Model Waveform Accuracy Requirements for Gravitational Wave Data Analysis

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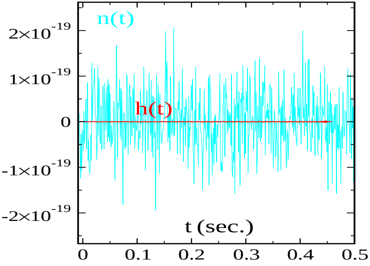
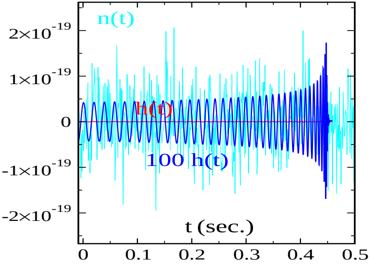
- How accurate must model waveforms be:
  - to prevent a significant rate of missed detections?
  - to prevent a significant accuracy loss for measurements?
  - to avoid unnecessary costs of achieving excess accuracy?

Gravitational Wave Data Analysis

- Gravitational wave signals are very weak.
- Current generation of detectors are fairly noisy (compared to the expected strengths of the signals.)
- Weakest detectable signal has signal-to-noise ratio $\rho \approx 8$.
- Figures illustrate a $\rho = 8$ signal from a binary black hole merger, compared to Initial LIGO noise.
- High quality gravitational waveforms are needed to allow these signals to be "seen" at all.





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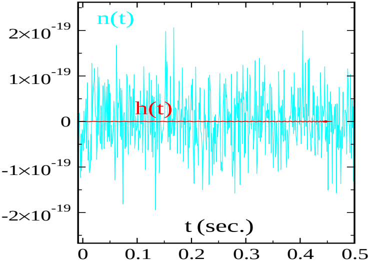
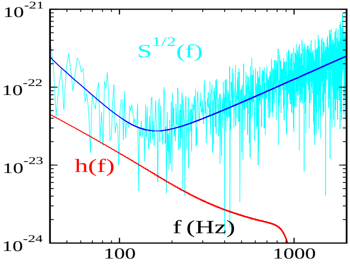
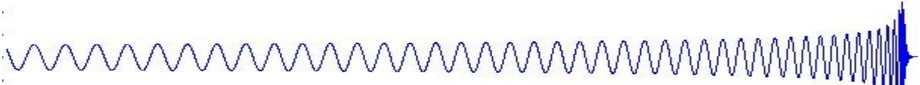




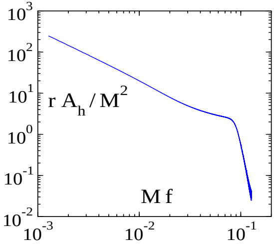
Basic GW Data Analysis:

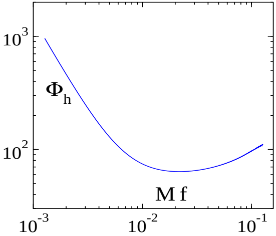
- Data analysis identifies and then measures the properties of signals in GW data by matching to model waveforms.



- Think of a waveform $h(t)$ as a vector, $\vec{h}$, whose components are the amplitudes of the waveform at each time, or equivalently at each frequency:

$$h(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt \equiv A_h(f) e^{i\Phi_h(f)}$$





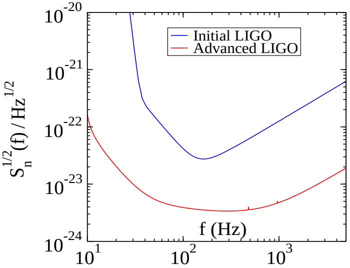
Basic GW Data Analysis II:

- Let $\vec{h}_e = h_e(f)$ denote the exact waveform for some source, and let $\vec{h}_m = h_m(f)$ denote a model of this waveform.
- Define a waveform inner product that weights frequency components in proportion to the detector's sensitivity:

$$\vec{h}_e \cdot \vec{h}_m = \langle h_e | h_m \rangle = \int_{-\infty}^{\infty} \frac{h_e^*(f) h_m(f) + h_e(f) h_m^*(f)}{S_n(f)} df,$$

where $S_n(f)$ is the power spectral density of the detector noise.

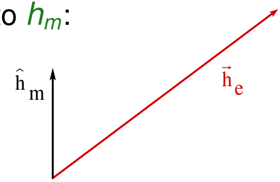
- This inner product is normalized so that $\rho = \sqrt{\langle h_e | h_e \rangle}$ is the optimal signal-to-noise ratio for detecting the waveform $\vec{h}_e$.



Basic GW Data Analysis III:

- Search for signals by projecting data onto model waveforms: ρ_m is the signal-to-noise ratio for $\vec{h}_e$ projected onto $\vec{h}_m$:

$$\rho_m \equiv \vec{h}_e \cdot \hat{h}_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}}.$$



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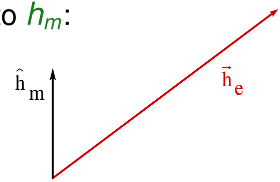

normalized so that $\langle \hat{h}_m | \hat{h}_m \rangle = 1$.

- A detection is made when $\vec{h}_e$ has a projected signal-to-noise ratio ρ_m that exceeds a predetermined threshold.
- Measured signal-to-noise ratio, ρ_m, is largest when the model waveform $\vec{h}_m$ is proportional to the exact $\vec{h}_e$;
  in this case ρ_m equals the optimal signal-to-noise ratio ρ:

$$\rho_m = \frac{\langle h_e | h_e \rangle}{\sqrt{\langle h_e | h_e \rangle}} = \sqrt{\langle h_e | h_e \rangle} = \rho = \sqrt{\int_{-\infty}^{\infty} \frac{2|h_e(f)|^2}{S_n(f)} df}.$$

Accuracy Standards for Detection

- The measured signal-to-noise ratio ρ_m for detecting the signal h_e is the projection of h_e onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

- Errors in model waveform, $h_m = h_e + \delta h$, result in reduction of ρ_m compared to the optimal signal-to-noise ratio ρ:

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- Evaluate this mismatch ϵ in terms of the waveform error:

$$\epsilon = \frac{\langle \delta h_\perp | \delta h_\perp \rangle}{2\langle h_e | h_e \rangle}, \quad \text{where} \quad \delta h_\perp = \delta h - \hat{h}_e \langle \hat{h}_e | \delta h \rangle.$$

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- The loss of detections can be limited to an acceptable level, by limiting the mismatch ϵ to an acceptable range: $\epsilon < \epsilon_{\max}$.
- Consequently model waveform accuracy must satisfy the requirement for detection: $\langle \delta h_\perp | \delta h_\perp \rangle < 2\epsilon_{\max}\rho^2$.

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
- Consider the one-parameter family of waveforms:

$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

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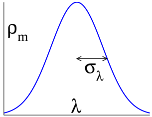

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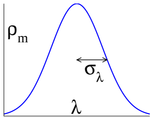

$$\frac{1}{\sigma_\lambda^2} = \left\langle \frac{\partial h}{\partial\lambda} \middle| \frac{\partial h}{\partial\lambda} \right\rangle = \langle\delta h|\delta h\rangle.$$

- If the parameter distance between the two waveforms, $(\Delta\lambda)^2 = 1$, is smaller than the variance σ_λ^2 for measuring that parameter, then the waveforms are indistinguishable.
- So h_m is indistinguishable from h_e if $1 < \sigma_\lambda^2 = 1/\langle\delta h|\delta h\rangle$, i.e., if $1 > \langle\delta h|\delta h\rangle$.

Accuracy Requirements for Advanced LIGO

- It is useful to define amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors:
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- The basic accuracy requirements can be written as

$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} = \sqrt{\frac{\langle\delta h|\delta h\rangle}{\langle h|h\rangle}} < \begin{cases} \eta_c/\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} & \text{detection,} \end{cases}$$

where the signal-weighted average errors are defined as

$$\overline{\delta\chi_m}^2 = \int_{-\infty}^{\infty} \delta\chi_m^2 \frac{2|h|^2}{\rho^2 S_n} df, \quad \text{and} \quad \overline{\delta\Phi_m}^2 = \int_{-\infty}^{\infty} \delta\Phi_m^2 \frac{2|h|^2}{\rho^2 S_n} df,$$

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- For Advanced LIGO, $\rho_{\max}$ could be as large as $\rho_{\max} \approx 100$, and calibration accuracy will (optimally) be comparable to model waveform accuracy, making $\eta_c \approx 1/2$, so

$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \frac{\eta_c}{\rho_{\max}} \approx 0.005 \text{ for measurement.}$$

Detection Accuracy Requirements for LIGO

- Accuracy requirement for detection depends on the parameter ϵ_{max}, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{\mathrm{max}} = 0.035$ limits the loss rate to about 10%.

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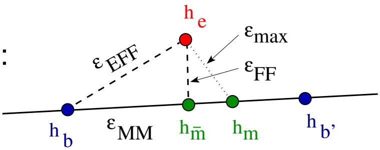
- Real searches are more complicated: comparing signals with a discrete template bank of model waveforms.



- For Initial LIGO, template banks are constructed with $\epsilon_{\text{MM}} = 0.03$, so $\epsilon_{\text{FF}} = \epsilon_{\text{EFF}} - \epsilon_{\text{MM}} = 0.035 - 0.03 = 0.005$.
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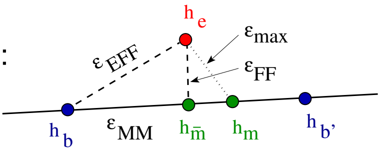

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- Accuracy requirement for BBH waveforms for detection in LIGO:

$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} \lesssim \sqrt{2\epsilon_{\mathsf{max}}} = 0.1 \quad \text{for detection.}$$

Time-Domain Accuracy Requirements

- The frequency-domain accuracy requirements can be converted to time-domain standards using the inequality:

$$\frac{\langle\delta h|\delta h\rangle}{\langle h|h\rangle} = \frac{2}{\rho^2}\int_{-\infty}^{\infty}\frac{|\delta h|^2}{S_n(f)}df \leq \frac{2\int_{-\infty}^{\infty}\left|\frac{d^k\delta h(t)}{dt^k}\right|^2 dt}{\rho^2\min[f^{2k}S_n(f)]} \equiv \frac{||\delta h||_k^2}{C_k^2||h||_k^2},$$

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where $$C_k = \frac{\rho}{\sqrt{2}||h||_k/\sqrt{\min[f^{2k}S_n(f)]}} \leq 1,$$

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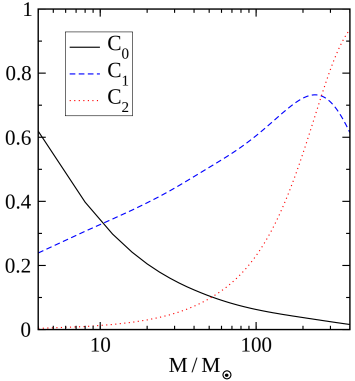
- It is straightforward then to create accuracy requirements based on the time domain norms $||\delta h||_k$:

$$\sqrt{\frac{\langle \delta h|\delta h\rangle}{\langle h|h\rangle}} \leq \frac{||\delta h||_k}{C_k||h||_k} < \begin{cases} \eta_c/\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} & \text{detection,} \end{cases}$$

Sufficient Time-Domain Requirements for LIGO

- The signal-to-noise quantity $C_k = \frac{\rho\sqrt{\min[f^{2k}S_n]}}{\sqrt{2}||h||_k} \leq 1$ has been evaluated for equal-mass non-spinning BBH waveforms using LIGO noise.



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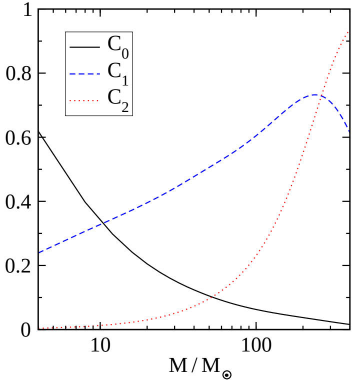
- It is sufficient to satisfy any one of these new time-domain standards, typically the one with the largest C_k:



$$\frac{||\delta h||_k}{||h||_k} \lesssim \begin{cases} C_k\eta_c/\rho_{\max} \approx 0.3\times 0.005 \approx 10^{-3} & \text{measurement,} \\ C_k\sqrt{2\epsilon_{\max}} \approx 0.3\times 0.1 \approx 0.03 & \text{detection.} \end{cases}$$

Summary

- A set of accuracy standards now exist for model waveforms,

$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} \leq \begin{cases} \eta_c/\rho_{\text{max}} & \text{measurement,} \\ \sqrt{2\epsilon_{\text{max}}} & \text{detection.} \end{cases}$$

- These standards are difficult (impossible?) to enforce directly. Easier and more reliable to enforce conditions based on certain time-domain norms of the waveform errors:

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- To Do List:
  - NR and analytic waveform simulation communities need to ensure their waveforms satisfy needed requirements.
  - Data analysis community needs to ensure the combined waveforms used in search and measurement templates are accurate enough.
  - Experimentalists need to ensure the calibration errors are sufficiently small.