Solving the Inverse Stellar Structure Problem

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- How can the equation of state of the matter in a star be determined from astronomical observations?
- This talk will focus on exploring the mathematical, rather than observational, aspects of this question.

Relativistic Stellar Structure Problem (SSP)

- Given an equation of state, $\epsilon = \epsilon(p)$, solve Einstein's equations,

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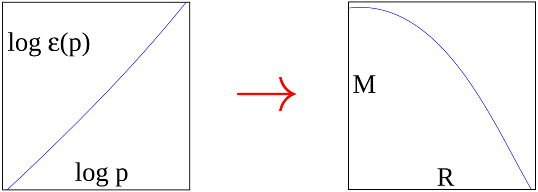
- Find the radius $p(R) = 0$ and mass $M = m(R)$ for each star.
- SSP can be thought of as a map from the equation of state $\epsilon = \epsilon(p)$ to the M-R curve $\{R(p_c), M(p_c)\}$.



Relativistic Inverse Stellar Structure Problem (SSP^{-1})

- When the equation of state is well understood – as in white dwarf stars – the standard stellar structure problem is useful.
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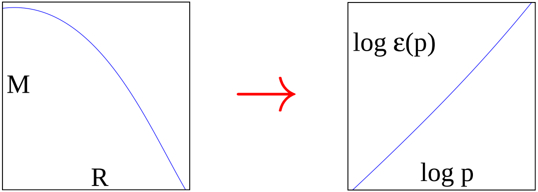
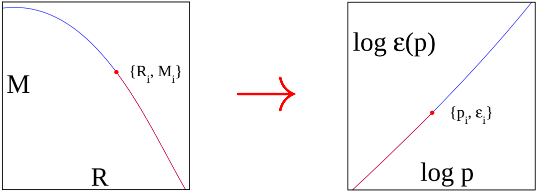
- When the equation of state is poorly known – as in neutron stars – the inverse stellar structure problem (SSP^{-1}) is more interesting.
- SSP^{-1} finds the equation of state $\epsilon = \epsilon(p)$ from a given mass-radius curve.
- SSP^{-1} can be thought of as the map from the M-R curve $\{R(p_c), M(p_c)\}$ to the equation of state $\epsilon = \epsilon(p)$.



Standard Solution to SSP^{-1}

- Assume the complete M-R curve is known, including the point $\{R_i, M_i\} = \{R(p_i), M(p_i)\}$.
- Assume the equation of state is known for $\epsilon \leq \epsilon_i = \epsilon(p_i)$.



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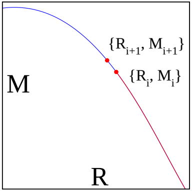

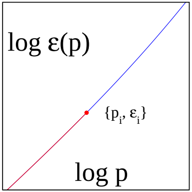

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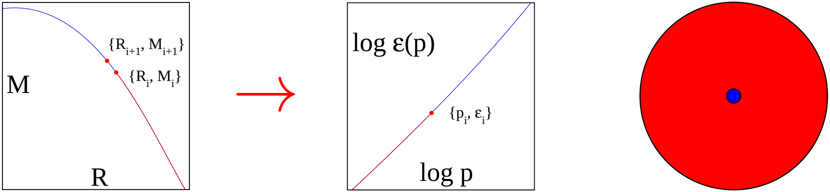


- Choose a new point on the M-R curve, $\{R_{i+1}, M_{i+1}\}$, having slightly larger central density.
- Integrate Einstein's equations,

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \qquad \frac{dp}{dr} = -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)},$$

through the outer parts of the star, to determine the mass and radius, $\{r_{i+1}, m_{i+1}\}$, of the small core with large densities $\epsilon \geq \epsilon_i$.

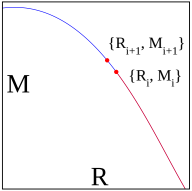
Standard Solution to SSP^{-1} II

- For very small cores, $\{r_{i+1}, m_{i+1}\}$, the solution to the OV equations is described by the power series solution:

$$m_{i+1} = \frac{4\pi}{3}\epsilon_{i+1} r_{i+1}^3 + \mathcal{O}(r_{i+1}^5),$$

$$p_i = p_{i+1} - \frac{2\pi}{3}(\epsilon_{i+1} + p_{i+1})(\epsilon_{i+1} + 3p_{i+1}) r_{i+1}^2 + \mathcal{O}(r_{i+1}^4).$$

- Invert these series to determine the central pressure and density, $\{p_{i+1}, \epsilon_{i+1}\}$, in terms of the known quantities, p_i, ϵ_i, m_{i+1}, r_{i+1}.



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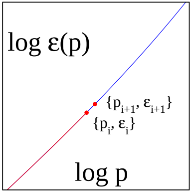

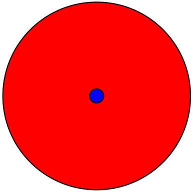

Can the Standard Solution to SSP^{-1} be Improved?

- Standard solution to the relativistic SSP^{-1} finds the equation of state, $\epsilon = \epsilon(p)$, represented as a table: $\{p_i, \epsilon_i\}$ for $i = 1, ..., N$.

- Standard solution has several weaknesses:
  - Solution converges (slowly) with the number of points, as N^{-p}.
  - Each new equation of state point, $\{p_i, \epsilon_i\}$, requires the knowledge of a separate new M-R curve point, $\{R_i, M_i\}$.
  - Accurate M-R curve points $\{R_i, M_i\}$ for neutron stars are scarce.

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- Can spectral methods provide better (*i.e.* more practical and more accurate) solutions to the SSP^{-1}?

- Can spectral methods provide interesting solutions to SSP^{-1} when only a few (*e.g.* two or three) M-R data points are available?

Outline for Solving SSP^{-1} Using Spectral Methods

- Assume the equation of state can be written in the form $\epsilon = \epsilon(p, \gamma_k)$, where the γ_k are a set of parameters.

  For example, the equation of state could be written as a spectral expansion, $\epsilon = \epsilon(p, \gamma_k) = \sum_k \gamma_k \Phi_k(p)$, where the $\Phi_k(p)$ are spectral basis functions, *e.g.* $\Phi_k(p) = e^{ikp}$, or $\Phi_k(p) = P_k(p)$.

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- Given a set of points from the "real" M-R curve, $\{R_i, M_i\}$, choose the parameters γ_k and p_i that minimize the difference measure:

$$\chi^2 = \frac{1}{N_{\rm stars}} \sum_{i=1}^{N_{\rm stars}} \left\{ \left[\log\left(\frac{R(p_i, \gamma_k)}{R_i}\right)\right]^2 + \left[\log\left(\frac{M(p_i, \gamma_k)}{M_i}\right)\right]^2 \right\}$$

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- Resulting γ_k for $k = 1, ..., N_{\gamma_k}$ determines an equation of state, $\epsilon = \epsilon(p, \gamma_k)$, that provides an approximate solution of SSP^{-1}.

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- Can the spectral parameters γ_k be determined accurately and robustly by matching model masses and radii $\{R(p_i, \gamma_k), M(p_i, \gamma_k)\}$ to given $\{R_i, M_i\}$ data?
- These questions are best answered using a somewhat different form of the standard stellar structure problem (SSP).
- Digress (briefly) now to describe this alternate formulation that provides a more efficient and more accurate way to solve the SSP.

Alternative Representations of the SSP

- The standard Oppenheimer-Volkoff (OV) representation of the SSP equations determines $m(r)$ and $p(r)$, given an equation of state of the form $\epsilon = \epsilon(p)$.
- The outer boundary of the star is the point where $p(R) = 0$. This condition is difficult to solve numerically because the pressure goes to zero non-linearly there: $p \propto (R - r)^{\Gamma_o/(\Gamma_0 - 1)}$.

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- This problem can be simplified by introducing the relativistic enthalpy $h(p) = \int_0^p dp'/[\epsilon(p') + p']$, and re-writing the OV equations in terms of it:

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- The surface of the star is now the point where $h(R) = 0$. This condition is easier to solve numerically because the enthalpy goes to zero linearly there: $h(r) \propto (R - r)$.

Alternative Representations of the SSP II

- Simplify again by swapping the roles of h and r:

$$\frac{dm}{dh} = -\frac{4\pi\epsilon(h)r^3(r-2m)}{m+4\pi r^3 p(h)}, \qquad \frac{dr}{dh} = -\frac{r(r-2m)}{m+4\pi r^3 p(h)}.$$

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- These alternative OV equations require that the equation of state, $\epsilon = \epsilon(p)$, be re-written as $\epsilon = \epsilon(h)$ and $p = p(h)$:
  - Start with the standard, $\epsilon = \epsilon(p)$.
  - Compute, $h(p) = \int_0^p dp'/[\epsilon(p') + p']$.
  - Invert to give $p = p(h)$.
  - Compose $\epsilon = \epsilon(p)$ with $p = p(h)$, to give $\epsilon = \epsilon(h) = \epsilon[p(h)]$.

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$$\frac{dp}{dh} = \epsilon + p, \qquad \frac{d\epsilon}{dh} = \frac{(\epsilon + p)^2}{p\,\Gamma(h)}.$$

Faithful Spectral Expansions of the Equation of State II

- Every equation of state is determined by the adiabatic index $\Gamma(h)$:

$$\frac{dp}{dh} = \epsilon + p, \qquad\qquad \frac{d\epsilon}{dh} = \frac{(\epsilon + p)^2}{p\,\Gamma(h)}.$$

- The solutions to these equations can be reduced to quadratures:

$$\begin{aligned}
p(h) &= p_0 \exp\left[\int_{h_0}^{h} \frac{e^{h'}\,dh'}{\mu(h')}\right],\\
\mu(h) &= \frac{p_0\, e^{h_0}}{\epsilon_0 + p_0} + \int_{h_0}^{h} \frac{\Gamma(h') - 1}{\Gamma(h')} e^{h'}\,dh',\\
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- Choosing, $\log\Gamma(h) = \sum_k \gamma_k \Phi_k(h)$, for any spectral basis functions, $\Phi_k(h)$, results in a faithful parametrized equation of state of the desired form: $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$.

Fitting Model Neutron-Star Equations of State

- How accurately and efficiently are realistic neutron-star equations of state represented by $\epsilon = \epsilon(h, \gamma_k)$ and $p = p(h, \gamma_k)$, when $\Gamma(h)$ is given by
$$\Gamma(h) = \exp\left\{\sum_{k=0}^{N_{\gamma_k}-1} \gamma_k \left[\log\left(\frac{h}{h_0}\right)\right]^k\right\} ?$$

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$$\Gamma(h) = \exp\left\{\sum_{k=0}^{N_{\gamma_k}-1} \gamma_k \left[\log\left(\frac{h}{h_0}\right)\right]^k\right\} ?$$
- Let $\{p_i, \epsilon_i, h_i\}$, for $i = 1, ..., N_{\rm EOS}$ denote one of the standard tabulated realistic neutron-star equations of state.
- Find the spectral parameters γ_k that minimize the fitting error:
$$\left(\Delta^{EOS}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\rm EOS}} \sum_{i=1}^{N_{\rm EOS}} \left[\log\left(\frac{\epsilon(h_i, \gamma_k)}{\epsilon_i}\right)\right]^2.$$

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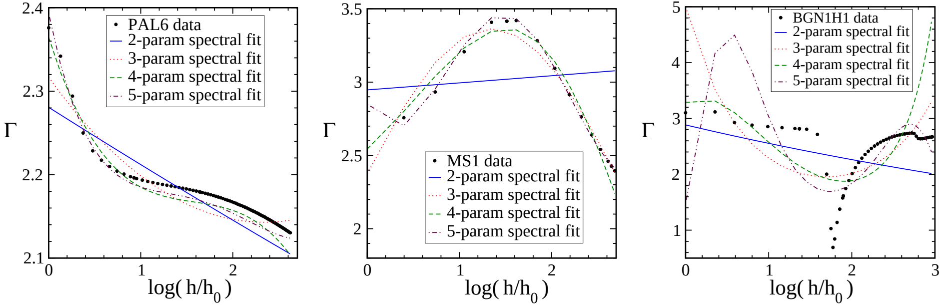
- The average values of these fitting errors, $\Delta^{EOS}_{N_{\gamma_k}}$, for 34 realistic neutron-star equations of state are:
$$\Delta^{EOS}_2 = 0.032, \qquad \Delta^{EOS}_3 = 0.017,$$
$$\Delta^{EOS}_4 = 0.012, \qquad \Delta^{EOS}_5 = 0.0089.$$

Spectral Fits of Model Neutron-Star Equations of State



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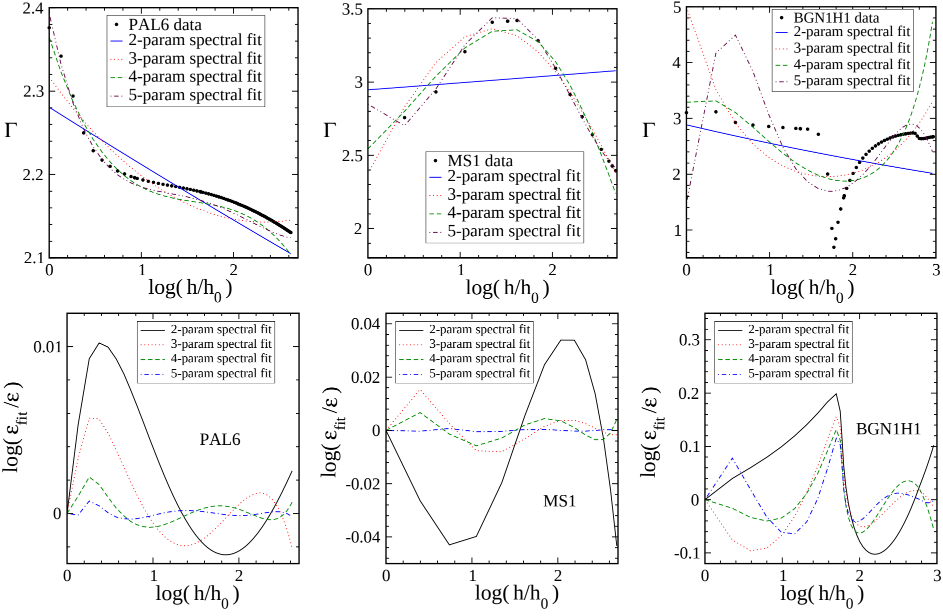

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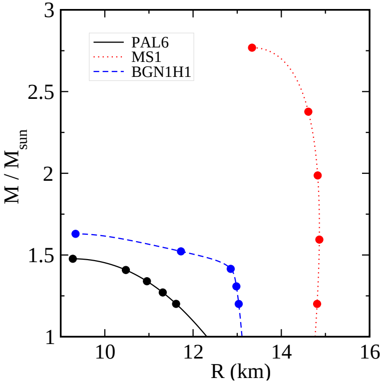
- Choose mock data points $\{R_i, M_i\}$ for neutron-star models computed with 34 realistic equations of state.



Spectral Solution of SSP^{-1} II

- Fix the spectral expansion coefficients γ_k by minimizing,

$$\chi^2 = \frac{1}{N_{\text{stars}}}\sum_{i=1}^{N_{\text{stars}}}\left\{\left[\log\left(\frac{M(h_c^i,\gamma_k)}{M_i}\right)\right]^2 + \left[\log\left(\frac{R(h_c^i,\gamma_k)}{R_i}\right)\right]^2\right\}.$$

  with respect to variations in γ_k, and variations in the central values of the enthalpy for each star, h_c^i.

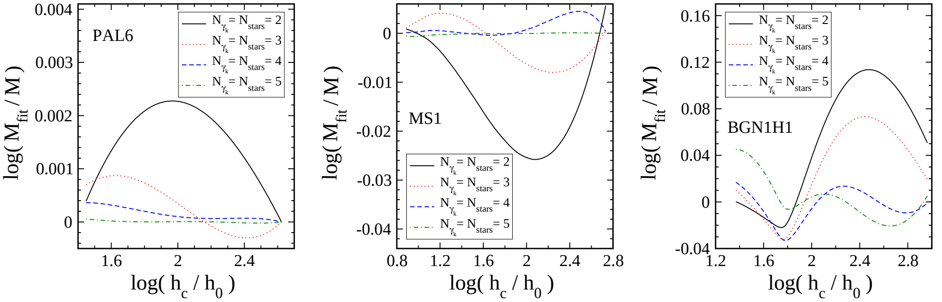
Spectral Solution of SSP^{-1} II

- Fix the spectral expansion coefficients γ_k by minimizing,

$$\chi^2 = \frac{1}{N_{\rm stars}} \sum_{i=1}^{N_{\rm stars}} \left\{ \left[\log\left(\frac{M(h_c^i, \gamma_k)}{M_i} \right) \right]^2 + \left[\log\left(\frac{R(h_c^i, \gamma_k)}{R_i} \right) \right]^2 \right\}.$$

  with respect to variations in γ_k, and variations in the central values of the enthalpy for each star, h_c^i.

- Compare the resulting M-R curve $\{R(h,\gamma_k), M(h,\gamma_k)\}$ with the exact curve from the known equation of state $\{R(h), M(h)\}$.



Spectral Solutions to SSP^{-1} III

- Next evaluate the equation of state fitting errors, $\Delta^{MR}_{N_{\gamma_k}}$,

$$\left(\Delta^{MR}_{N_{\gamma_k}}\right)^2 = \frac{1}{N_{\mathrm{EOS}}}\sum_{i=1}^{N_{\mathrm{EOS}}}\left[\log\left(\frac{\epsilon(h_i,\gamma_k)}{\epsilon_i}\right)\right]^2$$

  to determine how well the spectral expansion $\epsilon = \epsilon(h, \gamma_k)$, matches the exact neutron-star equation of state $\epsilon = \epsilon(h)$.

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- The average values of $\Delta^{MR}_{N_{\gamma_k}}$ (with $N_{\gamma_k}=N_{\rm stars}$) determined in this way for 34 realistic model equations of state are:

$$\begin{array}{ll}
\Delta^{MR}_2 = 0.039, & \Delta^{EOS}_2 = 0.032,\\
\Delta^{MR}_3 = 0.026, & \Delta^{EOS}_3 = 0.017,\\
\Delta^{MR}_4 = 0.017, & \Delta^{EOS}_4 = 0.012,\\
\Delta^{MR}_5 = 0.015, & \Delta^{EOS}_5 = 0.0089.
\end{array}$$

- The accuracy of these solutions to the SSP^{-1} is quite impressive, even though the number of M-R data used is very small.

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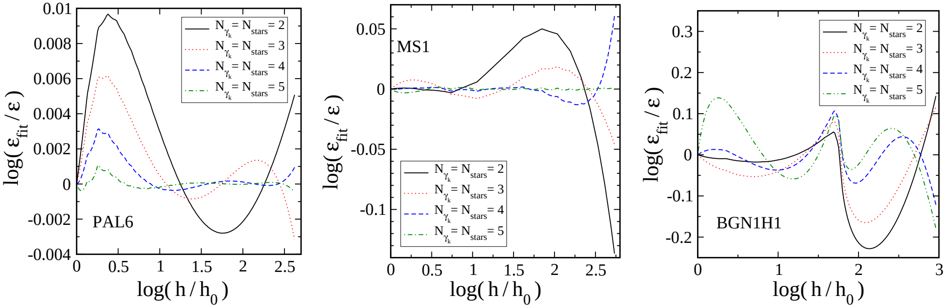
- The accuracy of these solutions to the SSP^{-1} is quite impressive, even though the number of M-R data used is very small.
- The convergence of $\Delta^{MR}_{N_{\gamma_k}}$ is not as good as $\Delta^{EOS}_{N_{\gamma_k}}$. Perhaps our χ^2 minimization finds local not global minima?

Spectral Solutions to SSP^{-1} IV

- Compare the spectral equation of state, $\epsilon(h, \gamma_k)$, determined by fitting the M-R data with the exact equation of state $\epsilon(h)$:



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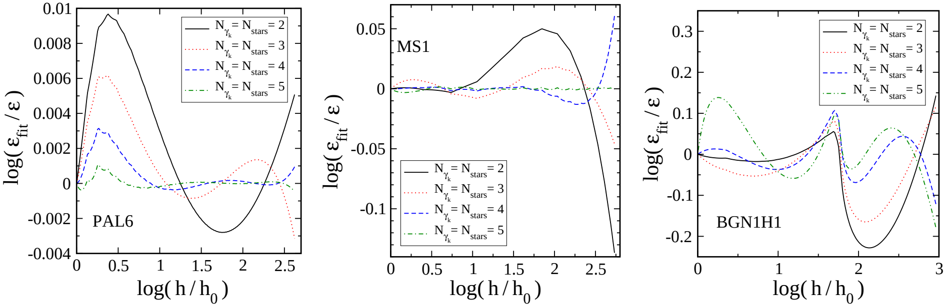

- **Conclusion:** The spectral approach provides a very promising way to determine the neutron-star equation of state from observed properties of these stars.