Solving Einstein's Equation Numerically I

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Center for Astrophysics and Space Sciences
University of California at San Diego

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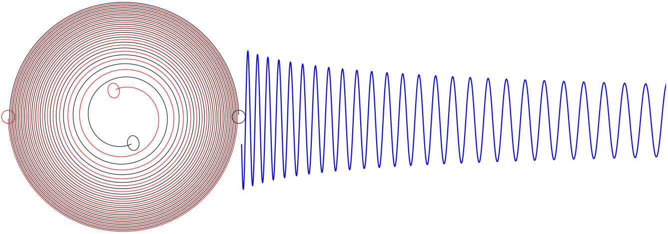

Overview

Lectures will discuss two broad classes of problems associated with solving Einstein's equation numerically:

- Spacetimes describing interesting souces of gravitational waves.
  - Binary black hole problem.
  - Gravitational waveform accuracy requirements for GW astronomy.
  - How to solve PDEs numerically.
  - Einstein's equations: hyperbolicity, constraints, gauge conditions, boundary conditions.
  - Feedback control for grid tracking.

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  - How to solve PDEs numerically.
  - Einstein's equations: hyperbolicity, constraints, gauge conditions, boundary conditions.
  - Feedback control for grid tracking.
- Spacetimes with interesting topological structures.
  - Multicube representations of manifolds.
  - Fixing the global differentiable structure.
  - Interface boundary conditions.
  - Examples of elliptic and hyperbolic numerical solutions.
  - Examples of Einstein evolutions on $R \times S^3$.
  - Constructing reference metrics on arbitrary multicube manifolds.

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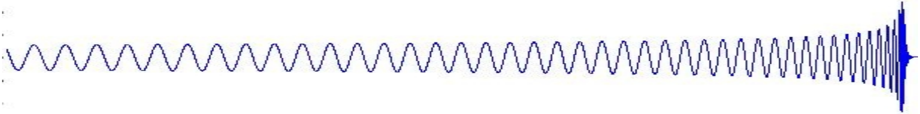

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- Strongest waves (and perhaps the most easily detectable waves) are emitted as the two black holes merge into a single hole.
- Full non-linear numerical relativity is needed to construct accurate model waveforms for these spacetimes.

Why Is Numerical Relativity So Difficult?

- Very big computational problem:
  - Must evolve ~ 50 dynamical fields (spacetime metric plus all first derivatives).
  - Must accurately resolve features on many scales from black hole horizons $r \sim GM/c^2$ to emitted waves $r \sim 100GM/c^2$.
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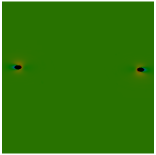
- Many representations of the Einstein equations have mathematically ill-posed initial value problems.
- Constraint violating instabilities destroy stable numerical solutions in many well-posed forms of the equations.



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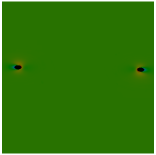

- Dynamics of binary black hole problem is driven by delicate adjustments to orbit due to emission of gravitational waves.

History of Numerical Solution of the BBH Problem:

- First Axisymmetric Head-On — Hahn & Lindquist (1964).
- Better Axisymmetric Head-On — Eppley & Smarr (1975-77).
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- Unequal masses – Goddard + Penn State groups (2006).
- Non-zero spins – Brownsville + AEI (2006-07).
- Post merger recoils (up to $\sim$ 4000 km/s)
  – Jena + AEI + Rochester (2007).
- Large mass ratios (1:10) – Jena (2009).
- Generic spins with precession – Rochester (2009).
- High precision inspiral + merger + ringdown waveforms
  – AEI + Caltech/Cornell (2009).
- Very large mass ratios (1:100) – Rochester (2010).
- Very high spins ($\chi \approx 0.95$) – Caltech/Cornell (2010).

State of the Art Numerical Relativity

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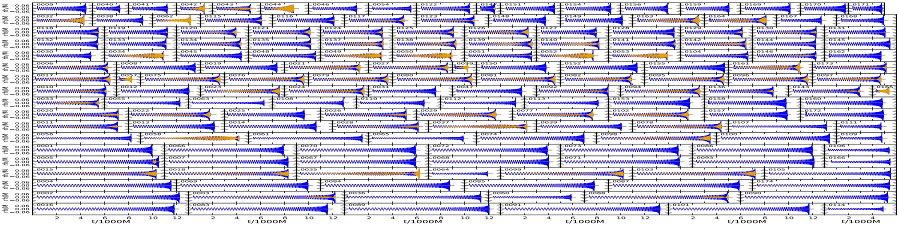

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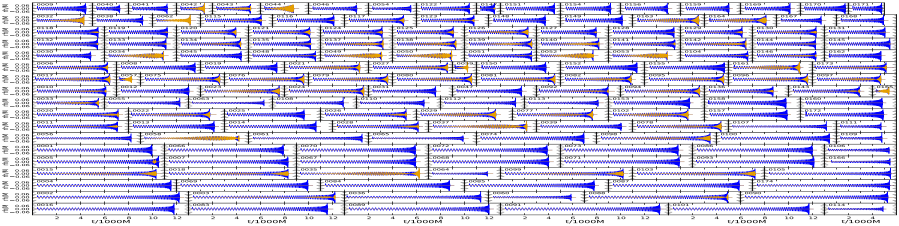
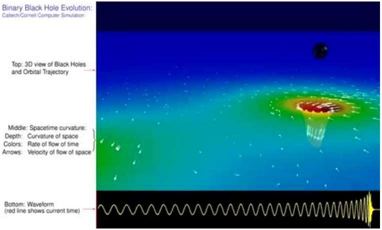


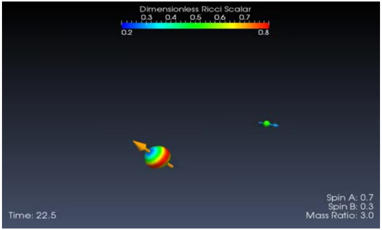
- 175 orbit simulation by Szilagyi and Blackman at Caltech.

Binary Black Hole Evolution Movies

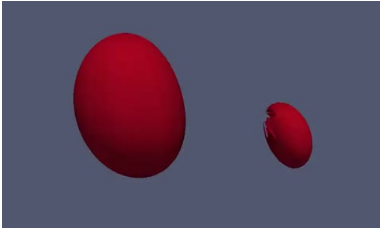
Equal Mass BBH Simulation



Better Equal Mass BBH Simulation



Generic BBH Simulation



Generic Event Horizon Merger

Overview

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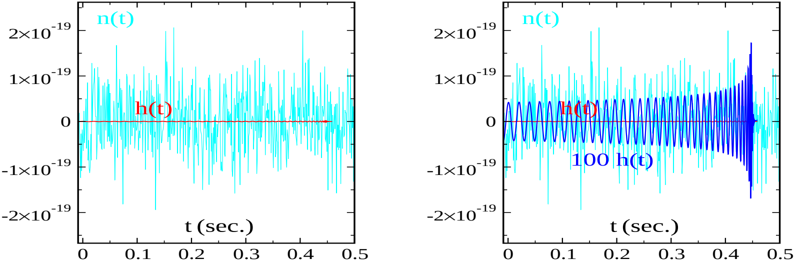
  - How and why to solve PDEs with spectral methods.
  - Einstein's equations: hyperbolicity, constraints, gauge conditions, boundary conditions.

Gravitational Wave Data Analysis

- Gravitational wave signals are very weak.
- Current generation of detectors are fairly noisy (compared to the expected strengths of the signals.)
- Weakest detectable signal has signal-to-noise ratio $\rho \approx 8$.
- Figures illustrate a $\rho = 8$ signal from a binary black hole merger, compared to Initial LIGO noise.
- High quality gravitational waveforms are needed to allow these signals to be "seen" at all.



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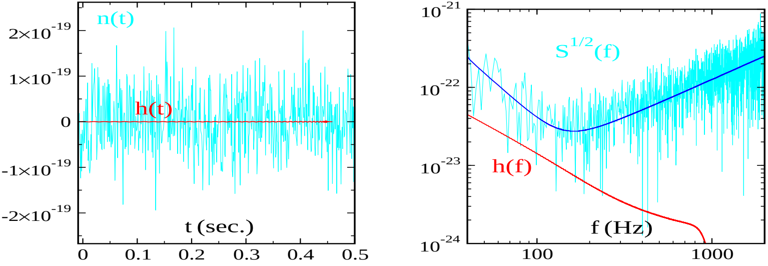
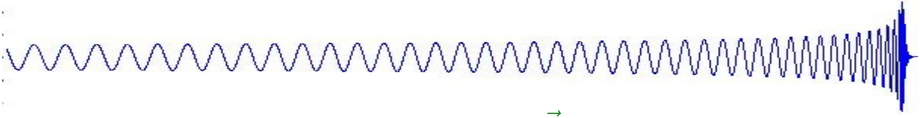


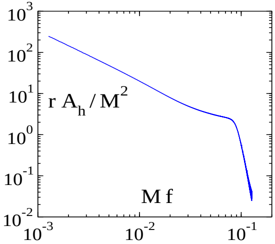
Basic GW Data Analysis:

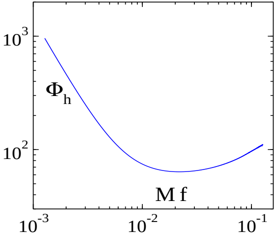
- Data analysis identifies and then measures the properties of signals in GW data by matching to model waveforms.



- Think of a waveform $h(t)$ as a vector, $\vec{h}$, whose components are the amplitudes of the waveform at each time, or equivalently at each frequency:

$$h(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt \equiv A_h(f) e^{i\Phi_h(f)}$$





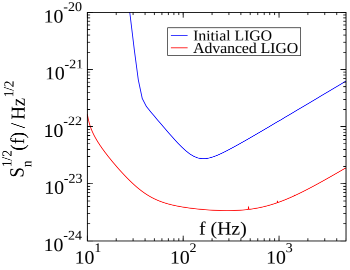
Basic GW Data Analysis II:

- Let $\vec{h}_e = h_e(f)$ denote the exact waveform for some source, and let $\vec{h}_m = h_m(f)$ denote a model of this waveform.
- Define a waveform inner product that weights frequency components in proportion to the detector's sensitivity:

$$\vec{h}_e \cdot \vec{h}_m = \langle h_e | h_m \rangle = \int_{-\infty}^{\infty} \frac{h_e^*(f) h_m(f) + h_e(f) h_m^*(f)}{S_n(f)} df,$$

where $S_n(f)$ is the power spectral density of the detector noise.

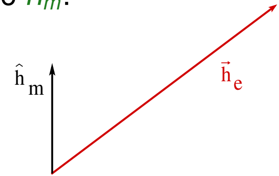
- This inner product is normalized so that $\rho = \sqrt{\langle h_e | h_e \rangle}$ is the optimal signal-to-noise ratio for detecting the waveform $\vec{h}_e$.



Basic GW Data Analysis III:

- Search for signals by projecting data onto model waveforms: ρ_m is the signal-to-noise ratio for $\vec{h}_e$ projected onto $\vec{h}_m$:

$$\rho_m \equiv \vec{h}_e \cdot \hat{h}_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\sqrt{\langle h_m | h_m \rangle}}.$$



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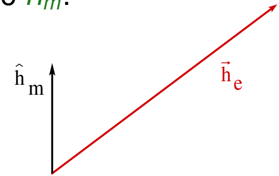

  normalized so that $\langle \hat{h}_m | \hat{h}_m \rangle = 1$.

- A detection is made when $\vec{h}_e$ has a projected signal-to-noise ratio ρ_m that exceeds a predetermined threshold.
- Measured signal-to-noise ratio, ρ_m, is largest when the model waveform $\vec{h}_m$ is proportional to the exact $\vec{h}_e$;
  in this case ρ_m equals the optimal signal-to-noise ratio ρ:

$$\rho_m = \frac{\langle h_e | h_e \rangle}{\sqrt{\langle h_e | h_e \rangle}} = \sqrt{\langle h_e | h_e \rangle} = \rho = \sqrt{\int_{-\infty}^{\infty} \frac{2|h_e(f)|^2}{S_n(f)} df}.$$

Accuracy Standards for Detection

- The measured signal-to-noise ratio ρ_m for detecting the signal h_e is the projection of h_e onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

- Errors in model waveform, $h_m = h_e + \delta h$, result in reduction of ρ_m compared to the optimal signal-to-noise ratio ρ:

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- Evaluate this mismatch ϵ in terms of the waveform error:

$$\epsilon = \frac{\langle \delta h_\perp|\delta h_\perp\rangle}{2\langle h_m|h_m\rangle}, \quad \text{where} \quad \delta h_\perp = \delta h - \hat{h}_m\langle \hat{h}_m|\delta h\rangle.$$

Accuracy Standards for Detection II

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- Consequently model waveform accuracy must satisfy the requirement for detection: $\langle \delta h_\perp | \delta h_\perp \rangle < 2\epsilon_{\rm max}\rho^2$.

Accuracy Standards for Measurement

- How close must two waveforms, $h_e(f)$ and $h_m(f)$, be to each other so that observations are unable to distinguish them?
- Consider the one-parameter family of waveforms:

$$h(\lambda, f) = h_e(f) + \lambda[h_m(f) - h_e(f)] = h_e(f) + \lambda\delta h(f)$$

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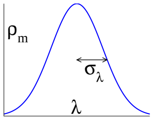

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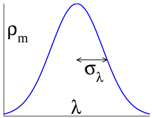

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- If the parameter distance between the two waveforms, $(\Delta\lambda)^2 = 1$, is smaller than the variance σ_λ^2 for measuring that parameter, then the waveforms are indistinguishable.
- So h_m is indistinguishable from h_e if $1 < \sigma_\lambda^2 = 1/\langle \delta h | \delta h \rangle$, i.e., if $1 > \langle \delta h | \delta h \rangle$.

Accuracy Requirements for Advanced LIGO

- It is useful to define amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors: $\delta h_m = h_e e^{\delta\chi_m + i\delta\Phi_m} - h_e \approx h_e(\delta\chi_m + i\delta\Phi_m)$.

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- The basic accuracy requirements can be written as

$$\sqrt{\overline{\delta\chi_m^2} + \overline{\delta\Phi_m^2}} = \sqrt{\frac{\langle\delta h|\delta h\rangle}{\langle h|h\rangle}} < \left\{ \begin{array}{ll} \eta_c/\rho_{\max} & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} & \text{detection,} \end{array} \right.$$

where the signal-weighted average errors are defined as

$$\overline{\delta\chi_m^2} = \int_{-\infty}^{\infty} \delta\chi_m^2 \frac{2|h|^2}{\rho^2 S_n} df, \quad \text{and} \quad \overline{\delta\Phi_m^2} = \int_{-\infty}^{\infty} \delta\Phi_m^2 \frac{2|h|^2}{\rho^2 S_n} df,$$

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and $0 < \eta_c \leq 1$ depends on the instrument calibration error.
- For Advanced LIGO, $\rho_{\rm max}$ could be as large as $\rho_{\rm max} \approx 100$, and calibration accuracy will (optimally) be comparable to model waveform accuracy, making $\eta_c \approx 1/2$, so

$$\sqrt{\overline{\delta\chi_m^2} + \overline{\delta\Phi_m^2}} < \frac{\eta_c}{\rho_{\rm max}} \approx 0.005 \ \text{ for measurement.}$$

Detection Accuracy Requirements for LIGO

- Accuracy requirement for detection depends on the parameter ϵ_{max}, the maximum allowed mismatch between an exact waveform and its model counterpart.
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{\mathrm{max}} = 0.035$ limits the loss rate to about 10%.

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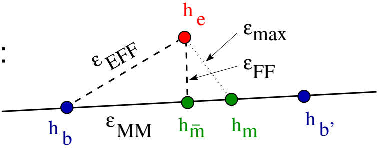
- Real searches are more complicated: comparing signals with a discrete template bank of model waveforms.



- For Initial LIGO, template banks are constructed with $\epsilon_{\text{MM}} = 0.03$, so $\epsilon_{\text{FF}} = \epsilon_{\text{EFF}} - \epsilon_{\text{MM}} = 0.035 - 0.03 = 0.005$.
- To ensure this condition, ϵ_{max} must be chosen so that $\epsilon_{\text{max}} \leq 0.005$.

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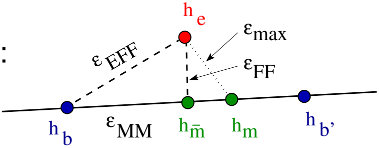
- The maximum mismatch is chosen to assure searches miss only a small fraction of real signals. The common choice $\epsilon_{\text{max}} = 0.035$ limits the loss rate to about 10%.
- Real searches are more complicated: comparing signals with a discrete template bank of model waveforms.



- For Initial LIGO, template banks are constructed with $\epsilon_{\text{MM}} = 0.03$, so $\epsilon_{\text{FF}} = \epsilon_{\text{EFF}} - \epsilon_{\text{MM}} = 0.035 - 0.03 = 0.005$.
- To ensure this condition, ϵ_{max} must be chosen so that $\epsilon_{\text{max}} \leq 0.005$.
- Accuracy requirement for BBH waveforms for detection in LIGO:

$$\sqrt{\overline{\delta\chi_m^2} + \overline{\delta\Phi_m^2}} \lesssim \sqrt{2\epsilon_{\text{max}}} = 0.1 \quad \text{for detection.}$$