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Comments on the Topology of Nonsingular Stellar Models†

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Abstract

We consider the question, what topologies of space–time are permitted by Einstein's equations as models of nonsingular stars. A number of results from the literature on singularity theorems are drawn together here to give a clear answer to this question. If a star evolves from nearly Newtonian initial conditions, of low densities and small space–time curvatures, to a nonsingular final state, then the topology of the space–time representing the star must be R^4. We also discuss some topological constraints on models that are not nearly Newtonian in the past, but may have evolved directly from the initial cosmic singularity.

I. Introduction

Professor Taub's contributions to the study of relativistic fluid mechanics form an important foundation on which any fundamental study of relativistic stellar structure must be based. We feel it is appropriate,

† Supported in part by the National Science Foundation.

ISBN 0-12-691380-3

therefore, to consider here another basic aspect of the study of relativistic stars: the topology of the space–time manifold representing a stellar model.

The endproducts of stellar evolution can be divided into two classes: those that contain space–time singularities and those that do not. If the cosmic censorship hypothesis [1] is true, the singular endproducts of stellar evolution are black holes. The final equilibrium black hole solutions to Einstein's equations are now completely understood due to the theorems of Israel [2], Carter [3], Hawking [4], and Robinson [5]. Therefore, it is of interest to inquire about the properties of the solutions to Einstein's equations that represent the nonsingular endpoints of stellar evolution. While some properties of these solutions are known (see Lindblom [6], for a review), there is much that is not known with certainty. We shall consider here one of the most fundamental aspects of this inquiry: What space–time topologies are possible for nonsingular solutions to Einstein's equations that represent stellar models?

Geometric theories of gravitation, such as general relativity, allow (in principle) a very large and diverse set of possible space–time topologies. One can imagine, for example, configurations of matter that have nontrivial spatial topologies such as those depicted in Fig. 1. One would like to know whether configurations of this sort are possible models for nonsingular stars. Is it possible that in regions of high space–time curvature, such as the inside of a neutron star, the topology of space–time there might be a "wormhole" as in Fig. 1a? What effect would such nontrivial topology have on the theorems that give an upper limit to the possible mass of a neutron star? Another nontrivial possibility is illustrated in Fig. 1b, and unlimited other possibilities exist. We argue here that configurations such as those in Fig. 1

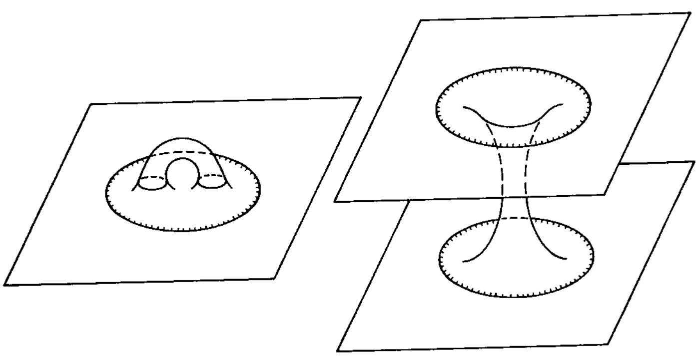

Fig. 1 Possible spatial topologies for stellar models. (a) represents a stellar model containing a nonsimply connected "wormhole" within it. The shaded line represents the surface of the star. (b) represents a model having two asymptotic regions connected by a "wormhole."

are not possible general relativistic stellar models having no singularities. We consider two separate cases. The first case corresponds to stars that originated from nearly Newtonian configurations that had low densities and small space–time curvatures. This case corresponds to the usual picture of the early stages of stellar evolution. For models of this type, we assume that past regions of the space–time have many of the asymptotic features of flat Minkowski space. These assumptions are justified for the description of normal stellar evolution because the effects of the overall curvature of our universe are quite small on distance–time scales that are still very large compared to the size–age of any star. However, it is possible that some primordial condensations of matter could have occurred very early in the history of our Universe. Perhaps these primordial stars now form the cores of quasars or even stranger as yet undiscovered objects. The possible topologies of these primordial stars represent the second case we wish to consider here.

II. The Topology of Initially Newtonian Stars

The first theorem that placed restrictions on the possible topology of stars, which began under nearly Newtonian circumstances, was given by Geroch [7]:

Theorem 1 If the space–time (M, g) is asymptotically simple and empty, then M is homeomorphic to R^4.

This theorem is extremely general and excludes nonstandard topologies provided the star never becomes extremely compact. But for compact objects, the assumption of asymptotic simplicity is stronger than is justifiable. Asymptotic simplicity requires that every null geodesic in the space–time begin and end at null infinity. This assumption would be violated by any spherical star of radius less than $3M$, because the exterior Schwarzschild geometry has circular null geodesics at $r = 3M$. Since it is possible to construct nonsingular models of radius smaller than $3M$ (see, e.g., Bondi [8]), we would hope to find results that would place restrictions on the topology of these highly compact objects as well.

Another theorem of Geroch [9] gives some topological information about a wider class of space–times than the asymptotically simple ones considered in Theorem 1.

Theorem 2 If a space–time (M, g) admits a Cauchy surface S, then M must be homeomorphic to $R \times S$.

In such a space–time, then, a star of initially trivial topology cannot change its topology even if it later becomes highly compact. However, again the assumptions on the space–time are stronger than are physically justifiable. The existence of a Cauchy surface in a space–time not only rules out the possibility of "naked singularities" (an acceptable omission for our present purposes) but also the possibility of other causal peculiarities such as closed timelike lines. One should not exclude, a priori, the possibility of bizarre causal behavior in a region of space–time having large curvatures and possible nontrivial topological structure.

In the case of a star that begins as a highly diffuse cloud of gas and dust (the usual picture), and which evolves to a nonsingular endpoint, we can show that bizarre causal behavior does not occur, i.e., that a Cauchy surface does exist. Early in the evolution of a star of this sort the density of matter is very low, and the geometry of space–time is nearly that of flat Minkowski space. For a space–time describing a star that begins in this way, it seems reasonable to assume that the space–time begins and remains asymptotically flat throughout the evolution of the star. Also, it seems reasonable to assume that any bizarre causal behavior occurs only during the late stages of stellar evolution when the densities and curvatures are high. Thus we assume the space–time has at least a partial Cauchy surface at sufficiently early times. The following theorem shows that a nonsingular stellar model that begins in this way cannot develop bizarre causal behavior as it evolves:

Theorem 3 If a space–time (M, g) *satisfies the following conditions:*

(a) (M, g) *is geodesically complete.*
(b) *The weak energy condition and Einstein's equations hold on* (M, g).
(c) *The generic condition holds on* (M, g).
(d) (M, g) *is weakly asymptotically simple and empty.*
(e) (M, g) *is partially asymptotically predictable from a partial Cauchy surface* S.

Then S *is a Cauchy surface for* (M, g).

The proof of this result is virtually identical to the proof of a closely related result by Tipler [10], and the interested reader is referred to the proof of his Theorem 1. (Another related result is given by Tipler [11], Theorem 6.) Assumptions (a)–(e) in our result are the technical restrictions that reflect the physical situation described qualitatively above. The precise mathematical meanings of these terms can be found in Hawking and Ellis [12] and Tipler [10]. For the reader unfamiliar with the language of "global techniques," we describe qualitatively the meaning of each assumption:

(a) Geodesic completeness is the condition that the star have no singularities.

(b) The weak energy condition requires that the density of matter must never be negative.

(c) The generic condition is a technical assumption that requires that every geodesic observer feel some tidal force sometime during its history. It is expected to be satisfied by physically realistic space–times.

(d) Weak asymptotic simplicity is the requirement that the space–time behave asymptotically (near null finity) like flat Minkowski space.

(e) The existence of a partial Cauchy surface, from which the space–time is partially asymptotically predictable, excludes any causal anomalies from early times in the history of the stars and from the asymptotic region of the space–time near spacelike infinity.

We can conclude from Theorems 2 and 3 that the space–time of a nonsingular star, which evolves from an initial state of low density, must have the topology $R \times S$. Moreover, we can determine what topology S must have. Consider one of the partial Cauchy surfaces S, which occurs very early in the evolution of the star when the densities are very low. It is reasonable to assume that the past directed null geodesics leaving this surface all reach past null infinity, since the space–time to the past of S is nearly the same as Minkowski space. Such a partial Cauchy surface is said to have an asymptotically simple past (see Hawking and Ellis [12], p. 316). By the same arguments used (by Geroch [7]) to prove Theorem 1, it follows that S must be homeomorphic to R^3, and consequently M must be homeomorphic to R^4. Thus we have:

Theorem 4 If a space–time (M, g), *having a partial Cauchy surface* S *with an asymptotically simple past, satisfies the assumptions of Theorem* 3, *then* M *is homeomorphic to* R^4.

Consequently, for any nonsingular stellar model in Einstein's theory that evolves from low density, nearly Newtonian, initial conditions must have the trivial topology R^4.

III. The Topology of Primordial Stars

We next consider the more difficult case of primordial stars, those (if any) that were formed immediately after the initial singularity of our universe, rather than from nearly Newtonian initial conditions. What assumptions can we make about the asymptotic features of the space–times describing these objects? The primordial objects, which we have in mind, are those that look from a distance like the exteriors of ordinary stars (clusters, or galaxies) but whose interiors might be bizarre, because they arose from the early stages of our universe rather than from nearly Newtonian conditions. Thus it is reasonable to assume that the asymptotic properties of the manifold are weakly asymptotically simple and empty, *except* possibly for some open neighborhood of past timelike infinity i_-. It may be, for example, that past null infinity $\mathscr{I}^-$ in these models is not complete in the past. Although there is no strong motivation for placing restrictions on possible bizarre causal behavior that may have been associated with these objects from the earliest times, it is reasonable to limit the spatial extent of any such peculiarities. Since we do not observe causal anomalies in the laboratory, any such anomalies associated with primordial stars should be confined spatially to regions of space–time near the interiors of these objects.

While no restrictions on the topologies of primordial stars (which satisfy only the above assumptions) have yet been found, some progress has been made under more restrictive conditions.

Theorem 5 If a space–time (M, g) satisfies the assumptions of Theorem 3 and the chronology condition, and admits an asymptotically regular partial Cauchy surface S, then M is homeomorphic to $R \times S$, S is simply connected, and the boundary of S is a single two-sphere.

This theorem is proved by combining the results of our Theorem 3 with the work of Gannon [13,14]. (See also the work of Lee [15,16].) We have introduced two new assumptions in this theorem. The chronology condition assumes the nonexistence of closed timelike lines. The condition of asymptotic regularity of the surface S is a condition that requires that S be "asymptotically flat" in an appropriate sense (see Gannon [13,14]). The use of the asymptotic regularity of S and weak asymptotic simplicity in the proof of this theorem involve only the properties of the space–time near spacelike infinity i_0 and future null infinity $\mathscr{I}^+$. Consequently these assumptions are acceptable for our purposes.

Theorem 5 shows that models of primordial stars, which are nonsingular and devoid of causal anomalies, must be simply connected (thus ruling out examples like Fig. 1a) and may have only one asymptotic region (thus ruling out examples like Fig. 1b). If the Poincaré conjecture is true, these restrictions on the topology of S are sufficient to show that S is homeomorphic to R^3. Even without the Poincaré conjecture, however, the result completely determines the structure of the topological groups of S. It follows, for example, that all of the homotopy groups are trivial: $\pi_k(S) = 1$, $k \geq 0$ (see Hempel [17]). Furthermore, one can conclude that the integral singular homology groups of S must be trivial: $H_k(S) = 0$, $k \geq 1$ (use the Hurewicz isomorphism theorem, see Spanier [18], p. 398). The remaining group $H_0(S)$ is isomorphic to the group of integers under addition, $H_0(S) = Z$. The integral singular cohomology groups can then be computed using Lefschetz duality and the singular cohomology (or homology) exact sequence (see

Hu [19]). The resulting groups are $H^0(S) = Z$ and $H^k(S) = 0$, $k \neq 0$. Thus S has the same homotopy, homology, and cohomology structure as R^3.

The weakness of Theorem 5, from the viewpoint of the study of primordial stars, is the overly restrictive causal assumptions: the chronology condition and the existence of a partial Cauchy surface. The theorem does show, however, that any nontrivial topology in these models must be accompanied by bizarre causal behavior as well.

Acknowledgment

We would like to thank James Hartle for helpful comments.

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