Generalized Harmonic Gauge Drivers

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- Gauge conditions are specified in the GH Einstein system by the gauge source function $H^a \equiv \nabla^c \nabla_c x^a$.
- How do you choose H^a corresponding to the familiar gauge conditions of numerical relativity — without destroying the hyperbolicity of the system?

Gauge Conditions and Hyperbolicity

- The principal parts of the GH Einstein equations may be written as

$$\psi^{cd}\partial_c\partial_d\psi_{ab} \;=\; \nabla_a H_b + \nabla_b H_a + Q_{ab}(\psi,\partial\psi),$$

  where ψ_{ab} is the spacetime metric.

- These equations are manifestly hyperbolic when H^a is specified as a function of x^a and ψ_{ab}: $H^a = H^a(x,\psi)$.

- In this case the principal parts of the GH Einstein system are simple wave operators on each component of ψ_{ab}:

$$\psi^{cd}\partial_c\partial_d\psi_{ab} \;=\; \hat{Q}_{ab}(x,\psi,\partial\psi).$$

Imposing Useful Gauge Conditions

- Specifying the gauge source function H^a places constraints on the derivatives of the spacetime metric:

$$H^a = \nabla^c \nabla_c x^a = \psi^{bc} \Gamma^a_{bc} \equiv \Gamma^a.$$

where Γ^a_{bc} is the Christoffel symbol.

- The quantity Γ^a depends on the time derivatives of the lapse N and shift N^i. For example,

$$\Gamma^t = N^{-3}\Big(\partial_t N - N^k \partial_k N + N^2 K\Big),$$

where K is the trace of the extrinsic curvature.

- One could impose the slicing condition $\partial_t N - N^k \partial_k N = -2NK$, for example, by setting $H^t = N^{-2}(N-2)K$.

- Unfortunately this choice has the form $H^a = H^a(x, \psi, \partial\psi)$ which destroys the hyperbolicity of the GH Einstein system.

Solution: Gauge Driver Equations

- Elevate H_a to the status of a dynamical field (Pretorius) and evolve it along with the spacetime metric ψ_{ab}:

$$\begin{aligned} \psi^{cd}\partial_c\partial_d H_a &= Q_a(x, H, \partial H, \psi, \partial\psi), \\ \psi^{cd}\partial_c\partial_d \psi_{ab} &= Q_{ab}(x, H, \partial H, \psi, \partial\psi), \end{aligned}$$

- Any gauge driver of this form produces a symmetric hyperbolic combined evolution system.
- Choose Q_a so that H_a evolves toward the desired gauge target F_a as the system evolves: $H_a \to F_a$.
- For example, consider the simple gauge driver:

$$\psi^{cd}\partial_c\partial_d H_a = Q_a = \mu^2(H_a - F_a) + 2\mu N^{-1}\partial_t H_a.$$

- For constant F_a, this gauge driver causes $H_a \to F_a + \mathcal{O}(e^{-\mu t})$.

Improved Gauge Driver Equations

- In the time independent (but spatially inhomogeneous) limit, the simple gauge driver evolution equation reduces to,

$$\psi^{jk}\partial_j\partial_k H_a = \mu^2(H_a - F_a).$$

so this equation does not achieve $H_a \to F_a$ unless $\partial_k H_a = 0$.

- The system can be improved by adding a time averaging field,

$$\partial_t\theta_a + \mu\theta_a = \psi^{jk}\partial_j\partial_k H_a,$$

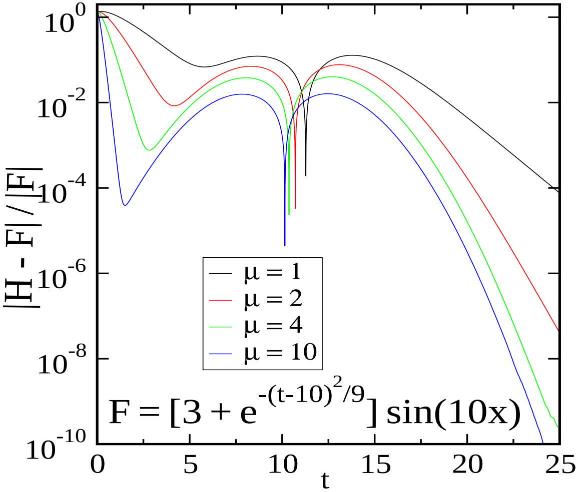
which is used to modify the gauge driver:

$$\psi^{cd}\partial_c\partial_d H_a = Q_a = \mu^2(H_a - F_a) + 2\mu N^{-1}\partial_t H_a + \mu\theta_a.$$

- The resulting gauge driver system is symmetric hyperbolic for all $F_a = F_a(x, \psi, \partial\psi)$, has solutions H_a that exponentially approach any time independent F_a when the background geometry is fixed, and reduces to $H_a = F_a$ in any time independent state.

Decoupled Gauge Driver Test:

- Test the gauge driver equation on a fixed (flat) background spacetime. Use a time and space dependent gauge target: $F = \left[3 + e^{-(t-10)^2/9}\right]\sin(10x)$.



Fully Coupled Gauge Driver Test:

- Test the gauge driver equation (with a conformal gamma driver target F_a) coupled to the GH Einstein equations for a perturbed Schwarzschild spacetime: $\delta N^i = 0.01\hat{r}^i\, Y_{30}\, e^{-(r-15)^2/9}$.