Model Waveform Accuracy Standards for Gravitational Wave Data Analysis

Lee Lindblom

Theoretical Astrophysics, Caltech

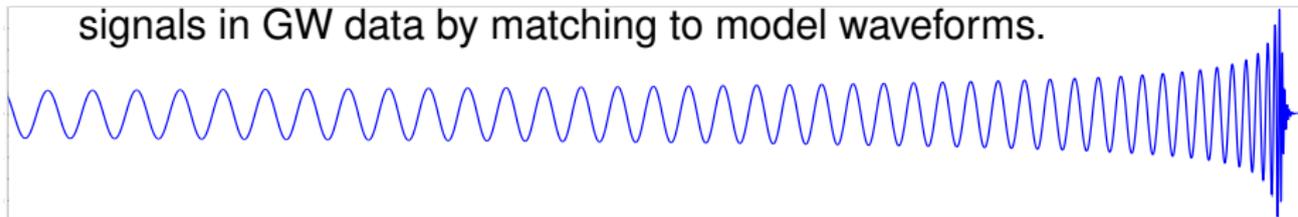Caltech/Cornell Video Conference — 17 July 2009

Collaborators:

Duncan Brown (Syracuse), Benjamin Owen (Penn State)

- How accurate must model waveforms be:
    - to prevent a significant rate of missed detections?
    - to prevent a significant accuracy loss for measurements?
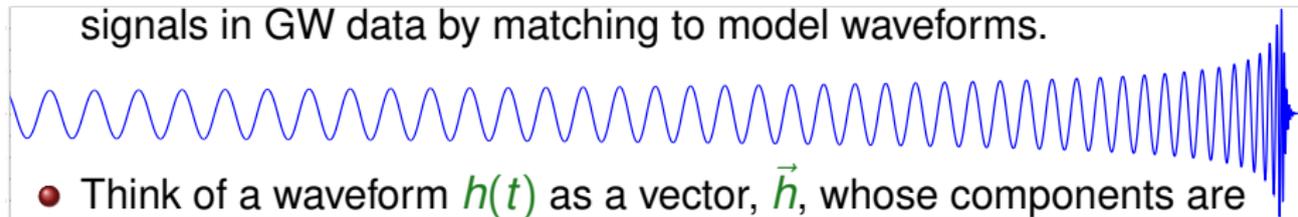   - to avoid the added cost of waveforms more accurate than needed?

Simple Introduction to GW Data Analysis:

- Data analysis identifies and then measures the properties of signals in GW data by matching to model waveforms.

Simple Introduction to GW Data Analysis:

- Data analysis identifies and then measures the properties of signals in GW data by matching to model waveforms.

- Think of a waveform $h(t)$ as a vector, $\vec{h}$, whose components are the amplitudes of the waveform at each frequency:

$$h(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi i f t}dt \equiv A_h(f)e^{i\Phi_h(f)}$$

Simple Introduction to GW Data Analysis:

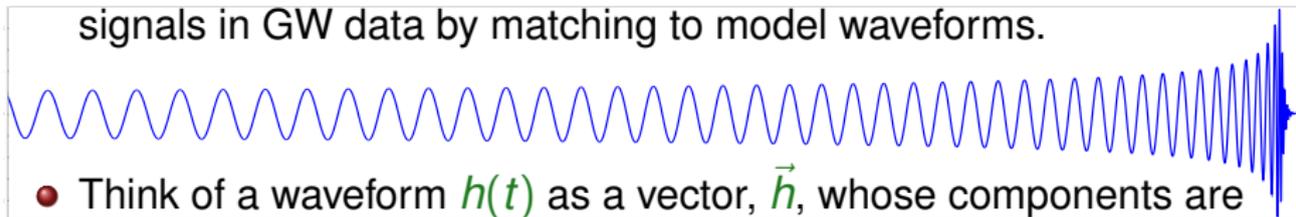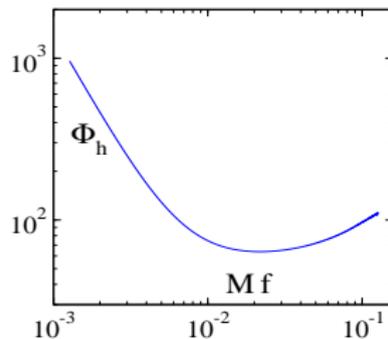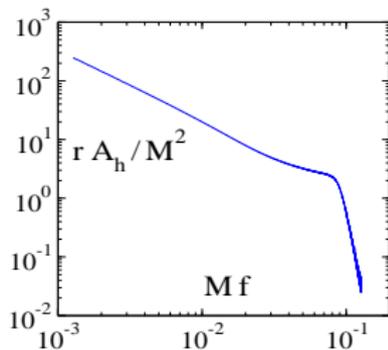- Data analysis identifies and then measures the properties of signals in GW data by matching to model waveforms.

- Think of a waveform $h(t)$ as a vector, $\vec{h}$, whose components are the amplitudes of the waveform at each frequency:

$$h(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi i f t}dt \equiv A_h(f)e^{i\Phi_h(f)}$$

Simple Introduction to GW Data Analysis II:

- Let $\vec{h}_e = h_e(f)$ denote the exact waveform for some source, and let $\vec{h}_m = h_m(f)$ denote a model of this waveform.

- Define a waveform inner product that weights components (frequencies) in proportion to the detector's sensitivity:

$$\vec{h}_e \cdot \vec{h}_m = \langle h_e | h_m \rangle = \int_{-\infty}^{\infty} \frac{h_e^*(f)h_m(f) + h_e(f)h_m^*(f)}{S_n(f)} df,$$

  where $S_n(f)$ is the power spectral density of the detector noise.

- This inner product is normalized so that $\rho = \sqrt{\langle h_e | h_e \rangle}$ is the optimal signal-to-noise ratio for detecting the waveform $\vec{h}_e$.
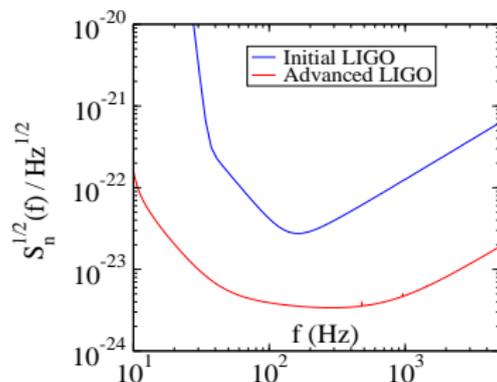
Simple Introduction to GW Data Analysis III:

- Project the signal $\vec{h}_e$ onto a model waveform, $\vec{h}_m$:

$$\rho_m \equiv \vec{h}_e \cdot \hat{h}_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}}.$$

  normalized so that $\langle \hat{h}_m | \hat{h}_m \rangle = 1$.

Simple Introduction to GW Data Analysis III:

- Project the signal $\vec{h}_e$ onto a model waveform, $\vec{h}_m$:

$$\rho_m \equiv \vec{h}_e \cdot \hat{h}_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}}.$$

normalized so that $\langle \hat{h}_m | \hat{h}_m \rangle = 1$.

- Search for signals by projecting data onto model waveforms: ρ_m is the signal-to-noise ratio for $\vec{h}_e$ projected onto $\vec{h}_m$.
- A detection is made when $\vec{h}_e$ has a projected signal-to-noise ratio ρ_m that exceeds a pre-determined threshold.
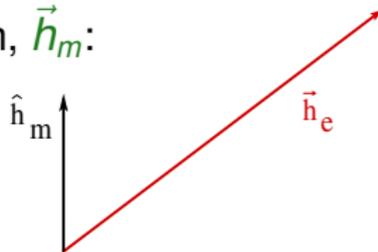
Simple Introduction to GW Data Analysis III:

- Project the signal $\vec{h}_e$ onto a model waveform, $\vec{h}_m$:

$$\rho_m \equiv \vec{h}_e \cdot \hat{h}_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}}.$$
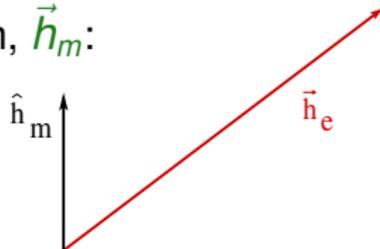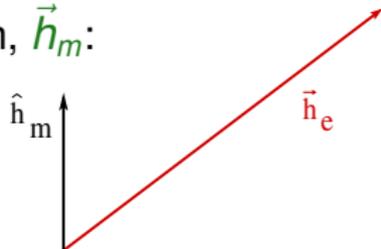
normalized so that $\langle \hat{h}_m | \hat{h}_m \rangle = 1$.

- Search for signals by projecting data onto model waveforms: ρ_m is the signal-to-noise ratio for $\vec{h}_e$ projected onto $\vec{h}_m$.

- A detection is made when $\vec{h}_e$ has a projected signal-to-noise ratio ρ_m that exceeds a pre-determined threshold.

- Measured signal-to-noise ratio, ρ_m, is largest when the model waveform $\vec{h}_m$ is proportional to the exact $\vec{h}_e$; in this case ρ_m equals the optimal signal-to-noise ratio ρ:

$$\rho_m = \frac{\langle h_e | h_e \rangle}{\sqrt{\langle h_e | h_e \rangle}} = \sqrt{\langle h_e | h_e \rangle} = \rho = \sqrt{\int_{-\infty}^{\infty} \frac{2|h_e(f)|^2}{S_n(f)} df}.$$

Simple Example of GW Data Analysis

- Consider the signal from an $10M_\odot + 10M_\odot$ optimally oriented black-hole binary system located at 135 Mpc.
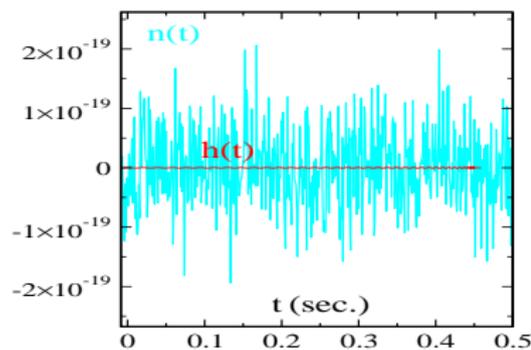
Simple Example of GW Data Analysis

- Consider the signal from an $10M_\odot + 10M_\odot$ optimally oriented black-hole binary system located at 135 Mpc.
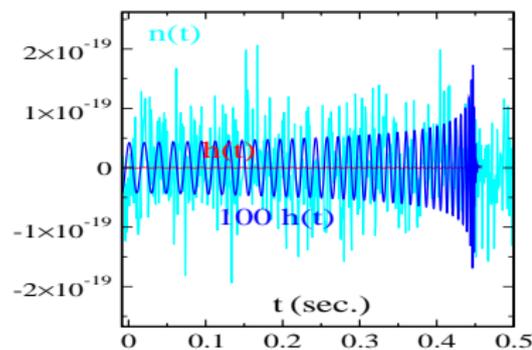
Simple Example of GW Data Analysis

- Consider the signal from an $10M_\odot + 10M_\odot$ optimally oriented black-hole binary system located at 135 Mpc.
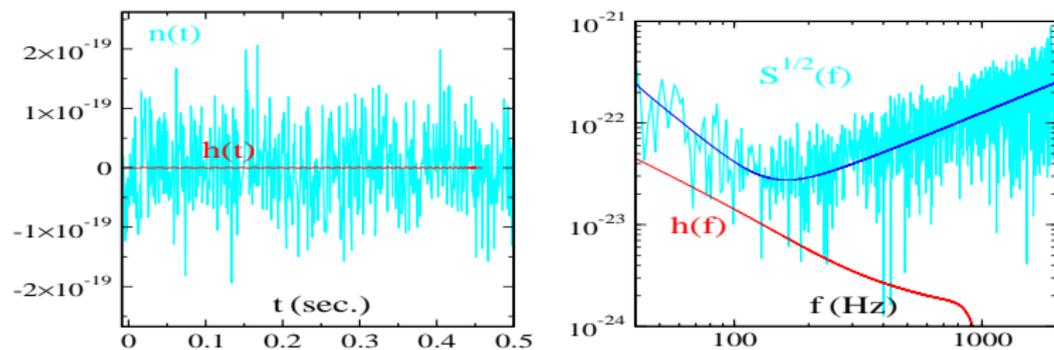
- Fourier transform the signal and model waveform.

Simple Example of GW Data Analysis

- Consider the signal from an $10 M_\odot + 10 M_\odot$ optimally oriented black-hole binary system located at 135 Mpc.
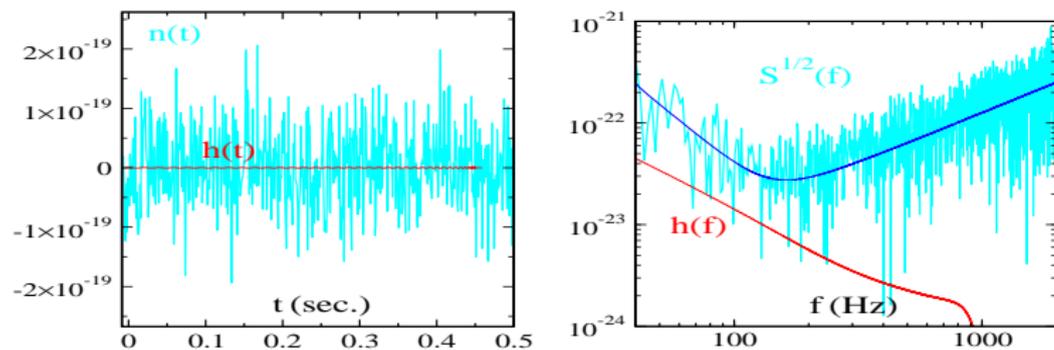
- Fourier transform the signal and model waveform.
- Optimal signal-to-noise ratio for detection by matching to an accurate model waveform:

$$\rho^2 = \langle h_e | h_e \rangle = 4 \int_0^\infty \frac{|h_e(f)|^2}{S_n(f)} df = 64.$$

How accurate must model waveforms be for GW data analysis?

- Derive model waveform accuracy requirements for ideal detectors:
  - Standards for detection.
  - Standards for measurement.

How accurate must model waveforms be for GW data analysis?

- Derive model waveform accuracy requirements for ideal detectors:
  - Standards for detection.
  - Standards for measurement.
- Influence of Detector Calibration Errors

How accurate must model waveforms be for GW data analysis?

- Derive model waveform accuracy requirements for ideal detectors:
  - Standards for detection.
  - Standards for measurement.
- Influence of Detector Calibration Errors
- Evaluate standards for the Advanced LIGO case.

How accurate must model waveforms be for GW data analysis?

- Derive model waveform accuracy requirements for ideal detectors:
    - Standards for detection.
    - Standards for measurement.
- Influence of Detector Calibration Errors
- Evaluate standards for the Advanced LIGO case.
- Possible misinterpretations and misapplications of the standards.

How accurate must model waveforms be for GW data analysis?

- Derive model waveform accuracy requirements for ideal detectors:
  - Standards for detection.
  - Standards for measurement.
- Influence of Detector Calibration Errors
- Evaluate standards for the Advanced LIGO case.
- Possible misinterpretations and misapplications of the standards.
- Transform standards into more user-friendly forms.

Accuracy Standards for Detection

- The measured signal-to-noise ratio ρ_m for detecting the signal h_e is the projection of h_e onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

Accuracy Standards for Detection

- The measured signal-to-noise ratio ρ_m for detecting the signal h_e is the projection of h_e onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

- Errors in model waveform, $h_m = h_e + \delta h$, result in reduction of ρ_m compared to the optimal signal-to-noise ratio ρ:

$$\rho_m = \rho\,(1 - \epsilon) = \langle h_e | h_e \rangle^{1/2}(1 - \epsilon).$$

Accuracy Standards for Detection

- The measured signal-to-noise ratio ρ_m for detecting the signal h_e is the projection of h_e onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

- Errors in model waveform, $h_m = h_e + \delta h$, result in reduction of ρ_m compared to the optimal signal-to-noise ratio ρ:

$$\rho_m = \rho \, (1 - \epsilon) = \langle h_e | h_e \rangle^{1/2} (1 - \epsilon).$$

- Evaluate this mismatch ϵ in terms of the waveform error:

$$\epsilon = \frac{\langle \delta h_\perp | \delta h_\perp \rangle}{2 \langle h_e | h_e \rangle}, \quad \text{where} \quad \delta h_\perp = \delta h - \hat{h}_e \langle \hat{h}_e | \delta h \rangle.$$

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1 - \epsilon)$.

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1 - \epsilon)$.

- The rate of detections is proportional to the volume of space where sources can be seen, so when model waveform errors exist the effective rate of detections is reduced by the amount:

$$\frac{R^3 - R^3(1 - \epsilon)^3}{R^3} = 1 - (1 - \epsilon)^3$$

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1 - \epsilon)$.

- The rate of detections is proportional to the volume of space where sources can be seen, so when model waveform errors exist the effective rate of detections is reduced by the amount:

$$\frac{R^3 - R^3(1 - \epsilon)^3}{R^3} = 1 - (1 - \epsilon)^3$$

- The loss of detections can be limited to an acceptable level, by limiting the mismatch ϵ to an acceptable range: $\epsilon < \epsilon_{\max}$.

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1 - \epsilon)$.

- The rate of detections is proportional to the volume of space where sources can be seen, so when model waveform errors exist the effective rate of detections is reduced by the amount:

$$\frac{R^3 - R^3(1 - \epsilon)^3}{R^3} = 1 - (1 - \epsilon)^3$$

- The loss of detections can be limited to an acceptable level, by limiting the mismatch ϵ to an acceptable range: $\epsilon < \epsilon_{\max}$.

- Consequently model waveform accuracy must satisfy the requirement for detection: $\langle \delta h_\perp | \delta h_\perp \rangle < 2\epsilon_{\max}\rho^2$.

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
- Consider the one-parameter family of waveforms:

$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
- Consider the one-parameter family of waveforms:

$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

- The variance for measuring the parameter λ is given by

$$\frac{1}{\sigma_\lambda^2} = \left\langle \frac{\partial h}{\partial \lambda} \middle| \frac{\partial h}{\partial \lambda} \right\rangle = \langle \delta h | \delta h \rangle,$$

where the noise weighted inner product is defined by

$$\langle h_e | h_m \rangle = \int_{-\infty}^{\infty} \frac{h_e^*(f)h_m(f) + h_e(f)h_m^*(f)}{S_n(f)} df.$$

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
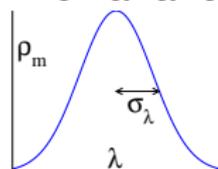- Consider the one-parameter family of waveforms:

$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

- The variance for measuring the parameter λ is given by

$$\frac{1}{\sigma_\lambda^2} = \left\langle \frac{\partial h}{\partial \lambda} \middle| \frac{\partial h}{\partial \lambda} \right\rangle = \langle \delta h | \delta h \rangle,$$

where the noise weighted inner product is defined by
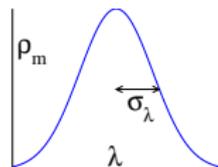
$$\langle h_e | h_m \rangle = \int_{-\infty}^{\infty} \frac{h_e^*(f)h_m(f) + h_e(f)h_m^*(f)}{S_n(f)} df.$$

- Two waveforms are indistinguishable iff the variance σ_λ^2 is larger than the parameter distance between the waveforms: $(\Delta\lambda)^2 = 1 < \sigma_\lambda^2 = 1/\langle \delta h | \delta h \rangle$, that is iff $1 > \langle \delta h | \delta h \rangle$.

Effects of Detector Calibration Error

- The raw electronic output of the detector, $v(f)$, is converted to the measured gravitational wave signal, $h(f)$, using the response function: $h(f) = R(f)v(f)$.

Effects of Detector Calibration Error

- The raw electronic output of the detector, $v(f)$, is converted to the measured gravitational wave signal, $h(f)$, using the response function: $h(f) = R(f)v(f)$.

- Errors in the measured response function produce errors in the inferred waveform:

$$h = Rv_e = (R_e + \delta R)\, v_e = h_e + \delta h_R,$$

or equivalently

$$\delta h_R = h_e e^{\delta \chi_R + i\delta\Phi_R} - h_e \approx h_e\left(\delta\chi_R + i\delta\Phi_R\right).$$

Effects of Detector Calibration Error

- The raw electronic output of the detector, $v(f)$, is converted to the measured gravitational wave signal, $h(f)$, using the response function: $h(f) = R(f)v(f)$.

- Errors in the measured response function produce errors in the inferred waveform:

$$h = Rv_e = (R_e + \delta R)\, v_e = h_e + \delta h_R,$$

  or equivalently

$$\delta h_R = h_e e^{\delta \chi_R + i \delta \Phi_R} - h_e \approx h_e \left(\delta \chi_R + i \delta \Phi_R\right).$$

- Errors in the measured response function also affect the measured power spectral density of the detector noise, $S_n(f) = e^{2\delta \chi_R(f)} S_e(f)$, with resulting effects on the measured signal-to-noise ratio ρ_m.

Effects of Detector Calibration Error II

- Evaluate the measured signal to noise-to-noise ratio:

$$
\begin{aligned}
\rho_m &= \frac{\langle h | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}} = \frac{\langle h_e + \delta h_R | h_e + \delta h_m \rangle}{\sqrt{\langle h_e + \delta h_m | h_e + \delta h_m \rangle}}, \\
&\approx \rho - \frac{1}{2\rho} \langle (\delta h_m - \delta h_R)_\perp | (\delta h_m - \delta h_R)_\perp \rangle,
\end{aligned}
$$

where

$$
(\delta h_m - \delta h_R)_\perp = \delta h_m - \delta h_R - \hat{h}_e \langle \hat{h}_e | \delta h_m - \delta h_R \rangle.
$$

Effects of Detector Calibration Error II

- Evaluate the measured signal to noise-to-noise ratio:

$$\begin{aligned}
\rho_m &= \frac{\langle h|h_m \rangle}{\sqrt{\langle h_m|h_m \rangle}} = \frac{\langle h_e + \delta h_R|h_e + \delta h_m \rangle}{\sqrt{\langle h_e + \delta h_m|h_e + \delta h_m \rangle}}, \\
&\approx \rho - \frac{1}{2\rho} \langle (\delta h_m - \delta h_R)_\perp | (\delta h_m - \delta h_R)_\perp \rangle,
\end{aligned}$$

where

$$(\delta h_m - \delta h_R)_\perp = \delta h_m - \delta h_R - \hat{h}_e \langle \hat{h}_e | \delta h_m - \delta h_R \rangle.$$

- Errors in the measured signal-to-noise ratio, $\delta \rho_m$, depend only on the difference between the waveform errors: $\delta h_m - \delta h_R$.

Effects of Detector Calibration Error II

- Evaluate the measured signal to noise-to-noise ratio:

$$\begin{aligned}
\rho_m &= \frac{\langle h | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}} = \frac{\langle h_e + \delta h_R | h_e + \delta h_m \rangle}{\sqrt{\langle h_e + \delta h_m | h_e + \delta h_m \rangle}}, \\
&\approx \rho - \frac{1}{2\rho} \langle (\delta h_m - \delta h_R)_\perp | (\delta h_m - \delta h_R)_\perp \rangle,
\end{aligned}$$

where

$$(\delta h_m - \delta h_R)_\perp = \delta h_m - \delta h_R - \hat{h}_e \langle \hat{h}_e | \delta h_m - \delta h_R \rangle.$$

- Errors in the measured signal-to-noise ratio, $\delta \rho_m$, depend only on the difference between the waveform errors: $\delta h_m - \delta h_R$.

- Waveform accuracy standards are therefore just the ideal detector ($\delta h_R = 0$) standards with δh_m replaced by $\delta h_m - \delta h_R$:
  $\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle < 1$ for measurement, and
  $\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle < 2\rho^2 \epsilon_{\max}$ for detection.

Effects of Detector Calibration Error III

- The combined accuracy requirements can be written as

$$\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle < \left\{ \begin{array}{ll} 1 & \text{measurement,} \\ 2\rho^2 \epsilon_{\max} & \text{detection.} \end{array} \right.$$

Effects of Detector Calibration Error III

- The combined accuracy requirements can be written as

$$\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle < \left\{ \begin{array}{ll} 1 & \text{measurement,} \\ 2\rho^2 \epsilon_{\max} & \text{detection.} \end{array} \right.$$

- Waveform modeling error, δh_m, is uncorrelated with calibration error, δh_R, so re-write the accuracy requirement using,

$$\sqrt{\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle} < \sqrt{\langle \delta h_m | \delta h_m \rangle} + \sqrt{\langle \delta h_R | \delta h_R \rangle},$$

which leads to the new accuracy requirements:

$$\sqrt{\langle \delta h_m | \delta h_m \rangle} + \sqrt{\langle \delta h_R | \delta h_R \rangle} < \left\{ \begin{array}{ll} 1 & \text{measurement,} \\ \sqrt{2\epsilon_{\max}}\rho & \text{detection.} \end{array} \right.$$

Effects of Detector Calibration Error III

- The combined accuracy requirements can be written as

$$\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle < \left\{ \begin{array}{ll} 1 & \text{measurement,} \\ 2\rho^2 \epsilon_{\max} & \text{detection.} \end{array} \right.$$

- Waveform modeling error, δh_m, is uncorrelated with calibration error, δh_R, so re-write the accuracy requirement using,

$$\sqrt{\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle} < \sqrt{\langle \delta h_m | \delta h_m \rangle} + \sqrt{\langle \delta h_R | \delta h_R \rangle},$$

which leads to the new accuracy requirements:

$$\sqrt{\langle \delta h_m | \delta h_m \rangle} + \sqrt{\langle \delta h_R | \delta h_R \rangle} < \left\{ \begin{array}{ll} 1 & \text{measurement,} \\ \sqrt{2\epsilon_{\max}}\rho & \text{detection.} \end{array} \right.$$

Effects of Detector Calibration Error III

- The combined accuracy requirements can be written as

$$\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle < \left\{ \begin{array}{ll} 1 & \text{measurement,} \\ 2\rho^2 \epsilon_{\max} & \text{detection.} \end{array} \right.$$

- Waveform modeling error, δh_m, is uncorrelated with calibration error, δh_R, so re-write the accuracy requirement using,

$$\sqrt{\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle} < \sqrt{\langle \delta h_m | \delta h_m \rangle} + \sqrt{\langle \delta h_R | \delta h_R \rangle},$$

  which leads to the new accuracy requirements:

$$\sqrt{\langle \delta h_m | \delta h_m \rangle} + \sqrt{\langle \delta h_R | \delta h_R \rangle} < \left\{ \begin{array}{ll} 1 & \text{measurement,} \\ \sqrt{2\epsilon_{\max}}\rho & \text{detection.} \end{array} \right.$$

- Choose the relative size of the errors based on cost, or ...?
  If comparable accuracy standards are adopted, then the waveform modeling standard becomes:

$$\sqrt{\langle \delta h_m | \delta h_m \rangle} < \left\{ \begin{array}{ll} 1/2 & \text{measurement,} \\ \sqrt{2\epsilon_{\max}}\rho - 1/2 & \text{detection.} \end{array} \right.$$

Accuracy Standards for LIGO

- It is useful to define the model waveform (logarithmic) amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors:
  $\delta h_m = h_e e^{\delta\chi_m + i\delta\Phi_m} - h_e \approx h_e(\delta\chi_m + i\delta\Phi_m)$.

Accuracy Standards for LIGO

- It is useful to define the model waveform (logarithmic) amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors:
  $\delta h_m = h_e e^{\delta\chi_m + i\delta\Phi_m} - h_e \approx h_e(\delta\chi_m + i\delta\Phi_m)$.

- The basic accuracy requirements can be written as

$$\frac{\sqrt{\langle\delta h|\delta h\rangle}}{\rho} = \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \left\{ \begin{array}{ll} 1/2\rho & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} - 1/2\rho_{\max} & \text{detection,} \end{array} \right.$$

  where the signal-weighted average errors are defined as

$$\overline{\delta\chi_m}^2 = \int_{-\infty}^{\infty} \delta\chi_m^2 \frac{2|h_e|^2}{\rho^2 S_n} df, \quad \text{and} \quad \overline{\delta\Phi_m}^2 = \int_{-\infty}^{\infty} \delta\Phi_m^2 \frac{2|h_e|^2}{\rho^2 S_n} df.$$

Accuracy Standards for LIGO

- It is useful to define the model waveform (logarithmic) amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors:
  $\delta h_m = h_e e^{\delta\chi_m + i\delta\Phi_m} - h_e \approx h_e(\delta\chi_m + i\delta\Phi_m)$.

- The basic accuracy requirements can be written as

  $$\frac{\sqrt{\langle\delta h|\delta h\rangle}}{\rho} = \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \left\{ \begin{array}{ll} 1/2\rho & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} - 1/2\rho_{\max} & \text{detection,} \end{array} \right.$$

  where the signal-weighted average errors are defined as

  $$\overline{\delta\chi_m}^2 = \int_{-\infty}^{\infty} \delta\chi_m^2 \frac{2|h_e|^2}{\rho^2 S_n} df, \quad \text{and} \quad \overline{\delta\Phi_m}^2 = \int_{-\infty}^{\infty} \delta\Phi_m^2 \frac{2|h_e|^2}{\rho^2 S_n} df.$$

- The most restrictive measurement standards are needed for the strongest gravitational wave signals. For Advanced LIGO the maximum signal-to-noise ratio unlikely larger than $\rho_{\max} \approx 100$.
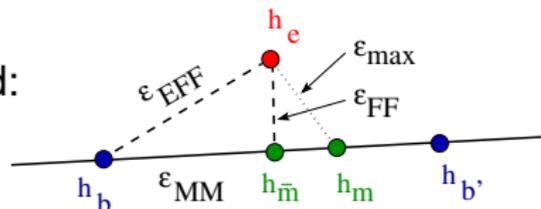
$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \frac{1}{2\rho_{\max}} \approx 0.005.$$

Detection Standards for LIGO

- Accuracy requirement for detection depends on the parameter ϵ_{max}, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{max} = 0.035$ limits the loss rate to about 10%.

Detection Standards for LIGO

- Accuracy requirement for detection depends on the parameter ϵ_{max}, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{max} = 0.035$ limits the loss rate to about 10%.
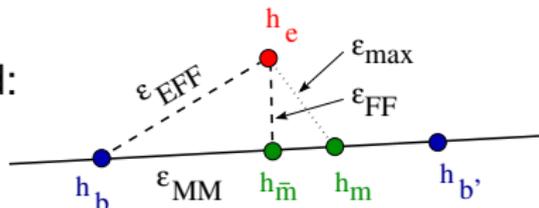- Real searches are more complicated: comparing signals with a discrete template bank of model waveforms.

- For Initial LIGO, template banks are constructed with $\epsilon_{MM} = 0.03$, so $\epsilon_{FF} = \epsilon_{EFF} - \epsilon_{MM} = 0.035 - 0.03 = 0.005$.

Detection Standards for LIGO

- Accuracy requirement for detection depends on the parameter ϵ_{max}, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{max} = 0.035$ limits the loss rate to about 10%.
- Real searches are more complicated: comparing signals with a discrete template bank of model waveforms.

- For Initial LIGO, template banks are constructed with $\epsilon_{MM} = 0.03$, so $\epsilon_{FF} = \epsilon_{EFF} - \epsilon_{MM} = 0.035 - 0.03 = 0.005$.
- To ensure this condition, ϵ_{max} must be chosen so that $\epsilon_{max} \leq 0.005$.
- Accuracy requirement for BBH waveforms for detection in LIGO:

$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \sqrt{2\epsilon_{max}} - 1/2\rho_{max} \approx 0.095.$$

Summary of Model Waveform Accuracy Standards

- The basic model waveform accuracy standards are:

$$\sqrt{\frac{\langle \delta h_m | \delta h_m \rangle}{\rho^2}} = \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \left\{ \begin{array}{ll} 1/2\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} - 1/2\rho_{\max} & \text{detection.} \end{array} \right.$$

Summary of Model Waveform Accuracy Standards

- The basic model waveform accuracy standards are:

$$\sqrt{\frac{\langle \delta h_m | \delta h_m \rangle}{\rho^2}} = \sqrt{\overline{\delta \chi_m}^2 + \overline{\delta \Phi_m}^2} < \left\{ \begin{array}{ll} 1/2\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} - 1/2\rho_{\max} & \text{detection.} \end{array} \right.$$

- Standards for LIGO and LISA:

	Measurement Requirement	Detection Requirement
General Detector	$1/2\rho_{\max}$	$\sqrt{2\epsilon_{\max}} - 1/2\rho_{\max}$
Advanced LIGO	0.005	0.095
LISA	0.0005	0.0995

Verifying NR waveform accuracy

- The standards place limits on the signal- and noise-weighted averages of the frequency-domain amplitude and phase errors:

$$\sqrt{\frac{\langle \delta h_m | \delta h_m \rangle}{\rho^2}} = \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \begin{cases} 1/\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} & \text{detection.} \end{cases}$$

- How can NR waveforms be checked against these standards?

Verifying NR waveform accuracy

- The standards place limits on the signal- and noise-weighted averages of the frequency-domain amplitude and phase errors:

$$\sqrt{\frac{\langle \delta h_m | \delta h_m \rangle}{\rho^2}} = \sqrt{\overline{\delta \chi_m}^2 + \overline{\delta \Phi_m}^2} < \begin{cases} 1/\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} & \text{detection.} \end{cases}$$

- How can NR waveforms be checked against these standards?
- Express the time-domain waveform in terms of an amplitude $A_e(t)$ and phase $\Phi_e(t)$ of the "exact" waveform,

$$h_e(t) = A_e(t) \cos \Phi_e(t),$$

plus errors,

$$h_m(t) = A_e(t) \left[1 + \delta\mu_\chi g_\chi(t)\right] \cos \left[\Phi_e(t) + \delta\mu_\Phi g_\Phi(t)\right],$$

where $\delta\mu_\chi$ and $\delta\mu_\Phi$ are the maximum amplitude and phase errors so that $|g_\chi(t)| \leq 1$ and $|g_\Phi(t)| \leq 1$.

Verifying NR waveform accuracy II

- Some NR groups have estimated the maximum time-domain waveform errors $\delta\mu_\chi$ and $\delta\mu_\Phi$, and compared them with the standards for $\overline{|\delta\chi_m|}$ and $\overline{|\delta\Phi_m|}$.
- Is this good enough?

Verifying NR waveform accuracy II

- Some NR groups have estimated the maximum time-domain waveform errors $\delta\mu_\chi$ and $\delta\mu_\Phi$, and compared them with the standards for $|\overline{\delta\chi_m}|$ and $|\overline{\delta\Phi_m}|$.
- Is this good enough?
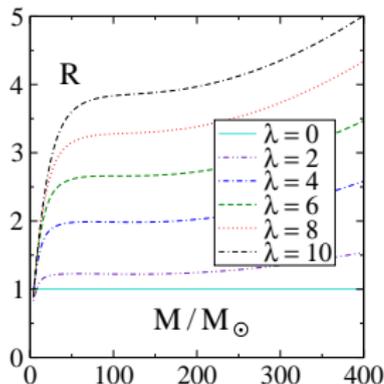- Consider a model waveform: $h_m(t)$ with errors of the form:

$$h_m(t) = A_e(t)\left[1 + \delta\mu_\chi g_\chi(t)\right]\cos\left[\Phi_e(t) + \delta\mu_\Phi g_\Phi(t)\right],$$

with $g_\chi = g_\Phi = \cos[\lambda\Phi_e(t)]$.

- Compute ratio of frequency- to time-domain error measures,

$$R = \sqrt{\frac{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2}{\delta\mu_\chi^2 + \delta\mu_\Phi^2}}$$

using the PN+Caltech/Cornell waveform for A_e and Φ_e.

Verifying NR waveform accuracy II

- Some NR groups have estimated the maximum time-domain waveform errors $\delta\mu_\chi$ and $\delta\mu_\Phi$, and compared them with the standards for $\overline{|\delta\chi_m|}$ and $\overline{|\delta\Phi_m|}$.

- Is this good enough?

- Consider a model waveform: $h_m(t)$ with errors of the form:

$$h_m(t) = A_e(t)\left[1 + \delta\mu_\chi g_\chi(t)\right]\cos\left[\Phi_e(t) + \delta\mu_\Phi g_\Phi(t)\right],$$
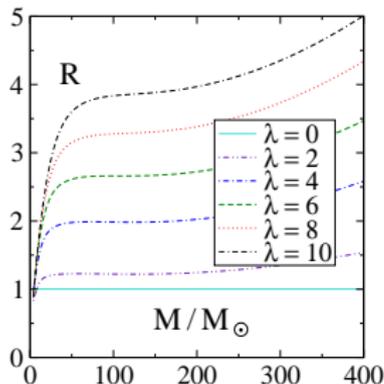
with $g_\chi = g_\Phi = \cos[\lambda\Phi_e(t)]$.

- Compute ratio of frequency- to time-domain error measures,

$$R = \sqrt{\frac{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2}{\delta\mu_\chi^2 + \delta\mu_\Phi^2}}$$

using the PN+Caltech/Cornell waveform for A_e and Φ_e.

- Bad News! Limiting $\delta\mu_\chi$ and $\delta\mu_\Phi$ to the standards is not sufficient.

Verifying NR waveform accuracy III

- Additional knowledge of the full waveform errors, $\delta\mu_\chi g_\chi(t)$ and $\delta\mu_\Phi g_\Phi(t)$, is needed. Unfortunately the exact time dependencies, $g_\chi(t)$ and $g_\Phi(t)$. will never be known.
- Is a partial knowledge of $g_\chi(t)$ and $g_\Phi(t)$ sufficient?

Verifying NR waveform accuracy III

- Additional knowledge of the full waveform errors, $\delta\mu_\chi g_\chi(t)$ and $\delta\mu_\Phi g_\Phi(t)$, is needed. Unfortunately the exact time dependencies, $g_\chi(t)$ and $g_\Phi(t)$. will never be known.
- Is a partial knowledge of $g_\chi(t)$ and $g_\Phi(t)$ sufficient?
- Probably the most we will ever know will be local-in-time error envelope-functions $G_\chi(t)$ and $G_\Phi(t)$, that satisfy

  $$|g_\chi(t)| \leq G_\chi(t) \leq 1, \quad \text{and} \quad |g_\Phi(t)| \leq G_\Phi(t) \leq 1.$$

- Do time-domain bounds imply frequency-domain bounds, i.e., does $|g(t)| \leq G(t)$ imply $|g(f)| \leq G(f)$?
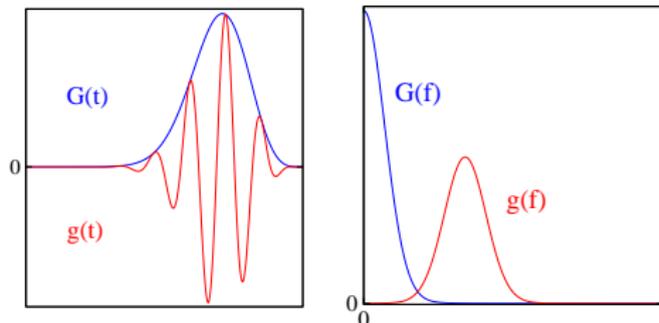
Verifying NR waveform accuracy III

- Additional knowledge of the full waveform errors, $\delta\mu_\chi g_\chi(t)$ and $\delta\mu_\Phi g_\Phi(t)$, is needed. Unfortunately the exact time dependencies, $g_\chi(t)$ and $g_\Phi(t)$. will never be known.
- Is a partial knowledge of $g_\chi(t)$ and $g_\Phi(t)$ sufficient?
- Probably the most we will ever know will be local-in-time error envelope-functions $G_\chi(t)$ and $G_\Phi(t)$, that satisfy

    $$|g_\chi(t)| \leq G_\chi(t) \leq 1, \quad \text{and} \quad |g_\Phi(t)| \leq G_\Phi(t) \leq 1.$$

- Do time-domain bounds imply frequency-domain bounds, i.e., does $|g(t)| \leq G(t)$ imply $|g(f)| \leq G(f)$?
- No!
- It is not possible to verify the accuracy of a waveform using a time-domain error-envelope function.

Verifying NR waveform accuracy II

- Some NR groups have estimated the maximum time-domain waveform errors $\delta\mu_\chi$ and $\delta\mu_\Phi$, and compared them with the standards for $|\overline{\delta\chi_m}|$ and $|\overline{\delta\Phi_m}|$.
- Is this good enough?
- Consider a model waveform: $h_m(t)$ with errors of the form:

$$h_m(t) = A_e(t)\left[1 + \delta\mu_\chi g_\chi(t)\right]\cos\left[\Phi_e(t) + \delta\mu_\Phi g_\Phi(t)\right],$$
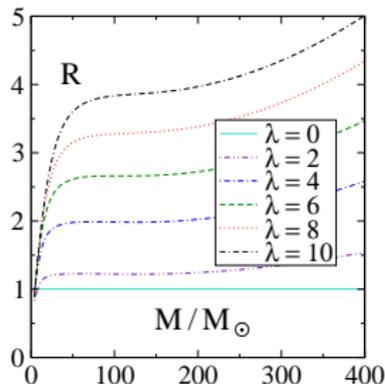
with $g_\chi = g_\Phi = \cos[\lambda\Phi_e(t)]$.

- Compute ratio of frequency- to time-domain error measures,

$$R = \sqrt{\frac{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2}{\delta\mu_\chi^2 + \delta\mu_\Phi^2}}$$

using the PN+Caltech/Cornell waveform for A_e and Φ_e.

- Bad News! Limiting $\delta\mu_\chi$ and $\delta\mu_\Phi$ to the standards is not sufficient.

Alternate Waveform Accuracy Requirements

- This seems like a disaster: error envelope functions are probably the most we will ever know about waveform errors, yet they do not provide useful estimates of the relevant error norms.

- Is it possible to construct an alternate waveform accuracy requirement that relies only on a bound, $|g(t)| \leq G(t) \leq 1$, of the time-domain waveform error?

Alternate Waveform Accuracy Requirements

- This seems like a disaster: error envelope functions are probably the most we will ever know about waveform errors, yet they do not provide useful estimates of the relevant error norms.

- Is it possible to construct an alternate waveform accuracy requirement that relies only on a bound, $|g(t)| \leq G(t) \leq 1$, of the time-domain waveform error?

- A local-in-time error envelope $G(t)$ does provide a bound on the L^2 norm of the frequency-domain waveform error:

$$
\begin{aligned}
\int_{-\infty}^{\infty} |g(f)|^2 df &= \int_{-\infty}^{\infty} |g(t)|^2 dt \\
&\leq \int_{-\infty}^{\infty} |G(t)|^2 dt.
\end{aligned}
$$

Alternate Waveform Accuracy Requirements

- This seems like a disaster: error envelope functions are probably the most we will ever know about waveform errors, yet they do not provide useful estimates of the relevant error norms.

- Is it possible to construct an alternate waveform accuracy requirement that relies only on a bound, $|g(t)| \leq G(t) \leq 1$, of the time-domain waveform error?

- A local-in-time error envelope $G(t)$ does provide a bound on the L^2 norm of the frequency-domain waveform error:

$$
\begin{aligned}
\int_{-\infty}^{\infty} |g(f)|^2 df &= \int_{-\infty}^{\infty} |g(t)|^2 dt \\
&\leq \int_{-\infty}^{\infty} |G(t)|^2 dt.
\end{aligned}
$$

- A waveform accuracy requirement based on L^2 norms, rather than the usual noise-weighted norm, could therefore be implemented using local-in-time error bounds

L^2 Norm Accuracy Standard

- We can derive an accuracy requirement based on L^2 norms:

$$\langle \delta h_m | \delta h_m \rangle = 2 \int_{-\infty}^{\infty} \frac{|\delta h_m|^2}{S_n(f)} df \leq \frac{2||\delta h_m(f)||^2}{\min S_n(f)},$$

where $||\delta h_m(f)||^2 = \int_{-\infty}^{\infty} |\delta h_m|^2 df$ is the L^2 norm of $\delta h_m(f)$.

L^2 Norm Accuracy Standard

- We can derive an accuracy requirement based on L^2 norms:

$$\langle \delta h_m | \delta h_m \rangle = 2 \int_{-\infty}^{\infty} \frac{|\delta h_m|^2}{S_n(f)} df \leq \frac{2||\delta h_m(f)||^2}{\min S_n(f)},$$

where $||\delta h_m(f)||^2 = \int_{-\infty}^{\infty} |\delta h_m|^2 df$ is the L^2 norm of $\delta h_m(f)$.

- We can therefore convert the basic accuracy requirements (on measurement in this case) into the following sufficient condition:

$$\sqrt{\langle \delta h_m | \delta h_m \rangle} \leq \frac{\sqrt{2}||\delta h_m(f)||}{\sqrt{\min S_n(f)}} < \frac{1}{2}.$$

L^2 Norm Accuracy Standard

- We can derive an accuracy requirement based on L^2 norms:

$$\langle \delta h_m | \delta h_m \rangle = 2 \int_{-\infty}^{\infty} \frac{|\delta h_m|^2}{S_n(f)} df \leq \frac{2||\delta h_m(f)||^2}{\min S_n(f)},$$

  where $||\delta h_m(f)||^2 = \int_{-\infty}^{\infty} |\delta h_m|^2 df$ is the L^2 norm of $\delta h_m(f)$.

- We can therefore convert the basic accuracy requirements (on measurement in this case) into the following sufficient condition:

$$\sqrt{\langle \delta h_m | \delta h_m \rangle} \leq \frac{\sqrt{2}||\delta h_m(f)||}{\sqrt{\min S_n(f)}} < \frac{1}{2}.$$

- This accuracy requirement requires the waveform h_m and its error-envelope estimate δh_m to have the proper scale.

- NR simulations only determine the scale invariant $r\, h_m / M$ and $r\delta h_m / M$, so what value of the scale r should be used?

L^2 Norm Accuracy Standards II

- A scale invariant accuracy standard can be constructed by introducing the obvious L^2 norm waveform scale:

$$\frac{||\delta h(f)||}{||h_m(f)||} = \frac{||\delta h(t)||}{||h_m(t)||} < \frac{\sqrt{\min\ S_n}}{2\sqrt{2}||h_m||}.$$

L^2 Norm Accuracy Standards II

- A scale invariant accuracy standard can be constructed by introducing the obvious L^2 norm waveform scale:

$$\frac{||\delta h(f)||}{||h_m(f)||} = \frac{||\delta h(t)||}{||h_m(t)||} < \frac{\sqrt{\min S_n}}{2\sqrt{2}||h_m||}.$$

- Unfortunately, the right side of this new condition depends on $||h_m||$, which must still be scaled properly.

L^2 Norm Accuracy Standards II

- A scale invariant accuracy standard can be constructed by introducing the obvious L^2 norm waveform scale:

$$\frac{||\delta h(f)||}{||h_m(f)||} = \frac{||\delta h(t)||}{||h_m(t)||} < \frac{\sqrt{\min S_n}}{2\sqrt{2}||h_m||}.$$

- Unfortunately, the right side of this new condition depends on $||h_m||$, which must still be scaled properly.

- Introduce the scale invariant quantity C, defined as

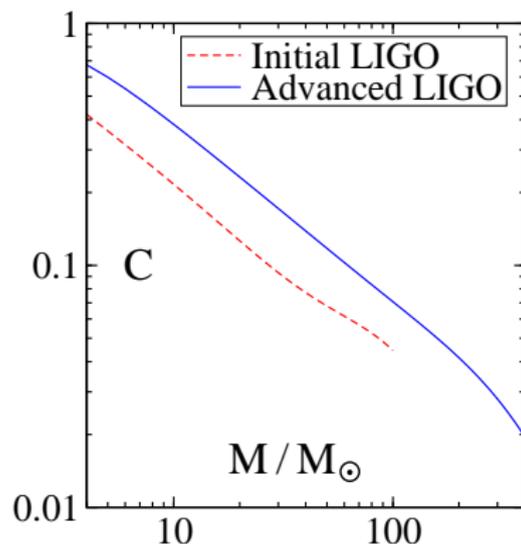$$C^2 = \frac{\rho^2}{2||h_m(f)||^2/\min S_n(f)} \leq 1,$$

and use it to re-write the accuracy standards,

$$\frac{||\delta h(f)||}{||h_m(f)||} = \frac{||\delta h(t)||}{||h_m(t)||} < \frac{C}{2\rho},$$

in a way that depends on the waveform scale only through the standard signal-to-noise ratio ρ.
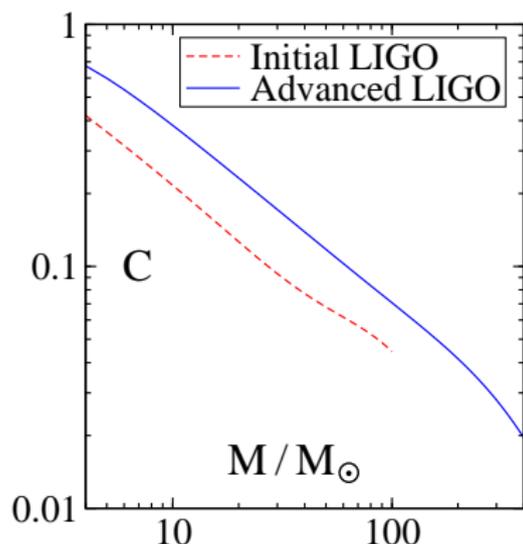
Sufficient Conditions for LIGO

- The signal-to-noise quantity
  $C^2 = \rho^2 \min S_n/2\|h_m\|^2 \leq 1$
  has been evaluated for
  equal-mass non-spinning BBH
  waveforms using LIGO noise.

Sufficient Conditions for LIGO

- The signal-to-noise quantity
  $C^2 = \rho^2 \min S_n/2\|h_m\|^2 \leq 1$
  has been evaluated for
  equal-mass non-spinning BBH
  waveforms using LIGO noise.

- Sufficient accuracy
  requirements for BBH
  waveforms for Advanced LIGO
  are therefore:

$$\frac{\|\delta h_m(t)\|}{\|h_m(t)\|} \lesssim \begin{cases} C/2\rho & \approx \frac{0.02}{200} & \approx 10^{-4} & \text{measurement,} \\ C\sqrt{2\epsilon_{\max}} & \approx 0.02 \times 0.1 \approx 2 \times 10^{-3} & \text{detection.} \end{cases}$$

Sufficient Conditions for LIGO

- The signal-to-noise quantity
  $C^2 = \rho^2 \min S_n/2 \|h_m\|^2 \leq 1$
  has been evaluated for
  equal-mass non-spinning BBH
  waveforms using LIGO noise.

- Sufficient accuracy
  requirements for BBH
  waveforms for Advanced LIGO
  are therefore:

$$\frac{\|\delta h_m(t)\|}{\|h_m(t)\|} \lesssim \begin{cases} C/2\rho & \approx \frac{0.02}{200} & \approx 10^{-4} & \text{measurement,} \\ C\sqrt{2\epsilon_{\max}} & \approx 0.02 \times 0.1 \approx 2 \times 10^{-3} & & \text{detection.} \end{cases}$$

- These requirements can be enforced as conditions on
  local-in-time bounds of the amplitude and phase errors:

$$\frac{\|\delta h_m(t)\|}{\|h_m(t)\|} \leq \sqrt{\frac{\int_{-\infty}^{\infty} A_m^2 \left(\delta\mu_\chi^2 G_\chi^2 + \delta\mu_\Phi^2 G_\Phi^2\right) dt}{\int_{-\infty}^{\infty} A_m^2 dt}} \lesssim \begin{cases} C/2\rho & \text{measurement} \\ C\sqrt{2\epsilon_{\max}} & \text{detection} \end{cases}$$
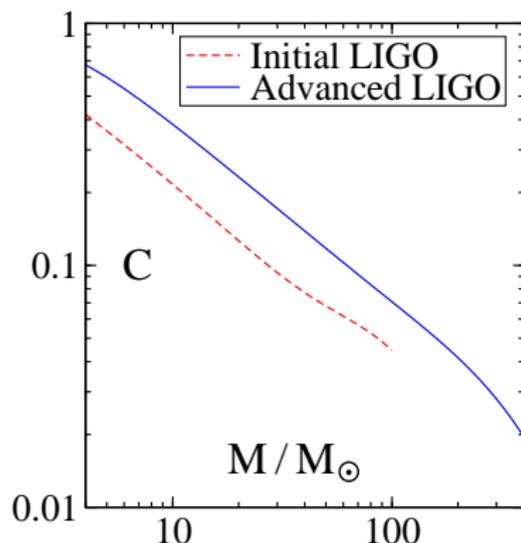
Summary and Questions

- A set of error bounds now exist for model waveforms to be useful in gravitational wave data analysis:

$$\frac{\sqrt{\langle \delta h_m | \delta h_m \rangle}}{\rho} = \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \begin{cases} 1/2\rho & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} & \text{detection.} \end{cases}$$

- These optimal bounds are difficult (impossible?) to enforce directly, so a set of somewhat stronger sufficient conditions on local-in-time error bounds have been derived:

$$\frac{||\delta h_m(t)||}{||h_m(t)||} \leq \sqrt{\frac{\int_{-\infty}^{\infty} A_m^2 \left(\delta\mu_\chi^2 G_\chi^2 + \delta\mu_\Phi^2 G_\Phi^2 \right) dt}{\int_{-\infty}^{\infty} A_m^2 dt}} \lesssim \begin{cases} C/2\rho & \text{measurement,} \\ C\sqrt{2\epsilon_{\max}} & \text{detection.} \end{cases}$$

- Do the search templates currently being used by LIGO satisfy these accuracy requirements?

- Do the waveforms produced by various NR groups satisfy these accuracy requirements?