Spectral Representations of Neutron-Star Equations of State

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- Why?
- What?
- How?
- How?

Why Equation of State Representations?

- Neutron-star models are solutions of Einstein's equations,

$$\begin{aligned} \frac{dm}{dr} &= 4\pi r^2 \epsilon, \\ \frac{dp}{dr} &= -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)}, \end{aligned}$$

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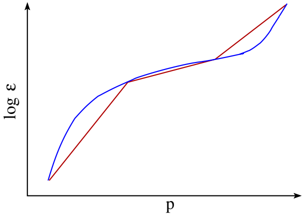
  which require a knowledge of the equation of state, $\epsilon = \epsilon(p)$.
- Neutron-star matter can not be duplicated in the laboratory, so its equation of state is not well known.
- Accurate representations of the equation of state, $\epsilon = \epsilon(p, \lambda_k)$, are needed to construct accurate stellar models.
- Some equation of state parameters, λ_k, should be measurable from neutron-star observations.

Parametric Representations of Equations of State

- Parametric representations $\epsilon = \epsilon(p, \lambda_k)$ can be constructed by connecting simple power law curves:



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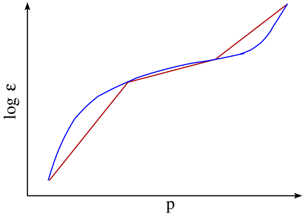

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- Neutron-star equations of state can be represented with reasonable accuracy in this way using only a few (3 or 4) parameters (Vuille & Ipser 1999, Read, et. al 2009).

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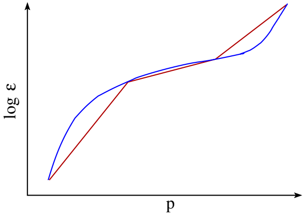

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- Neutron-star equations of state can be represented with reasonable accuracy in this way using only a few (3 or 4) parameters (Vuille & Ipser 1999, Read, et. al 2009).
- Spectral methods provide a more efficient way to construct parametric representations: $\epsilon(p) = \sum_k \lambda_k \Phi_k(p)$.
- Spectral expansions of smooth functions typically converge exponentially (errors scale as $e^{-\kappa N}$ for large N), so fewer parameters are typically needed for given accuracy.

Faithful Spectral Representations

- Physical equations of state, $\epsilon = \epsilon(p)$, are positive monotonic-increasing functions.
- Simple spectral representations, $\epsilon = \epsilon(p, \lambda_k) = \sum_k \lambda_k \Phi_k(p)$, require complicated conditions on λ_k to enforce positivity, etc.
- Faithful representations are needed: where every choice of λ_k corresponds to a possible physical equation of state, and every equation of state can be represented by such an expansion.

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- The adiabatic index $\Gamma(p)$ must be positive, but need not be monotonic. $\log \Gamma(p)$ is unrestricted, and so standard spectral expansions are faithful:

$$\Gamma(p) = \frac{\epsilon + p}{p} \frac{dp}{d\epsilon} = \exp\left[\sum_k \lambda_k \Phi_k(p)\right].$$

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- The equation of state $\epsilon(p)$ is determined by solving

$$\frac{d\epsilon(p)}{dp} = \frac{\epsilon(p) + p}{p\,\Gamma(p)},$$

once the adiabatic index $\Gamma(p)$ is specified.

How to Construct Faithful Representations

- Given a spectral expansion for $\log\Gamma(p) = \sum_k \lambda_k \Phi_k(p)$, a faithful parametric representation of the equation of state is therefore,

$$\begin{aligned}\epsilon(p) &= \frac{\epsilon_0}{\mu(p)} + \frac{1}{\mu(p)}\int_{p_0}^{p}\frac{\mu(p')}{\Gamma(p')}dp',\\ \mu(p) &= \exp\left[-\int_{p_0}^{p}\frac{dp'}{p'\Gamma(p')}\right].\end{aligned}$$

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- The following simple expansion works well (for $p \geq p_0$),

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- For a given eos $\{\epsilon_i, p_i\}$ choose the parameters λ_k that minimize

$$\Delta_\epsilon^2 = \frac{1}{N}\sum_{i=1}^{N}\left\{\left[\log\left(\frac{\epsilon(p_i,\lambda_k)}{\epsilon_i}\right)\right]^2\right\}.$$

How Well Do They Work?

- Test effectiveness of spectral representations for realistic equations of state. Fix the λ_k by minimizing

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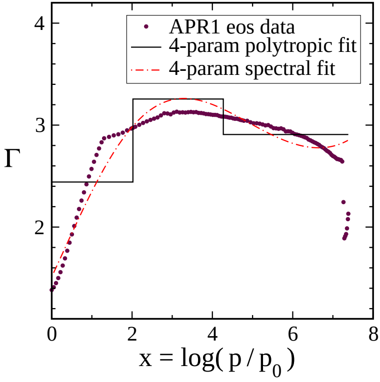

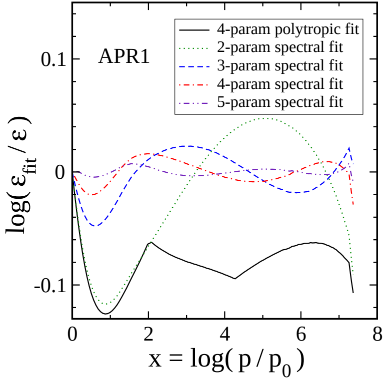

Summary

- Spectral representations of 34 neutron-star equations of state were constructed using $N = \{2, 3, 4, 5\}$ spectral parameters.
- The average values of $\Delta_\epsilon^2 = \frac{1}{N}\sum_{i=1}^{N}\left\{\left[\log\left(\frac{\epsilon(p_i,\lambda_k)}{\epsilon_i}\right)\right]^2\right\}$ for these fits were $\Delta_\epsilon = \{0.029, 0.015, 0.011, 0.008\}$.
- Graph showing residuals for individual equation of state fits: