta_{bdf}\right) \\ &\quad -2\psi^{cd}\tilde R^e{}_{cd(a}\psi_{b)e}+\gamma_0\left[2\delta^c_{(a}t_{b)}-\psi_{ab}t^c\right]\left(H_c+\Delta_c\right). \end{aligned}$$

Covariant Representations of Einstein's Equation II

- A first-order representation of the Einstein equations can be obtained from this covariant generalized harmonic representation by choosing dynamical fields:

  $$u^\alpha = \{\psi_{ab}, \Pi_{ab} = -t^c\tilde\nabla_c\psi_{ab}, \Phi_{iab} = \tilde\nabla_i\psi_{ab}\},$$

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which are tensors with respect to spatial coordinate transformations.
- The first order equation that arises from the definition of Π_{ab}, $t^c\tilde\nabla_c\psi_{ab} = -\Pi_{ab}$ is now covariant, as is the equation for $t^c\tilde\nabla_c\Phi_{iab}$ that follows from the covariant constraint evolution equation, $t^c\tilde\nabla_c C_{iab} = -\gamma_2 C_{iab}$, where $C_{iab} = \Phi_{iab} - \tilde\nabla_i\psi_{ab}$.

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- The resulting first-order Einstein evolution system, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\tilde\nabla_k u^\beta = F^\alpha(u)$, is symmetric-hyperbolic and covariant with respect to spatial coordinate transformations.
- The characteristic speeds and fields of this covariant system have the same forms as the standard ones in terms of the dynamical fields ψ_{ab}, Π_{ab} and Φ_{iab}. These fields are now tensors, however, so the actual characteristic fields are somewhat different.

Testing the Einstein Solver: Static Universe on S^3

- Metric initial data is taken from the "Einstein Static Universe" geometry:

$$ds^2 = -dt^2 + R_3^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right],$$

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- This metric solves Einstein's equation with cosmological constant and complex scalar field source on a manifold with spatial topology S^3.

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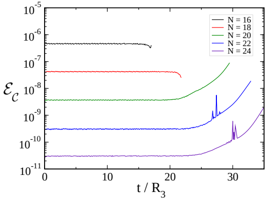
\end{aligned}
$$

- This metric solves Einstein’s equation with cosmological constant and complex scalar field source on a manifold with spatial topology S^3.
- Evolution of these initial data is the static universe geometry, if the cosmological constant is chosen to be $\Lambda = 1/R_3^2$, and the complex scalar field is $\varphi = \varphi_0 e^{i\mu t}$ with $\mu^2|\varphi_0|^2 = 1/4\pi R_3^2$.

Testing the Einstein Solver: Static Universe on S^3 II

- Monitor how well the numerical solutions satisfy the Einstein system by evaluating the norm of the various constraints:

$$\mathcal{E}_{\mathcal{C}} = \sqrt{\frac{\int \sum |\mathcal{C}|^2 \sqrt{g} d^3x}{\int \sum |\partial_i u|^2 \sqrt{g} d^3x}}.$$



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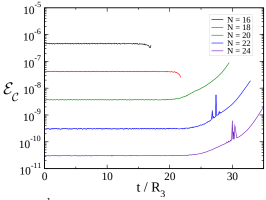
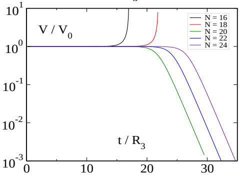
- Monitor the physical volume V of the S^3 in comparison with the Einstein Static Solution value $V_0 = 2\pi^2 R_3^3$:





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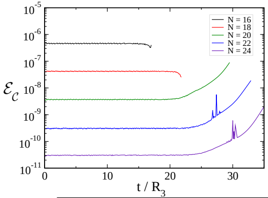

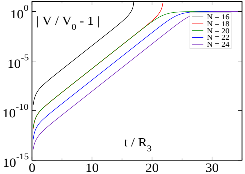

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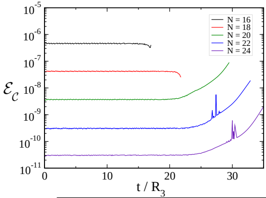

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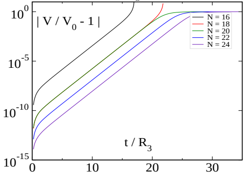
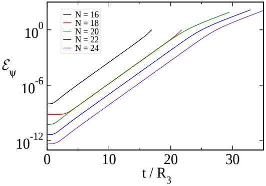


- Monitor the accuracy of numerical metric solution by evaluating the norm of its error, $\Delta\psi_{ab} = \psi_{Nab} - \psi_{Aab}$:

$$\mathcal{E}_\psi = \sqrt{\frac{\int \sum_{ab} |\Delta\psi_{ab}|^2 \sqrt{g} d^3x}{\int \sum_{ab} |\psi_{ab}|^2 \sqrt{g} d^3x}}.$$



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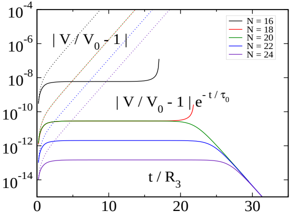

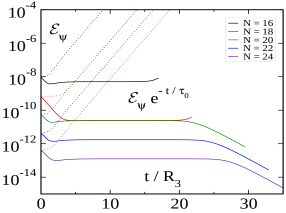

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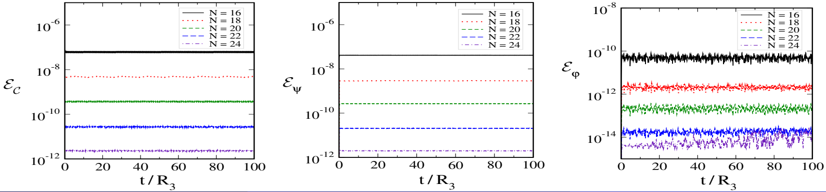

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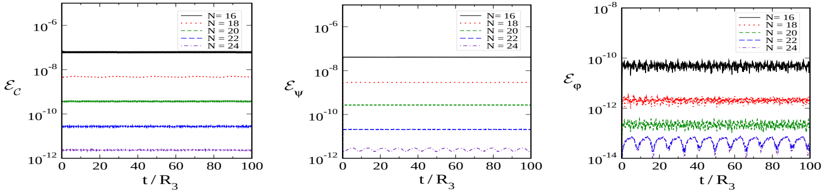

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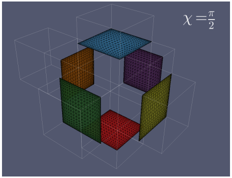

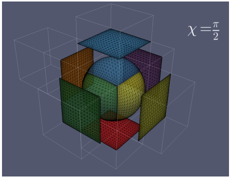

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