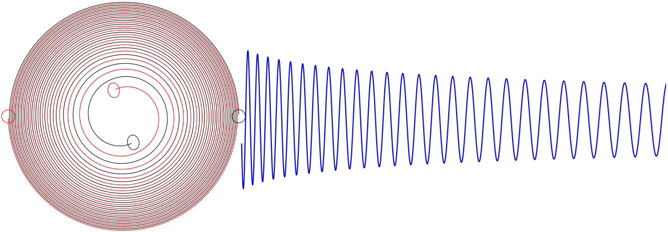
Solving Einstein's Equation Numerically II

Lee Lindblom

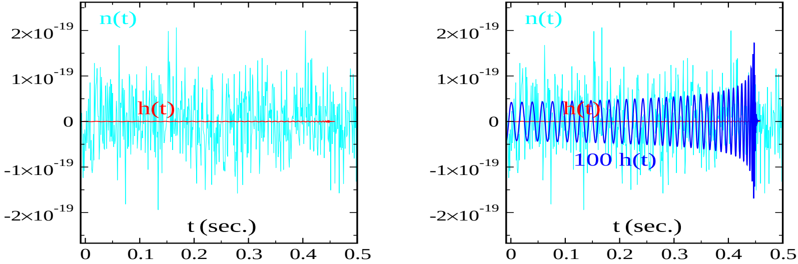
Center for Astrophysics and Space Sciences
University of California at San Diego

Mathematical Sciences Center Lecture Series
Tsinghua University – 14 November 2014



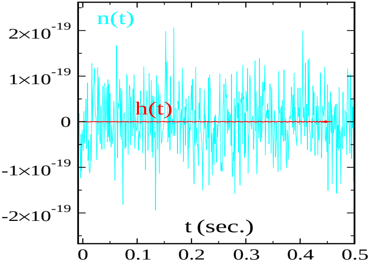
Gravitational Wave Data Analysis

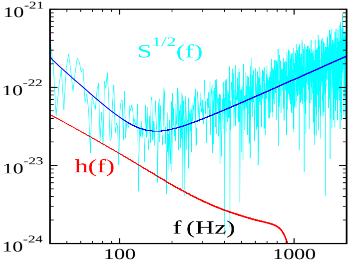
- Gravitational wave signals are very weak.
- Current generation of detectors are fairly noisy (compared to the expected strengths of the signals.)
- Weakest detectable signal has signal-to-noise ratio $\rho \approx 8$.
- Figures illustrate a $\rho = 8$ signal from a binary black hole merger, compared to Initial LIGO noise.
- High quality gravitational waveforms are needed to allow these signals to be “seen” at all.



Gravitational Wave Data Analysis

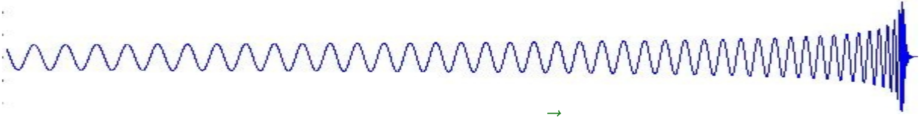
- Gravitational wave signals are very weak.
- Current generation of detectors are fairly noisy (compared to the expected strengths of the signals.)
- Weakest detectable signal has signal-to-noise ratio $\rho \approx 8$.
- Figures illustrate a $\rho = 8$ signal from a binary black hole merger, compared to Initial LIGO noise.
- High quality gravitational waveforms are needed to allow these signals to be "seen" at all.





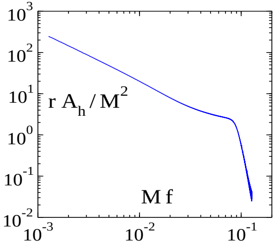
Basic GW Data Analysis:

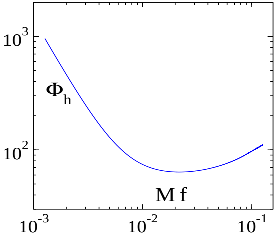
- Data analysis identifies and then measures the properties of signals in GW data by matching to model waveforms.



- Think of a waveform $h(t)$ as a vector, $\vec{h}$, whose components are the amplitudes of the waveform at each time, or equivalently at each frequency:

$$h(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt \equiv A_h(f) e^{i\Phi_h(f)}$$





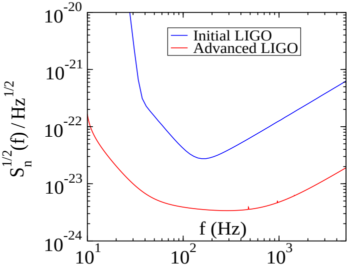
Basic GW Data Analysis II:

- Let $\vec{h}_e = h_e(f)$ denote the exact waveform for some source, and let $\vec{h}_m = h_m(f)$ denote a model of this waveform.
- Define a waveform inner product that weights frequency components in proportion to the detector's sensitivity:

$$\vec{h}_e \cdot \vec{h}_m = \langle h_e | h_m \rangle = \int_{-\infty}^{\infty} \frac{h_e^*(f) h_m(f) + h_e(f) h_m^*(f)}{S_n(f)} df,$$

where $S_n(f)$ is the power spectral density of the detector noise.

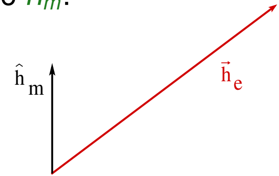
- This inner product is normalized so that $\rho = \sqrt{\langle h_e | h_e \rangle}$ is the optimal signal-to-noise ratio for detecting the waveform $\vec{h}_e$.



Basic GW Data Analysis III:

- Search for signals by projecting data onto model waveforms: ρ_m is the signal-to-noise ratio for $\vec{h}_e$ projected onto $\vec{h}_m$:

$$\rho_m \equiv \vec{h}_e \cdot \hat{h}_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}}.$$

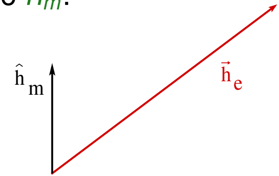


normalized so that $\langle \hat{h}_m | \hat{h}_m \rangle = 1$.

Basic GW Data Analysis III:

- Search for signals by projecting data onto model waveforms: ρ_m is the signal-to-noise ratio for $\vec{h}_e$ projected onto $\vec{h}_m$:

$$\rho_m \equiv \vec{h}_e \cdot \hat{h}_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}}.$$



normalized so that $\langle \hat{h}_m | \hat{h}_m \rangle = 1$.

- A detection is made when $\vec{h}_e$ has a projected signal-to-noise ratio ρ_m that exceeds a predetermined threshold.
- Measured signal-to-noise ratio, ρ_m, is largest when the model waveform $\vec{h}_m$ is proportional to the exact $\vec{h}_e$;
in this case ρ_m equals the optimal signal-to-noise ratio ρ:

$$\rho_m = \frac{\langle h_e | h_e \rangle}{\sqrt{\langle h_e | h_e \rangle}} = \sqrt{\langle h_e | h_e \rangle} = \rho = \sqrt{\int_{-\infty}^{\infty} \frac{2|h_e(f)|^2}{S_n(f)} df}.$$

Accuracy Standards for Detection

- The measured signal-to-noise ratio ρ_m for detecting the signal h_e is the projection of h_e onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

- Errors in model waveform, $h_m = h_e + \delta h$, result in reduction of ρ_m compared to the optimal signal-to-noise ratio ρ:

$$\rho_m = \rho\,(1 - \epsilon).$$

Accuracy Standards for Detection

- The measured signal-to-noise ratio ρ_m for detecting the signal h_e is the projection of h_e onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

- Errors in model waveform, $h_m = h_e + \delta h$, result in reduction of ρ_m compared to the optimal signal-to-noise ratio ρ:

$$\rho_m = \rho\,(1 - \epsilon).$$

- Evaluate this mismatch ϵ in terms of the waveform error:

$$\epsilon = \frac{\langle \delta h_\perp | \delta h_\perp \rangle}{2\langle h_m | h_m \rangle}, \quad \text{where} \quad \delta h_\perp = \delta h - \hat{h}_m \langle \hat{h}_m | \delta h \rangle.$$

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1 - \epsilon)$.

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1-\epsilon)$.
- The rate of detections is proportional to the volume of space where sources can be seen. So when model waveform errors exist, the rate of detections is reduced by the amount:

$$\frac{R^3 - R^3(1-\epsilon)^3}{R^3} = 1 - (1-\epsilon)^3 \approx 3\epsilon$$

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1-\epsilon)$.
- The rate of detections is proportional to the volume of space where sources can be seen. So when model waveform errors exist, the rate of detections is reduced by the amount:

$$\frac{R^3 - R^3(1-\epsilon)^3}{R^3} = 1 - (1-\epsilon)^3 \approx 3\epsilon$$

- The loss of detections can be limited to an acceptable level, by limiting the mismatch ϵ to an acceptable range: $\epsilon < \epsilon_{\max}$.

Accuracy Standards for Detection II

- If the maximum range for detecting a signal using an exact model waveform is R, then the effective range for detections using an inexact model waveform will be $R(1-\epsilon)$.
- The rate of detections is proportional to the volume of space where sources can be seen. So when model waveform errors exist, the rate of detections is reduced by the amount:

$$\frac{R^3 - R^3(1-\epsilon)^3}{R^3} = 1 - (1-\epsilon)^3 \approx 3\epsilon$$

- The loss of detections can be limited to an acceptable level, by limiting the mismatch ϵ to an acceptable range: $\epsilon < \epsilon_{\max}$.
- Consequently model waveform accuracy must satisfy the requirement for detection: $\langle \delta h_\perp | \delta h_\perp \rangle < 2\epsilon_{\max}\rho^2$.

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
- Consider the one-parameter family of waveforms:

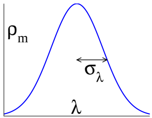
$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
- Consider the one-parameter family of waveforms:

$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

- The variance for measuring the parameter λ is given by



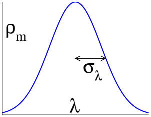
$$\frac{1}{\sigma_\lambda^2} = \left\langle \frac{\partial h}{\partial \lambda} \middle| \frac{\partial h}{\partial \lambda} \right\rangle = \langle \delta h | \delta h \rangle.$$

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
- Consider the one-parameter family of waveforms:

$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

- The variance for measuring the parameter λ is given by



$$\frac{1}{\sigma_\lambda^2} = \left\langle \frac{\partial h}{\partial \lambda} \middle| \frac{\partial h}{\partial \lambda} \right\rangle = \langle \delta h | \delta h \rangle.$$

- If the parameter distance between the two waveforms, $(\Delta\lambda)^2$, is smaller than the variance σ_λ^2 for measuring that parameter, then the waveforms are indistinguishable.
- So h_m is indistinguishable from h_e if $1 = \Delta\lambda^2 < \sigma_\lambda^2 = 1/\langle \delta h | \delta h \rangle$, i.e., if $1 > \langle \delta h | \delta h \rangle$.

Accuracy Requirements for Advanced LIGO

- It is useful to define amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors: $\delta h_m = h_e e^{\delta\chi_m + i\delta\Phi_m} - h_e \approx h_e(\delta\chi_m + i\delta\Phi_m)$.

Accuracy Requirements for Advanced LIGO

- It is useful to define amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors: $\delta h_m = h_e e^{\delta\chi_m + i\delta\Phi_m} - h_e \approx h_e(\delta\chi_m + i\delta\Phi_m)$.
- The basic accuracy requirements can be written as

$$\sqrt{\overline{\delta\chi_m^2} + \overline{\delta\Phi_m^2}} = \sqrt{\frac{\langle\delta h|\delta h\rangle}{\langle h|h\rangle}} < \begin{cases} \eta_c/\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} & \text{detection,} \end{cases}$$

where the signal-weighted average errors are defined as

$$\overline{\delta\chi_m^2} = \int_{-\infty}^{\infty} \delta\chi_m^2 \frac{2|h|^2}{\rho^2 S_n} df, \quad \text{and} \quad \overline{\delta\Phi_m^2} = \int_{-\infty}^{\infty} \delta\Phi_m^2 \frac{2|h|^2}{\rho^2 S_n} df,$$

and $0 < \eta_c \leq 1$ depends on the instrument calibration error.

Accuracy Requirements for Advanced LIGO

- It is useful to define amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors: $\delta h_m = h_e e^{\delta\chi_m + i\delta\Phi_m} - h_e \approx h_e(\delta\chi_m + i\delta\Phi_m)$.
- The basic accuracy requirements can be written as

$$\sqrt{\overline{\delta\chi_m^2} + \overline{\delta\Phi_m^2}} = \sqrt{\frac{\langle \delta h | \delta h \rangle}{\langle h | h \rangle}} < \left\{ \begin{array}{ll} \eta_c/\rho_{\rm max} & \text{measurement,} \\ \sqrt{2\epsilon_{\rm max}} & \text{detection,} \end{array} \right.$$

where the signal-weighted average errors are defined as

$$\overline{\delta\chi_m^2} = \int_{-\infty}^{\infty} \delta\chi_m^2 \frac{2|h|^2}{\rho^2 S_n} df, \quad \text{and} \quad \overline{\delta\Phi_m^2} = \int_{-\infty}^{\infty} \delta\Phi_m^2 \frac{2|h|^2}{\rho^2 S_n} df,$$

and $0 < \eta_c \leq 1$ depends on the instrument calibration error.

- For Advanced LIGO, $\rho_{\rm max}$ could be as large as $\rho_{\rm max} \approx 100$, and calibration accuracy will (optimally) be comparable to model waveform accuracy, making $\eta_c \approx 1/2$, so

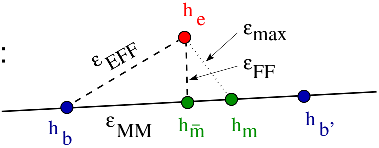
$$\sqrt{\overline{\delta\chi_m^2} + \overline{\delta\Phi_m^2}} < \frac{\eta_c}{\rho_{\rm max}} \approx 0.005 \ \text{ for measurement.}$$

Detection Accuracy Requirements for LIGO

- Accuracy requirement for detection depends on the parameter $\epsilon_{\rm max}$, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{\rm max} = 0.035$ limits the loss rate to about 10%.

Detection Accuracy Requirements for LIGO

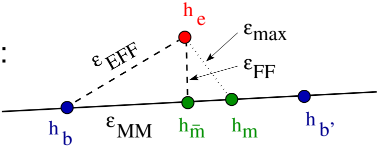
- Accuracy requirement for detection depends on the parameter ϵ_{max}, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{\mathsf{max}} = 0.035$ limits the loss rate to about 10%.
- Real searches are more complicated: comparing signals with a discrete template bank of model waveforms.



- For Initial LIGO, template banks are constructed with $\epsilon_{\mathrm{MM}} = 0.03$, so $\epsilon_{\mathrm{FF}} = \epsilon_{\mathrm{EFF}} - \epsilon_{\mathrm{MM}} = 0.035 - 0.03 = 0.005$.
- To ensure this condition, ϵ_{max} must be chosen so that $\epsilon_{\mathsf{max}} \le 0.005$.

Detection Accuracy Requirements for LIGO

- Accuracy requirement for detection depends on the parameter ϵ_{max}, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{\text{max}} = 0.035$ limits the loss rate to about 10%.
- Real searches are more complicated: comparing signals with a discrete template bank of model waveforms.



- For Initial LIGO, template banks are constructed with $\epsilon_{\text{MM}} = 0.03$, so $\epsilon_{\text{FF}} = \epsilon_{\text{EFF}} - \epsilon_{\text{MM}} = 0.035 - 0.03 = 0.005$.
- To ensure this condition, ϵ_{max} must be chosen so that $\epsilon_{\text{max}} \leq 0.005$.
- Accuracy requirement for BBH waveforms for detection in LIGO:

$$\sqrt{\overline{\delta\chi_m^2} + \overline{\delta\Phi_m^2}} \lesssim \sqrt{2\epsilon_{\text{max}}} = 0.1 \quad \text{for detection.}$$

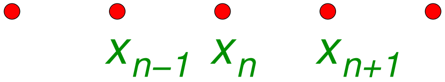
Overview

- Spacetimes describing interesting souces of gravitational waves.
  - Binary black hole problem.
  - Gravitational waveform accuracy requirements for GW astronomy.
  - How and why to solve PDEs with spectral methods.
  - Einstein's equations: hyperbolicity, constraints, gauge conditions, boundary conditions.

Numerical Solution of Evolution Equations

$$\partial_t u = Q(u, \partial_x u, x, t).$$

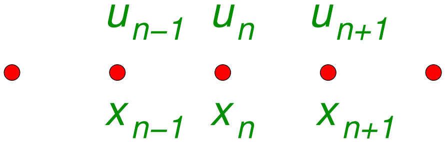
- Choose a grid of spatial points, x_n.



Numerical Solution of Evolution Equations

$$\partial_t u = Q(u, \partial_x u, x, t).$$

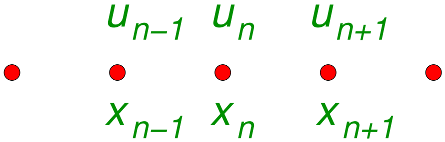
- Choose a grid of spatial points, x_n.
- Evaluate the function u on this grid: $u_n(t) = u(x_n, t)$.



Numerical Solution of Evolution Equations

$$\partial_t u = Q(u, \partial_x u, x, t).$$

- Choose a grid of spatial points, x_n.
- Evaluate the function u on this grid: $u_n(t) = u(x_n, t)$.

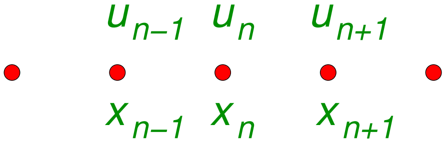


- Approximate the spatial derivatives at the grid points
$\partial_x u(x_n) = \sum_k D_{n\,k} u_k$.

Numerical Solution of Evolution Equations

$$\partial_t u = Q(u, \partial_x u, x, t).$$

- Choose a grid of spatial points, x_n.
- Evaluate the function u on this grid: $u_n(t) = u(x_n, t)$.

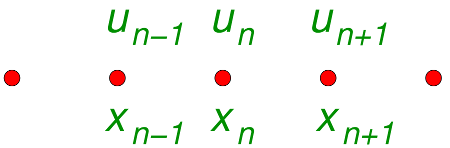


- Approximate the spatial derivatives at the grid points $\partial_x u(x_n) = \sum_k D_{n\,k} u_k$.
- Evaluate Q at the grid points x_n in terms of the u_k: $\bar{Q}(u_k, x_n, t)$.

Numerical Solution of Evolution Equations

$$\partial_t u = Q(u, \partial_x u, x, t).$$

- Choose a grid of spatial points, x_n.
- Evaluate the function u on this grid: $u_n(t) = u(x_n, t)$.



- Approximate the spatial derivatives at the grid points
  $\partial_x u(x_n) = \sum_k D_{n\,k} u_k$.
- Evaluate Q at the grid points x_n in terms of the u_k: $\bar{Q}(u_k, x_n, t)$.
- Solve the coupled system of ordinary differential equations,

$$\frac{du_n(t)}{dt} = \bar{Q}[u_k(t), x_n, t],$$

using standard numerical methods (e.g. Runge-Kutta).

Basic Numerical Methods

- Different numerical methods use different ways of choosing the grid points x_n, and different expressions for the spatial derivatives $\partial_x u(x_n) = \sum_k D_{n\,k} u_k$.

Basic Numerical Methods

- Different numerical methods use different ways of choosing the grid points x_n, and different expressions for the spatial derivatives
$$\partial_x u(x_n) = \sum_k D_{n\,k} u_k.$$
- Most numerical groups use finite difference methods:
  - Uniformly spaced grids: $x_n - x_{n-1} = \Delta x =$ constant.
  - Use Taylor expansions to obtain approximate expressions for the derivatives, e.g.,
$$\partial_x u(x_n) = \frac{u_{n+1} - u_{n-1}}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

Basic Numerical Methods

- Different numerical methods use different ways of choosing the grid points x_n, and different expressions for the spatial derivatives
$$\partial_x u(x_n) = \sum_k D_{n\,k} u_k.$$
- Most numerical groups use finite difference methods:
  - Uniformly spaced grids: $x_n - x_{n-1} = \Delta x =$ constant.
  - Use Taylor expansions to obtain approximate expressions for the derivatives, e.g.,
$$\partial_x u(x_n) = \frac{u_{n+1} - u_{n-1}}{2\Delta x} + \mathcal{O}(\Delta x^2).$$
- Grid spacing decreases as the number of grid points N increases, $\Delta x \sim 1/N$. Errors in finite difference methods scale as N^{-p}.
- Most groups now use finite difference codes with $p = 6$ or $p = 8$.

Basic Numerical Methods II

- A few groups (Caltech/Cornell, Meudon) use spectral methods.

Basic Numerical Methods II

- A few groups (Caltech/Cornell, Meudon) use spectral methods.
- Represent functions as finite sums: $u(x,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) e^{ikx}$.
- Choose grid points x_n to allow efficient (and exact) inversion of the series: $\tilde{u}_k(t) = \sum_{n=0}^{N-1} w_n \, u(x_n, t) e^{-ikx_n}$.

Basic Numerical Methods II

- A few groups (Caltech/Cornell, Meudon) use spectral methods.
- Represent functions as finite sums: $u(x,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) e^{ikx}$.
- Choose grid points x_n to allow efficient (and exact) inversion of the series: $\tilde{u}_k(t) = \sum_{n=0}^{N-1} w_n \, u(x_n,t) e^{-ikx_n}$.
- Obtain derivative formulas by differentiating the series: $\partial_x u(x_n,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) \partial_x e^{ikx_n} = \sum_{m=0}^{N-1} D_{n\,m} \, u(x_m,t)$.

Basic Numerical Methods II

- A few groups (Caltech/Cornell, Meudon) use spectral methods.
- Represent functions as finite sums: $u(x,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) e^{ikx}$.
- Choose grid points x_n to allow efficient (and exact) inversion of the series: $\tilde{u}_k(t) = \sum_{n=0}^{N-1} w_n\, u(x_n,t) e^{-ikx_n}$.
- Obtain derivative formulas by differentiating the series: $\partial_x u(x_n,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) \partial_x e^{ikx_n} = \sum_{m=0}^{N-1} D_{n\,m}\, u(x_m,t)$.
- Errors in spectral methods are dominated by the size of $\tilde{u}_N$.
- Estimate the errors (e.g. for Fourier series of *smooth* functions):

$$\tilde{u}_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x) e^{-iNx} dx \le \frac{1}{N^p} \max \left| \frac{d^p u(x)}{dx^p} \right| .$$

Basic Numerical Methods II

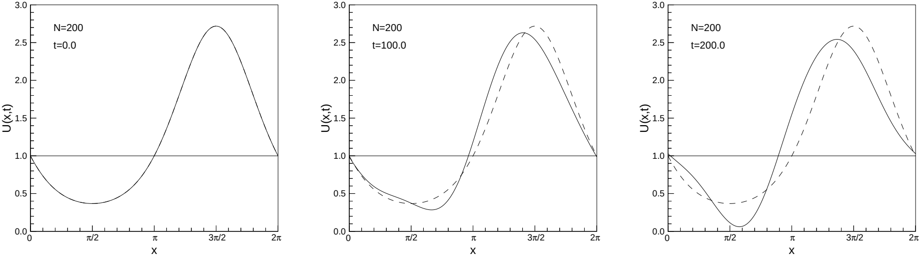
- A few groups (Caltech/Cornell, Meudon) use spectral methods.
- Represent functions as finite sums: $u(x,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) e^{ikx}$.
- Choose grid points x_n to allow efficient (and exact) inversion of the series: $\tilde{u}_k(t) = \sum_{n=0}^{N-1} w_n\, u(x_n, t) e^{-ikx_n}$.
- Obtain derivative formulas by differentiating the series: $\partial_x u(x_n, t) = \sum_{k=0}^{N-1} \tilde{u}_k(t) \partial_x e^{ikx_n} = \sum_{m=0}^{N-1} D_{n\,m}\, u(x_m, t)$.
- Errors in spectral methods are dominated by the size of $\tilde{u}_N$.
- Estimate the errors (e.g. for Fourier series of *smooth* functions):

$$\tilde{u}_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x) e^{-iNx} dx \leq \frac{1}{N^p} \max \left| \frac{d^p u(x)}{dx^p} \right| .$$

- Errors in spectral methods decrease faster than any power N^p.
- This means that a given level of accuracy can be achieved using many fewer grid points with spectral methods.

Comparing Different Numerical Methods

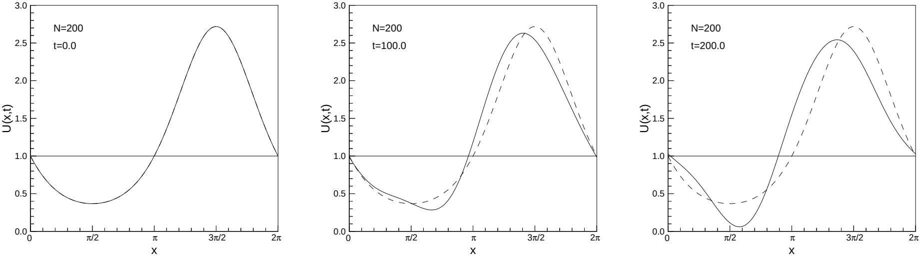
- Wave propagation with second-order finite difference method:



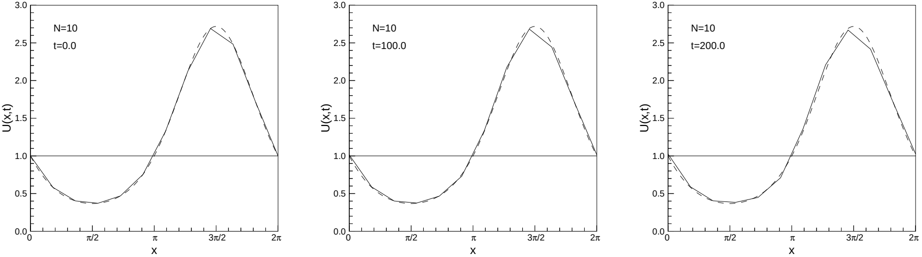
Figures from Hesthaven, Gottlieb, & Gottlieb (2007).

Comparing Different Numerical Methods

- Wave propagation with second-order finite difference method:



- Wave propagation with spectral method:



Figures from Hesthaven, Gottlieb, & Gottlieb (2007).

Overview

- Spacetimes describing interesting souces of gravitational waves.
  - Binary black hole problem.
  - Gravitational waveform accuracy requirements for GW astronomy.
  - How and why to solve PDEs with spectral methods.
  - Einstein's equations: hyperbolicity, constraints, gauge conditions, boundary conditions.

General Relativity Theory

- Einstein's theory of gravitation, general relativity theory, is a geometrical theory in which gravitational effects are described as geometrical structures on spacetime.
- The fundamental "gravitational" field is the spacetime metric ψ_{ab}, a symmetric ($\psi_{ab} = \psi_{ba}$) non-degenerate ($\psi_{ab} v^b = 0 \;\Rightarrow\; v^a = 0$) tensor field.
- The tensor ψ^{ab} is the inverse metric, i.e. $\psi^{ac}\psi_{cb} = \delta^a{}_b$.
- The metric and inverse metric are used to define the dual transformations between vector and co-vector fields, e.g. $v_a = \psi_{ab} v^b$ and $w^a = \psi^{ab} w_b$.

General Relativity Theory II

- The spacetime metric ψ_{ab} is determined by Einstein's equation:
  $$R_{ab} - \tfrac{1}{2} R \psi_{ab} = 8\pi T_{ab},$$
  where R_{ab} is the Ricci curvature tensor associated with ψ_{ab}, $R = \psi^{ab} R_{ab}$ is the scalar curvature, and T_{ab} is the stress-energy tensor of the matter present in spacetime.

General Relativity Theory II

- The spacetime metric ψ_{ab} is determined by Einstein's equation:
  $$R_{ab} - \tfrac{1}{2}R\psi_{ab} = 8\pi T_{ab},$$
  where R_{ab} is the Ricci curvature tensor associated with ψ_{ab}, $R = \psi^{ab}R_{ab}$ is the scalar curvature, and T_{ab} is the stress-energy tensor of the matter present in spacetime.
- For "vacuum" spacetimes (like binary black hole systems) $T_{ab} = 0$, so Einstein's equations can be reduced to $R_{ab} = 0$.
- For spacetimes containing matter (like neutron-star binary systems) a suitable matter model must be used, e.g. the perfect fluid approximation $T_{ab} = (\epsilon + p)u_a u_b + p\psi_{ab}$.

General Relativity Theory II

- The spacetime metric ψ_{ab} is determined by Einstein's equation:
  $$R_{ab} - \tfrac{1}{2}R\psi_{ab} = 8\pi T_{ab},$$
  where R_{ab} is the Ricci curvature tensor associated with ψ_{ab}, $R = \psi^{ab}R_{ab}$ is the scalar curvature, and T_{ab} is the stress-energy tensor of the matter present in spacetime.
- For "vacuum" spacetimes (like binary black hole systems) $T_{ab} = 0$, so Einstein's equations can be reduced to $R_{ab} = 0$.
- For spacetimes containing matter (like neutron-star binary systems) a suitable matter model must be used, e.g. the perfect fluid approximation $T_{ab} = (\epsilon + p)u_a u_b + p\psi_{ab}$.
- The Ricci curvature R_{ab} is determined by derivatives of the metric:
  $$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{bc} + \Gamma^c{}_{cd}\Gamma^d{}_{ab} - \Gamma^c{}_{ad}\Gamma^d{}_{bc},$$
  where $\Gamma^c{}_{ab} = \tfrac{1}{2}\psi^{cd}(\partial_a\psi_{db} + \partial_b\psi_{da} - \partial_d\psi_{ab})$.
- Einstein's equations are second-order pde's that (should, hopefully) determine the spacetime metric, e.g. in vacuum
  $$R_{ab}(\partial\partial\psi, \partial\psi, \psi) = 0.$$

General Relativity Theory III

- Einstein's equations are second-order PDEs that (should, hopefully) determine the spacetime metric, e.g. in vacuum

$$R_{ab}(\partial\partial\psi, \partial\psi, \psi) = 0.$$

General Relativity Theory III

- Einstein's equations are second-order PDEs that (should, hopefully) determine the spacetime metric, e.g. in vacuum

$$R_{ab}(\partial\partial\psi, \partial\psi, \psi) = 0.$$

- What are the properties of these PDEs?
- How do we go about solving them?
- What are the appropriate boundary and/or initial data needed to determine a unique solution to these equations?

General Relativity Theory III

- Einstein's equations are second-order PDEs that (should, hopefully) determine the spacetime metric, e.g. in vacuum

$$R_{ab}(\partial\partial\psi, \partial\psi, \psi) = 0.$$

- What are the properties of these PDEs?
- How do we go about solving them?
- What are the appropriate boundary and/or initial data needed to determine a unique solution to these equations?
- The important fundamental ideas needed to understand these questions are:
  - gauge freedom,
  - constraints.

General Relativity Theory III

- Einstein's equations are second-order PDEs that (should, hopefully) determine the spacetime metric, e.g. in vacuum

$$R_{ab}(\partial\partial\psi, \partial\psi, \psi) = 0.$$

- What are the properties of these PDEs?
- How do we go about solving them?
- What are the appropriate boundary and/or initial data needed to determine a unique solution to these equations?
- The important fundamental ideas needed to understand these questions are:
  - gauge freedom,
  - constraints.
- Maxwell's equations are a simpler system in which these same fundamental issues play analogous roles.

Gauge and Hyperbolicity in Electromagnetism

- The usual representation of the vacuum Maxwell equations split into evolution equations and constraints:

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0,$$
$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0.$$

These equations are often written in the more compact 4-dimensional form $\nabla^a F_{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$, where F_{ab} has components $\vec{E}$ and $\vec{B}$.

Gauge and Hyperbolicity in Electromagnetism

- The usual representation of the vacuum Maxwell equations split into evolution equations and constraints:

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0,$$
$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0.$$

  These equations are often written in the more compact 4-dimensional form $\nabla^a F_{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$, where F_{ab} has components $\vec{E}$ and $\vec{B}$.
- Maxwell's equations can be solved in part by introducing a vector potential $F_{ab} = \nabla_a A_b - \nabla_b A_a$. This reduces the system to the single equation: $\nabla^a \nabla_a A_b - \nabla_b \nabla^a A_a = 0.$

Gauge and Hyperbolicity in Electromagnetism

- The usual representation of the vacuum Maxwell equations split into evolution equations and constraints:

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0,$$
$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \qquad \nabla \cdot \vec{B} = 0.$$

  These equations are often written in the more compact 4-dimensional form $\nabla^a F_{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$, where F_{ab} has components $\vec{E}$ and $\vec{B}$.
- Maxwell's equations can be solved in part by introducing a vector potential $F_{ab} = \nabla_a A_b - \nabla_b A_a$. This reduces the system to the single equation: $\nabla^a \nabla_a A_b - \nabla_b \nabla^a A_a = 0.$
- This form of the equations can be made manifestly hyperbolic by choosing the gauge correctly, e.g., let $\nabla^a A_a = H(x, t, A)$, giving:

$$\nabla^a \nabla_a A_b = \left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right) A_b = \nabla_b H.$$

Gauge and Hyperbolicity in General Relativity

- The spacetime Ricci curvature tensor can be written as:

$$R_{ab} \;=\; -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi, \partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi_{ad}\psi^{bc}\Gamma^d{}_{bc}$.

- Like Maxwell's equations, these equation can not be solved without specifying suitable gauge conditions.

Gauge and Hyperbolicity in General Relativity

- The spacetime Ricci curvature tensor can be written as:

$$R_{ab} = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi, \partial\psi),$$

where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi_{ad}\psi^{bc}\Gamma^d{}_{bc}$.

- Like Maxwell's equations, these equation can not be solved without specifying suitable gauge conditions.
- The gauge freedom in general relativity theory is the freedom to represent the equations using any coordinates x^a on spacetime.
- Solving the equations requires some specific choice of coordinates be made. Gauge conditions are used to impose the desired choice.

Gauge and Hyperbolicity in General Relativity

- The spacetime Ricci curvature tensor can be written as:

$$R_{ab} = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi,\partial\psi),$$

  where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi_{ad}\psi^{bc}\Gamma^d{}_{bc}$.

- Like Maxwell's equations, these equation can not be solved without specifying suitable gauge conditions.
- The gauge freedom in general relativity theory is the freedom to represent the equations using any coordinates x^a on spacetime.
- Solving the equations requires some specific choice of coordinates be made. Gauge conditions are used to impose the desired choice.
- One way to impose the needed gauge conditions is to specify H^a, the source term for a wave equation for each coordinate x^a:

$$H^a = \nabla^c\nabla_c x^a = \psi^{bc}(\partial_b\partial_c x^a - \Gamma^e{}_{bc}\partial_e x^a) = -\Gamma^a,$$

  where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

Gauge Conditions in General Relativity

- Specifying coordinates by the *generalized harmonic* (GH) method is accomplished by choosing a gauge-source function $H^a(x,\psi)$, e.g. $H^a = \psi^{ab}H_b(x)$, and requiring that

$$H^a(x,\psi) = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b\psi_{dc} + \partial_c\psi_{db} - \partial_d\psi_{bc}).$$

Gauge Conditions in General Relativity

- Specifying coordinates by the *generalized harmonic* (GH) method is accomplished by choosing a gauge-source function $H^a(x,\psi)$, e.g. $H^a = \psi^{ab}H_b(x)$, and requiring that

$$H^a(x,\psi) = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b\psi_{dc} + \partial_c\psi_{db} - \partial_d\psi_{bc}).$$

- Recall that the spacetime Ricci tensor is given by

$$R_{ab} \;=\; -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi,\partial\psi).$$

- The Generalized Harmonic Einstein equation is obtained by replacing $\Gamma_a = \psi_{ab}\Gamma^b$ with $-H_a(x,\psi) = -\psi_{ab}H^b(x,\psi)$:

$$R_{ab} - \nabla_{(a}\left[\Gamma_{b)} + H_{b)}\right] = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(\psi,\partial\psi).$$

Gauge Conditions in General Relativity

- Specifying coordinates by the *generalized harmonic* (GH) method is accomplished by choosing a gauge-source function $H^a(x,\psi)$, e.g. $H^a = \psi^{ab}H_b(x)$, and requiring that
$$H^a(x,\psi) = -\Gamma^a = -\tfrac{1}{2}\psi^{ad}\psi^{bc}(\partial_b\psi_{dc} + \partial_c\psi_{db} - \partial_d\psi_{bc}).$$
- Recall that the spacetime Ricci tensor is given by
$$R_{ab} \;=\; -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + Q_{ab}(\psi,\partial\psi).$$
- The Generalized Harmonic Einstein equation is obtained by replacing $\Gamma_a = \psi_{ab}\Gamma^b$ with $-H_a(x,\psi) = -\psi_{ab}H^b(x,\psi)$:
$$R_{ab} - \nabla_{(a}\left[\Gamma_{b)} + H_{b)}\right] = -\tfrac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(\psi,\partial\psi).$$
- The vacuum GH Einstein equation, $R_{ab} = 0$ with $\Gamma_a + H_a = 0$, is therefore manifestly hyperbolic, having the same principal part as the scalar wave equation:
$$0 = \nabla_a\nabla^a\Phi = \psi^{ab}\partial_a\partial_b\Phi + Q(\partial\Phi).$$