Solving Einstein's Equation Numerically IV

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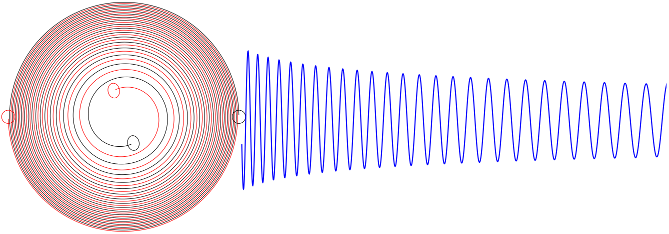

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- From a pragmatic physicist's point of view, hyperbolic means anything that acts like the wave equation, i.e. any system of equations having a well posed initial-boundary value problem.
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- Symmetric hyperbolic systems are one class of equations for which suitable well-posedness theorems exist, and which are general enough to include Einstein's equations together with most of the other dynamical field equations used by physicists.
- Evolution equations of the form,

$$\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u,x,t)\partial_k u^\beta = F^\alpha(u,x,t),$$

for a collection of dynamical fields u^α, are called symmetric hyperbolic if there exists a positive definite $S_{\alpha\beta}$ having the property that $S_{\alpha\gamma}A^{k\,\gamma}{}_\beta \equiv A^k_{\alpha\beta} = A^k_{\beta\alpha}$.

Example: Scalar Wave Equation

- Consider the scalar wave equation in flat space, expressed in terms of arbitrary spatial coordinates:
$$0 = -\partial_t^2\psi + \nabla^k\nabla_k\psi = -\partial_t^2\psi + g^{k\ell}(\partial_k\partial_\ell\psi - \Gamma^n_{k\ell}\partial_n\psi).$$

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- Define the first-order dynamical fields, $u^\alpha = \{\psi, \Pi, \Phi_k\}$, which satisfy the following evolution equations:
  $$\partial_t\psi = -\Pi, \qquad \partial_t\Pi + \nabla^k\Phi_k = 0, \qquad \partial_t\Phi_k + \nabla_k\Pi = 0.$$

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  $$\partial_t\psi = -\Pi, \qquad \partial_t\Pi + \nabla^k\Phi_k = 0, \qquad \partial_t\Phi_k + \nabla_k\Pi = 0.$$
- The principal part of this system, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta\partial_k u^\beta \simeq 0$, is given:

$$\partial_t\begin{pmatrix}\psi\\ \Pi\\ \Phi_x\\ \Phi_y\\ \Phi_z\end{pmatrix} + \begin{pmatrix}0&0&0&0&0\\ 0&0&g^{xx}&g^{xy}&g^{xz}\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix}\partial_x\begin{pmatrix}\psi\\ \Pi\\ \Phi_x\\ \Phi_y\\ \Phi_z\end{pmatrix} + \ldots \simeq 0$$

Example: Scalar Wave Equation II

- The symmetrizer for the first-order scalar field system is:

$$dS^2 = S_{\alpha\beta} du^\alpha du^\beta = \Lambda^2 d\psi^2 + d\Pi^2 + g^{mn} d\Phi_m d\Phi_n.$$

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- Check the symmetrization of the characteristic matrices:

$$S_{\alpha\gamma}A^{x\,\gamma}{}_\beta = \begin{pmatrix} \Lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & g^{xx} & g^{xy} & g^{xz} \\ 0 & 0 & g^{yx} & g^{yy} & g^{yz} \\ 0 & 0 & g^{zx} & g^{zy} & g^{zz} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g^{xx} & g^{xy} & g^{xz} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g^{xx} & g^{xy} & g^{xz} \\ 0 & g^{xx} & 0 & 0 & 0 \\ 0 & g^{yx} & 0 & 0 & 0 \\ 0 & g^{zx} & 0 & 0 & 0 \end{pmatrix} = A^x_{\alpha\beta}$$

First Order Generalized Harmonic Evolution System

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- We denote the principal part of these first order systems using the notation $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta \partial_k u^\beta \simeq 0$. The principal part of the first-order GH Einstein equations can be written as:

$$\begin{aligned} \partial_t \psi_{ab} - N^k \partial_k \psi_{ab} &= -N\,\Pi_{ab}, \\ \partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + N g^{ki} \partial_k \Phi_{iab} &\simeq 0, \\ \partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} &\simeq 0, \end{aligned}$$

where $\Phi_{kab} = \partial_k \psi_{ab}$.

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\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + N g^{ki} \partial_k \Phi_{iab} &\simeq 0, \\
\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} &\simeq 0,
\end{aligned}$$

where $\Phi_{kab} = \partial_k \psi_{ab}$.

- This system has two immediate problems:
    - This system has new constraints, $\mathcal{C}_{kab} = \partial_k \psi_{ab} - \Phi_{kab}$, that tend to grow exponentially during numerical evolutions.
    - This system is not linearly degenerate, so it is possible (likely?) that shocks will develop (e.g. the components that determine the shift evolution have the form $\partial_t N^i - N^k \partial_k N^i \simeq 0$).

A 'New' Generalized Harmonic Evolution System

- We can correct these problems by adding additional multiples of the constraints to the evolution system:

$$\partial_t\psi_{ab} - (1+\gamma_1)N^k\partial_k\psi_{ab} = -N\Pi_{ab} - \gamma_1 N^k\Phi_{kab},$$
$$\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} - \gamma_1\gamma_2 N^k\partial_k\psi_{ab} \simeq -\gamma_1\gamma_2 N^k\Phi_{kab},$$
$$\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} \simeq -\gamma_2 N\Phi_{iab}.$$

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$$\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} \simeq -\gamma_2 N\Phi_{iab}.$$

- This 'new' generalized-harmonic evolution system has several nice properties:
  - This system is symmetric hyperbolic for all values of γ_1 and γ_2.
  - The Φ_{iab} evolution equation can be written in the form, $\partial_t\mathcal{C}_{iab} - N^k\partial_k\mathcal{C}_{iab} \simeq -\gamma_2 N\mathcal{C}_{iab}$, so the new constraints are damped when $\gamma_2 > 0$.
  - This system is linearly degenerate for $\gamma_1 = -1$ (and so shocks should not form from smooth initial data).

Constraint Evolution for the New GH System

- The evolution of the constraints, $c^A = \{\mathcal{C}_a, \mathcal{C}_{kab}, \mathcal{F}_a \approx t^c\partial_c\mathcal{C}_a, \mathcal{C}_{ka} \approx \partial_k\mathcal{C}_a, \mathcal{C}_{klab} = \partial_{[k}\mathcal{C}_{l]ab}\}$ are determined by the evolution of the fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$:
$$\partial_t c^A + A^{k\,A}{}_B(u)\partial_k c^B = F^A{}_B(u, \partial u)\, c^B.$$

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$$\partial_t c^A + A^{k\,A}{}_B(u)\partial_k c^B = F^A{}_B(u, \partial u)\, c^B.$$

- This constraint evolution system is symmetric hyperbolic with principal part:

$$\begin{aligned}
\partial_t\mathcal{C}_a &\simeq 0,\\
\partial_t\mathcal{F}_a - N^k\partial_k\mathcal{F}_a - Ng^{ij}\partial_i\mathcal{C}_{ja} &\simeq 0,\\
\partial_t\mathcal{C}_{ia} - N^k\partial_k\mathcal{C}_{ia} - N\partial_i\mathcal{F}_a &\simeq 0,\\
\partial_t\mathcal{C}_{iab} - (1+\gamma_1)N^k\partial_k\mathcal{C}_{iab} &\simeq 0,\\
\partial_t\mathcal{C}_{ijab} - N^k\partial_k\mathcal{C}_{ijab} &\simeq 0.
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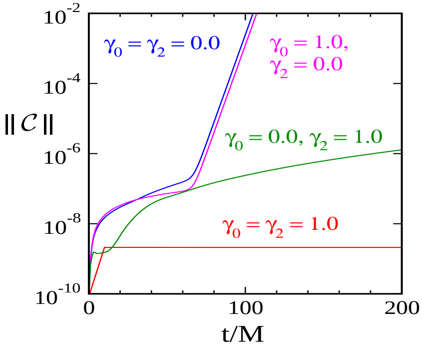
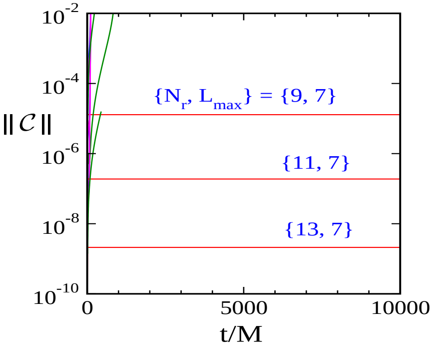
\end{aligned}$$
- An analysis of this system shows that all of the constraints are damped in the WKB limit when $\gamma_0 > 0$ and $\gamma_2 > 0$. So, this system has constraint suppression properties that are similar to those of the Pretorius (and Gundlach, et al.) system.

Numerical Tests of the New GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of our GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.





- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

Boundary Condition Basics

- We impose boundary conditions on first-order hyperbolic evolution systems, $\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\partial_k u^\beta = F^\alpha(u)$ in the following way (where in our case $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$):

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- We first find the eigenvectors of the characteristic matrix $n_k A^{k\,\alpha}{}_\beta$ at each boundary point:

$$e^{\hat\alpha}{}_\alpha\; n_k A^{k\,\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta,$$

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- Finally we impose a boundary condition on each incoming characteristic field (*i.e.* every field with $v_{(\hat\alpha)} < 0$), and impose no condition on any outgoing field (*i.e.* any field with $v_{(\hat\alpha)} \geq 0$).
- At internal boundaries (i.e. interfaces between computational subdomains) use outgoing characteristics of one subdomain to fix data for incoming characteristics of neighboring subdomain.

Characteristic Fields and Characteristic Speeds

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- If $V^\alpha = S^{\alpha\beta} e^{\hat\alpha}{}_\beta$ is one of the eigenvectors of $n_k A^{k\,\alpha}{}_\beta$ with eigenvalue $v_{(\hat\alpha)}$, then the short wavelength dispersion relation for this perturbation becomes: $\omega = k\, v_{(\hat\alpha)}$ for this perturbation.

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- The phase velocity of this perturbation is just the characteristic speed $v_{(\hat\alpha)}$.
  - If $v_{(\hat\alpha)} < 0$, this represents an incoming perturbation that requires a boundary condition.
  - If $v_{(\hat\alpha)} \geq 0$, this represents an outgoing wave that is determined completely by the fields inside the boundary, so no boundary condition is allowed.

Characteristic Fields for the Scalar Field System

- Consider the scalar field equation $\nabla^a\nabla_a\psi = 0$ on a spacetime with fixed metric $ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$. The first order form of this system (including constraint damping):

$$
\begin{aligned}
\partial_t\psi - N^k(1+\gamma_1)\partial_k\psi &\simeq 0,\\
\partial_t\Pi - N^k\partial_k\Pi + Ng^{ki}\partial_k\Phi_i &\simeq 0,\\
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- The characteristic fields $u^{\hat\alpha} = e^{\hat\alpha}_{\beta} u^\beta$ for this system consist of the fields $u^{\hat\alpha} = \{u^{\hat 0}, u^{\hat 1\pm}, u^{\hat 2}_i\}$, given by

$$u^{\hat 0} = \psi, \qquad u^{\hat 1\pm} = \Pi \pm n^k\Phi_k - \gamma_2\psi, \qquad u^{\hat 2}_i = (\delta_i^k - n^k n_i)\Phi_k,$$

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- The coordinate characteristic speeds associated with these fields are $v_{(\hat 0)} = -(1+\gamma_1)n_kN^k$ for the field $u^{\hat 0}$, $v_{(\hat 1\pm)} = -n^kN_k \pm N$ for the fields $u^{\hat 1\pm}$, and $v_{(\hat 2)} = -n_kN^k$ for the fields $u^{\hat 2}_i$.

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- A boundary condition must be imposed on each characteristic field whose characteristic speed is negative on that boundary.
- A boundary condition may not be imposed on any characteristic field whose characteristic speed is non-negative on that boundary.

Characteristic Fields for the Einstein System

- The characteristic fields $u^{\hat\alpha} = e^{\hat\alpha}_{\beta} u^\beta$ for the generalized harmonic version of the Einstein evolution equations look very much like their scalar field counterparts: $u^{\hat\alpha} = \{u^{\hat 0}_{ab}, u^{\hat 1\pm}_{ab}, u^{\hat 2}_{iab}\}$, given by

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- The coordinate characteristic speeds associated with these fields also have the same forms as those for the scalar field system: $v_{(\hat 0)} = -(1+\gamma_1) n_k N^k$ for the fields $u^{\hat 0}_{ab}$, $v_{(\hat 1\pm)} = -n^k N_k \pm N$ for the fields $u^{\hat 1\pm}_{ab}$, and $v_{(\hat 2)} = -n_k N^k$ for the fields $u^{\hat 2}_{iab}$.

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Characteristic Fields for the Einstein System

- The characteristic fields $u^{\hat\alpha} = e^{\hat\alpha}_{\beta} u^\beta$ for the generalized harmonic version of the Einstein evolution equations look very much like their scalar field counterparts: $u^{\hat\alpha} = \{u^{\hat 0}_{ab}, u^{\hat 1\pm}_{ab}, u^{\hat 2}_{iab}\}$, given by

$$\begin{aligned} u^{\hat 0}_{ab} &= \psi_{ab}, \\ u^{\hat 1\pm}_{ab} &= \Pi_{ab} \pm n^k \Phi_{kab} - \gamma_2 \psi_{ab}, \\ u^{\hat 2}_{iab} &= (\delta^k_i - n^k n_i)\Phi_{kab}, \end{aligned}$$

- The coordinate characteristic speeds associated with these fields also have the same forms as those for the scalar field system: $v_{(\hat 0)} = -(1+\gamma_1) n_k N^k$ for the fields $u^{\hat 0}_{ab}$, $v_{(\hat 1\pm)} = -n^k N_k \pm N$ for the fields $u^{\hat 1\pm}_{ab}$, and $v_{(\hat 2)} = -n_k N^k$ for the fields $u^{\hat 2}_{iab}$.
- A boundary condition must be imposed on each characteristic field whose characteristic speed is negative on that boundary.
- A boundary condition may not be imposed on any characteristic field whose characteristic speed is non-negative on that boundary.

Evolutions of a Perturbed Schwarzschild Black Hole

- The simplest boundary conditions that correspond (roughly) to "no incoming waves" set $u^{\hat\alpha} = 0$ for each incoming field, or $d_t u^{\hat\alpha} \equiv e^{\hat\alpha}{}_\beta \partial_t u^\beta = 0$ for fields that include static "Coulomb" parts.

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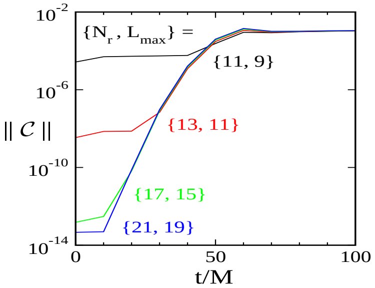
- The outgoing waves interact with the boundary of the computational domain and produce constraint violations.



Play Ψ_4 Movie

Play Constraint Movie

Constraint Preserving Boundary Conditions

- Construct the characteristic fields, $\hat{c}^{\hat{A}} = e^{\hat{A}}{}_{A}c^{A}$, associated with the constraint evolution system, $\partial_t c^A + A^{k\,A}{}_{B}\partial_k c^B = F^{A}{}_{B}c^B$.

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$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$

- Set boundary conditions on the fields $\hat{u}^-$ by requiring

$$d_\perp \hat{u}^- = -\hat{F}(u, d_\parallel u).$$

Constraint Characteristic Fields

- The characteristic fields associated with the constraint evolution system, and their associated characteristic speeds for the first-order Einstein system are:

$$
\begin{aligned}
c_a^{\hat{0}\pm} &= \mathcal{F}_a \mp n^k \mathcal{C}_{ka} \approx t^c \partial_c \mathcal{C}_a \mp n^k \partial_k \mathcal{C}_a, & v_{(\hat{0}\pm)} &= -n_k N^k \pm N, \\
c_a^{\hat{1}} &= \mathcal{C}_a, & v_{(\hat{1})} &= 0, \\
c_{ia}^{\hat{2}} &= P^k{}_i \mathcal{C}_{ka} \approx (\delta^k{}_i - n^k n_i) \partial_k \mathcal{C}_a, & v_{(\hat{2})} &= -n_k N^k, \\
c_{iab}^{\hat{3}} &= \mathcal{C}_{iab}, & v_{(\hat{3})} &= -(1+\gamma_1) n_k N^k, \\
c_{ijab}^{\hat{4}} &= \mathcal{C}_{ijab} = 2\partial_{[j} \mathcal{C}_{i]ab}, & v_{(\hat{4})} &= -n_k N^k.
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c^{\hat{1}}_a &= \mathcal{C}_a, & v_{(\hat{1})} &= 0,\\
c^{\hat{2}}_{ia} &= P^k{}_i\mathcal{C}_{ka} \approx (\delta^k{}_i - n^kn_i)\partial_k\mathcal{C}_a, & v_{(\hat{2})} &= -n_kN^k,\\
c^{\hat{3}}_{iab} &= \mathcal{C}_{iab}, & v_{(\hat{3})} &= -(1+\gamma_1)n_kN^k,\\
c^{\hat{4}}_{ijab} &= \mathcal{C}_{ijab} = 2\partial_{[j}\mathcal{C}_{i]ab}, & v_{(\hat{4})} &= -n_kN^k.
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- The constraint characteristic fields $c^{\hat{0}-}_a$, $c^{\hat{3}}_{iab}$ and $c^{\hat{4}}_{ijab}$ have the same characteristic speeds as the principal dynamical fields $u^{\hat{1}-}_{ab}$, $u^{\hat{0}}_{ab}$ and $u^{\hat{2}}_{iab}$ respectively. These constraint fields will be incoming under the same conditions as these dynamical fiels.

Constraint Characteristic Fields II

- Fortunately, the incoming constraint characteristic fields, $c^{\hat{0}-}_a$, $c^{\hat{3}}_{iab}$ and $c^{\hat{4}}_{ikab}$, can be expressed in terms of the corresponding principal dynamical characteristic fields:

$$
\begin{aligned}
c^{\hat{0}-}_a &\approx \sqrt{2}\left[k^{(c}\psi^{d)}{}_a - \tfrac{1}{2}k_a\psi^{cd}\right] d_\perp u^{\hat{1}-}_{cd},\\
n^k c^{\hat{3}}_{kab} &\approx d_\perp u^{\hat{0}}_{ab},\\
n^k c^{\hat{4}}_{kiab} &\approx d_\perp u^{\hat{2}}_{iab},
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  where $k^a = (t^a - n^a)/\sqrt{2}$ is the ingoing null vector.

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c_a^{\hat{0}-} &\approx \sqrt{2}\left[k^{(c}\psi^{d)}{}_a - \tfrac{1}{2}k_a\psi^{cd}\right] d_\perp u_{cd}^{\hat{1}-}, \\
n^k c_{kab}^{\hat{3}} &\approx d_\perp u_{ab}^{\hat{0}}, \\
n^k c_{kiab}^{\hat{4}} &\approx d_\perp u_{iab}^{\hat{2}},
\end{aligned}
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  where $k^a = (t^a - n^a)/\sqrt{2}$ is the ingoing null vector.

- Setting these incoming characteristic constraint fields to zero therefore provides boundary conditions on the normal derivatives $d_\perp u^{\hat{\alpha}} = e^{\hat{\alpha}}{}_\beta n^k \partial_k u^\beta$ of some of the primary dynamical characteristic fields.

Physical Boundary Conditions

- The Weyl curvature tensor C_{abcd} satisfies a system of evolution equations from the Bianchi identities: $\nabla_{[a}C_{bc]de} = 0$.
- The characteristic fields of this system corresponding to physical gravitational waves are the quantities:

$$\hat{w}^{\pm}_{ab} = (P_a{}^c P_b{}^d - \tfrac{1}{2} P_{ab}P^{cd})(t^e \mp n^e)(t^f \mp n^f)C_{cedf},$$

where t^a is a unit timelike vector, n^a a unit spacelike vector (with $t^a n_a = 0$), and $P_{ab} = \psi_{ab} + t_a t_b - n_a n_b$.

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- The incoming field $\hat{w}^-_{ab}$ can be expressed in terms of the characteristic fields of the primary evolution system:

$$\hat{w}^-_{ab} = d_\perp u^{\hat{1}-}_{ab} + \hat{F}_{ab}(u, d_\parallel u).$$

- We impose boundary conditions on the physical graviational wave degrees of freedom then by setting:

$$d_\perp u^{\hat{1}-}_{ab} = -\hat{F}_{ab}(u, d_\parallel u) + \hat{w}^-_{ab}|_{t=0}.$$

Imposing Neumann-like Boundary Conditions

- Consider Neumann-like boundary conditions of the form

$$e^{\hat{\alpha}}{}_{\beta} n^k \partial_k u^{\beta} \equiv d_{\perp} u^{\hat{\alpha}} = d_{\perp} u^{\hat{\alpha}}|_{\mathrm{BC}}.$$

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- The spatial derivatives of u^γ in this expression can be re-written:
$$e^{\hat\alpha}{}_\beta A^{k\,\beta}{}_\gamma\partial_k u^\gamma = v_{(\hat\alpha)} e^{\hat\alpha}{}_\gamma n^k\partial_k u^\gamma + e^{\hat\alpha}{}_\beta A^{\ell\,\beta}{}_\gamma(\delta^k{}_\ell - n^k n_\ell)\partial_k u^\gamma.$$

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- We impose these Neumann-like boundary conditions by changing the appropriate components of the evolution equations at the boundary to:
$$d_t u^{\hat\alpha} = D_t u^{\hat\alpha} + v_{(\hat\alpha)}\left(d_\perp u^{\hat\alpha} - d_\perp u^{\hat\alpha}|_{\mathrm{BC}}\right).$$