Generalized Harmonic Evolutions of Binary Black Hole Spacetimes

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AMS Meeting :: New Orleans :: 7 January 2007

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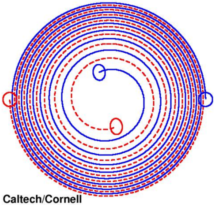

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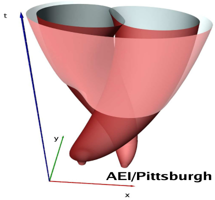

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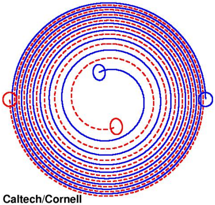

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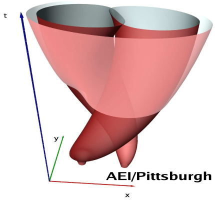

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- Frans Pretorius performed first numerical BBH inspiral, merger and ringdown calculations in 2005. Pretorius Inspiral Movies
- Caltech/Cornell collaboration and the AEI/Pittsburgh collaboration perform successful BBH simulations in 2006 using GH methods.
- Outline of this talk:
  - Review Generalized Harmonic (GH) form of the Einstein system.
  - Constraint damping.
  - Boundary conditions.

Methods of Specifying Spacetime Coordinates

- We often decompose the 4-metric into its 3+1 parts: $ds^2 = \psi_{ab}dx^a dx^b = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$. The lapse N and shift N^i specify how coordinates are laid out on a spacetime manifold: $\vec{n} = \partial_\tau = (\partial_t - N^k\partial_k)/N$.

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- An alternate way to specify the coordinates is through the gauge source function H^a:
- Let H^a denote the function obtained by the action of the covariant scalar wave operator on the coordinates x^a:

$$H^a \equiv \nabla^c\nabla_c x^a = -\Gamma^a,$$

where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

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where $\Gamma^a = \psi^{bc}\Gamma^a{}_{bc}$ and ψ_{ab} is the 4-metric.

- Specifying coordinates by the *generalized harmonic* (GH) method can be accomplished by choosing a gauge-source function $H_a(x,\psi) = \psi_{ab}H^b$, and requiring that $H_a(x,\psi) = -\Gamma_a = -\psi_{ab}\psi^{cd}\Gamma^b_{cd}$.

Important Properties of the GH Method

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

$$R_{ab} = -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + F_{ab}(\psi, \partial\psi),$$

where ψ_{ab} is the 4-metric, and $\Gamma_a = \psi^{bc}\Gamma_{abc}$. The vacuum Einstein equation, $R_{ab} = 0$, has the same principal part as the scalar wave equation when $H_a(x, \psi) = -\Gamma_a$ is imposed.

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- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, $\mathcal{C}_a = 0$, where

$$\mathcal{C}_a = H_a + \Gamma_a,$$

  depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, $\mathcal{M}_a = 0$, are determined by the derivatives of the gauge constraint $\mathcal{C}_a$:

$$\mathcal{M}_a \equiv \Big[R_{ab} - \frac{1}{2}\psi_{ab}R\Big]n^b = \Big[\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\nabla^c\mathcal{C}_c\Big]n^b.$$

Constraint Damping Generalized Harmonic System

- Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[n_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\, n^c\, \mathcal{C}_c\right],$$

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  where n^a is a unit timelike vector field. Since $\mathcal{C}_a = H_a + \Gamma_a$ depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.
- Evolution of the constraints $\mathcal{C}_a$ follow from the Bianchi identities:

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^c\left[n_{(c}\mathcal{C}_{a)}\right] + \mathcal{C}^c\nabla_{(c}\mathcal{C}_{a)} - \frac{1}{2}\gamma_0\, n_a\mathcal{C}^c\mathcal{C}_c.$$

  This is a damped wave equation for $\mathcal{C}_a$, that drives all small short-wavelength constraint violations toward zero as the system evolves (for $\gamma_0 > 0$).

First Order Generalized Harmonic Evolution System

- Kashif Alvi (2002) derived a nice (symmetric hyperbolic) first-order form for the generalized-harmonic evolution system:

$$
\begin{aligned}
\Phi_{kab} &= \partial_k \psi_{ab}, \\
\partial_t \psi_{ab} - N^k \partial_k \psi_{ab} &= -N\,\Pi_{ab}, \\
\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + N g^{ki} \partial_k \Phi_{iab} &\simeq 0, \\
\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} &\simeq 0.
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- This system has two immediate problems:
  - This system has new constraints, $\mathcal{C}_{kab} = \partial_k \psi_{ab} - \Phi_{kab}$, that tend to grow exponentially during numerical evolutions.
  - This system is not linearly degenerate, so it is possible (likely) that shocks will develop (e.g. the shift evolution equation is of the form $\partial_t N^i - N^k \partial_k N^i \simeq 0$).

Improved First-Order GH Evolution System

- We can correct these problems by adding additional multiples of the constraints to the evolution system:

$$\partial_t\psi_{ab} - (1+\gamma_1)N^k\partial_k\psi_{ab} = -N\Pi_{ab} - \gamma_1 N^k\Phi_{kab},$$
$$\partial_t\Pi_{ab} - N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} - \gamma_1\gamma_2 N^k\partial_k\psi_{ab} \simeq -\gamma_1\gamma_2 N^k\Phi_{kab},$$
$$\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} \simeq -\gamma_2 N\Phi_{iab}.$$

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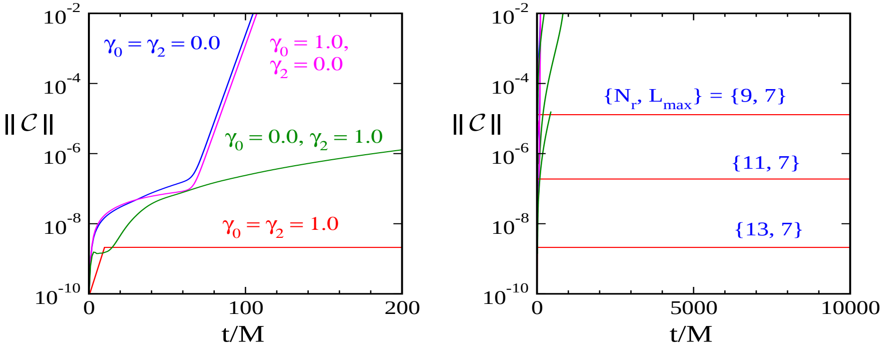
$$\partial_t\Phi_{iab} - N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - \gamma_2 N\partial_i\psi_{ab} \simeq -\gamma_2 N\Phi_{iab}.$$

- This improved GH evolution system has several nice properties:
  - This system is linearly degenerate for $\gamma_1 = -1$ (and so shocks should not form from smooth initial data).
  - The Φ_{iab} evolution equation can be written in the form, $\partial_t\mathcal{C}_{iab} - N^k\partial_k\mathcal{C}_{iab} \simeq -\gamma_2 N\mathcal{C}_{iab}$, so the new constraints are damped when $\gamma_2 > 0$.
  - This system is symmetric hyperbolic for all values of γ_1 and γ_2.

Numerical Tests of the First-Order GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.



- The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.

Boundary Conditions

- Boundary conditions are straightforward to formulate for first-order hyperbolic evolutions systems,

$$\partial_t u^\alpha + A^{k\,\alpha}{}_\beta(u)\partial_k u^\beta = F^\alpha(u).$$

For the GH system $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$.

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- Find the eigenvectors of the characteristic matrix $s_k A^{k\,\alpha}{}_\beta$ at each boundary point:

$$e^{\hat\alpha}{}_\alpha\, s_k A^{k\,\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta,$$

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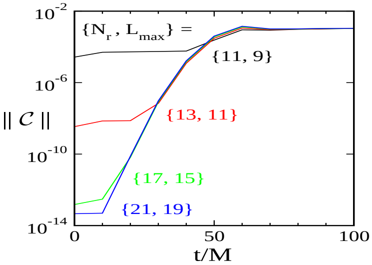
$$u^{\hat\alpha} = e^{\hat\alpha}{}_\alpha u^\alpha.$$
- A boundary condition must be imposed on each incoming characteristic field (*i.e.* every field with $v_{(\hat\alpha)} < 0$), and must not be imposed on any outgoing field (*i.e.* any field with $v_{(\hat\alpha)} > 0$).

Evolutions of a Perturbed Schwarzschild Black Hole

- A black-hole spacetime is perturbed by an incoming gravitational wave that excites quasi-normal oscillations.
- Use boundary conditions that *Freeze* the remaining incoming characteristic fields.
- The resulting outgoing waves interact with the boundary of the computational domain and produce constraint violations.



Lapse Movie Constraint Movie

Constraint Evolution for the First-Order GH System

- The evolution of the constraints, $c^A = \{\mathcal{C}_a, \mathcal{C}_{kab}, \mathcal{M}_a \approx n^c\partial_c\mathcal{C}_a, \mathcal{C}_{ka} \approx \partial_k\mathcal{C}_a, \mathcal{C}_{klab} = \partial_{[k}\Phi_{l]ab}\}$ are determined by the evolution of the fields $u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\}$:
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$$\partial_t c^A + A^{k\,A}{}_B(u)\partial_k c^B = F^A{}_B(u,\partial u)\, c^B.$$
- This constraint evolution system is symmetric hyperbolic with principal part:
$$\begin{aligned}
\partial_t\mathcal{C}_a &\simeq 0,\\
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\partial_t\mathcal{C}_{ia} - N^k\partial_k\mathcal{C}_{ia} - N\partial_i\mathcal{M}_a &\simeq 0,\\
\partial_t\mathcal{C}_{iab} - (1+\gamma_1)N^k\partial_k\mathcal{C}_{iab} &\simeq 0,\\
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\partial_t\mathcal{C}_{ijab} - N^k\partial_k\mathcal{C}_{ijab} &\simeq 0.
\end{aligned}$$
- An analysis of this system shows that all of the constraints are damped in the WKB limit when $\gamma_0 > 0$ and $\gamma_2 > 0$. So, this system has constraint suppression properties that are similar to those of the Pretorius (and Gundlach, et al.) system.

Constraint Preserving Boundary Conditions

- Construct the characteristic fields, $\hat{c}^{\hat{A}} = e^{\hat{A}}{}_{A} c^{A}$, associated with the constraint evolution system, $\partial_t c^A + A^{k\,A}{}_B \partial_k c^B = F^A{}_B c^B$.

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- The incoming characteristic fields mush vanish on the boundaries, $\hat{c}^- = 0$, if the influx of constraint violations is to be prevented.
- The constraints depend on the primary evolution fields (and their derivatives). We find that $\hat{c}^-$ for the GH system can be expressed:

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$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$

- Set boundary conditions on the fields $\hat{u}^-$ by requiring

$$d_\perp \hat{u}^- = -\hat{F}(u, d_\parallel u).$$

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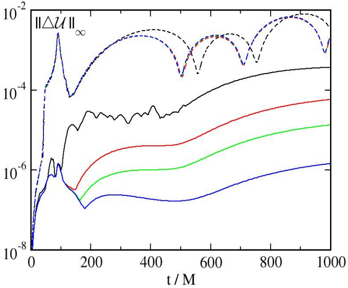
- Oliver Rinne (2006) used Fourier-Laplace analysis to show that these BC satisfy the Kreiss (1970) condition which is necessary for well-posedness (but not sufficient for this type of BC).
- Help Wanted! New analysis methods are needed to prove (or disprove) complete well-posedness for this type of BC.

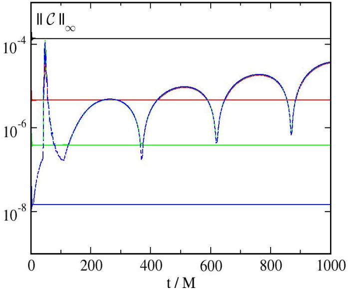
Numerical Tests of Boundary Conditions

- Compare the solution obtained on a "small" computational domain with a reference solution obtained on a "large" domain where the boundary is not in causal contact with the comparison region.

Solution Differences



Constraints



- Solutions using "Freezing" BC (dashed curves) have differences and constraints that do not converge to zero.
- Solutions using constraint preserving and physical BC (solid curves) have much smaller differences and constraints that converge to zero.

Summary

- Generalized Harmonic method produces manifestly hyperbolic representations of the Einstein equations for any choice of coordinates.
- Constraint damping makes the modified GH equations stable for numerical simulations.
- Constraint preserving boundary conditions have been implemented and tested for the GH system.
- Binary black hole simulations have been successfully performed using GH methods by several groups using very different numerical methods.