A New Generalized Harmonic Evolution System

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This note describes recent work on finding a formulation of the Einstein equations suitable for constructing stable numerical evolutions. The formulation described here specifies the coordinate degrees of freedom with a generalized harmonic gauge source function rather than with the usual lapse and shift. This type of formulation appears to have played a critical role in the very impressive binary black hole evolutions performed recently by Pretorius. This note analyzes why this type of formulation is so effective for numerical work, describes a recent extension of the system that makes it possible to construct boundary conditions (including constraint-preserving boundary conditions), and describes numerical tests that demonstrate the effectiveness of the new equations and boundary conditions.

Two properties have made harmonic or generalized harmonic (GH) coordinates an important tool throughout the history of general relativity theory. The first property is well known: this method of specifying the coordinates transforms the principal parts of the Einstein equations into a manifestly hyperbolic form, in which each component of the metric is acted on by the standard second-order wave operator. The second property is not as widely appreciated: this method of specifying coordinates fundamentally transforms the constraints of the theory. This new form of the constraints makes it possible to modify the evolution equations in a way that prevents small constraint violations from growing during numerical evolutions—without changing the physical solutions of the system and without changing the fundamental hyperbolic structure of the equations. The purpose of this note is to explore these important properties and to describe how the GH evolution system has been extended in a way that makes it very useful for numerical computations.

Coordinates are fixed in the generalized harmonic (GH) method by specifying a gauge source function H_a, defined as the action of the scalar-wave operator on the coordinate functions x^a:

$$H_a \equiv \psi_{ab}\nabla^c\nabla_a x^b = -\psi^{bc}\Gamma_{abc} \equiv -\Gamma_a, \tag{1}$$

where ψ_{ab} is the spacetime metric and Γ_{abc} is the usual Christoffel symbol. The coordinates are fixed in this approach by requiring that $\Gamma_a = -H_a$, where $H_a = H_a(x, \psi)$ is a prescribed function of the coordinates x^a and the metric ψ_{ab}. The choice $H_a = 0$ corresponds to standard harmonic coordinates; the existence of solutions to the inhomogeneous wave equation, Eq. (1), implies the existence of such coordinates more generally. Choosing the coordinates in this way has two important consequences. The first is well known: the vacuum Einstein equations have a simple manifestly hyperbolic structure when expressed in GH coordinates. The Ricci curvature tensor can be written as

$$R_{ab} = -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)} + \psi^{cd}\psi^{ef}\left(\partial_e\psi_{ca}\partial_f\psi_{db} - \Gamma_{ace}\Gamma_{bdf}\right), \tag{2}$$

in any coordinate system, where $\nabla_a\Gamma_b \equiv \partial_a\Gamma_b - \psi^{cd}\Gamma_{cab}\Gamma_d$. In GH coordinates, $\Gamma_a = -H_a$, so the only second-derivative term remaining in the Ricci tensor is $\psi^{cd}\partial_c\partial_d\psi_{ab}$. Therefore, in GH coordinates the vacuum Einstein equations, $R_{ab} = 0$, form a manifestly hyperbolic system,

$$\psi^{cd}\partial_c\partial_d\psi_{ab} = -2\nabla_{(a}H_{b)} + 2\,\psi^{cd}\psi^{ef}\left(\partial_e\psi_{ca}\partial_f\psi_{db} - \Gamma_{ace}\Gamma_{bdf}\right), \tag{3}$$

for any choice of guage source function H_a.[1]

The second consequence of using GH coordinates is less widely appreciated: The constraints of the system are profoundly transformed. The vacuum Einstein equations, Eq. (3), can also be written in the more covariant form

$$0 = R_{ab} - \nabla_{(a}C_{b)}, \tag{4}$$

where $C_a = H_a + \Gamma_a$. The condition $C_a = 0$ is the primary constraint of this system, while the standard Hamiltonian and momentum constraints $\mathcal{M}_a = G_{ab}t^b$ (where t^a is the unit normal to a Cauchy surface) are determined by the derivatives of C_a: $\mathcal{M}_a = t^b(\nabla_{(a}C_{b)} - \frac{1}{2}\psi_{ab}\nabla^c C_c)$. This means that the primary constraints depend on the first but not the second derivatives of the metric.

Adding multiples of the constraints to the Einstein equations is known to have a significant effect on the growth rates of constraint violating solutions.[2] However, adding multiples of the Hamiltonian and momentum constraints has not been found to be very effective in controlling the growth of constraint violating solutions. This is because the addition of those constraints changes the principal part of the equations, so constraints can be added only in very restricted ways consistent with the hyperbolic structure of the equations. In contrast, arbitrary multiples of the gauge constraint $\mathcal{C}_a$ can be added to the system, Eq. (4), without effecting the hyperbolic structure at all. Pretorius,[3] based on the suggestion of Gundlach, et al.,[4] used a modified evolution system that included the following additional gauge constraint terms designed to suppress the growth of the constraints:

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\left[t_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}\, t^c\mathcal{C}_c\right]. \tag{5}$$

The Bianchi identities then imply that $\mathcal{C}_a$ satisfies the damped wave equation,

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^b[t_{(b}\mathcal{C}_{a)}] + \mathcal{C}^b\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\gamma_0\, t_a\mathcal{C}^b\mathcal{C}_b, \tag{6}$$

which exponentially suppresses all small short-wavelength constraint violations when the parameter γ_0 is positive.[4] This constraint suppressing feature of the modified generalized harmonic system, Eq. (5), contributed significantly to the success of Pretorius' impressive binary black-hole evolutions.[3,5]

We have recently extended the modified generalized harmonic evolution system, Eq. (5), to a first-order symmetric-hyperbolic form. (See Ref. [6] for the details.) The vacuum Einstein system expressed in this new GH first-order form is given by

$$\partial_t\psi_{ab} - (1+\gamma_1)N^k\partial_k\psi_{ab} = -N\Pi_{ab} - \gamma_1 N^i\Phi_{iab}, \tag{7}$$

$$\begin{aligned}\partial_t\Pi_{ab} &- N^k\partial_k\Pi_{ab} + Ng^{ki}\partial_k\Phi_{iab} - \gamma_1\gamma_2 N^k\partial_k\psi_{ab}\\ &= 2N\psi^{cd}\left(g^{ij}\Phi_{ica}\Phi_{jdb} - \Pi_{ca}\Pi_{db} - \psi^{ef}\Gamma_{ace}\Gamma_{bdf}\right)\\ &\quad -2N\nabla_{(a}H_{b)} - \frac{1}{2}Nt^ct^d\Pi_{cd}\Pi_{ab} - Nt^c\Pi_{ci}g^{ij}\Phi_{jab}\\ &\quad +N\gamma_0\left[2\delta^c{}_{(a}t_{b)} - \psi_{ab}t^c\right](H_c+\Gamma_c) - \gamma_1\gamma_2 N^i\Phi_{iab},\end{aligned} \tag{8}$$

$$\begin{aligned}\partial_t\Phi_{iab} &- N^k\partial_k\Phi_{iab} + N\partial_i\Pi_{ab} - N\gamma_2\partial_i\psi_{ab}\\ &= \frac{1}{2}Nt^ct^d\Phi_{icd}\Pi_{ab} + Ng^{jk}t^c\Phi_{ijc}\Phi_{kab} - N\gamma_2\Phi_{iab},\end{aligned} \tag{9}$$

where the dynamical field Π_{ab} is defined by Eq. (7), and Φ_{iab} is defined by $\Phi_{iab} = \partial_i\psi_{ab}$. We use the lapse N, shift N^i, and spatial metric g_{ij} (the standard functions of ψ_{ab}) to simplify the principal parts of Eqs. (7)–(9). The terms on the right sides of Eqs. (7)–(9) are algebraic functions of the dynamical fields. The connection terms Γ_{cab} appearing on the right side of Eq. (8) are computed using the standard definition of Γ_{abc}, with the partial derivatives of ψ_{ab} determined from the dynamical fields by

$$\partial_t \psi_{ab} = -N\Pi_{ab} + N^i \Phi_{iab}, \tag{10}$$

$$\partial_i \psi_{ab} = \Phi_{iab}. \tag{11}$$

The parameter γ_0 that appears in these expressions is the one used by Pretorius in Eq. (5). The parameter γ_1 was introduced to control the characteristic speed of the field ψ_{ab}. And the parameter γ_2 was introduced to suppress the growth of the new constraint $\mathcal{C}_{kab} = \partial_k \psi_{ab} - \Phi_{iab}$ that arises in this first-order form of the equations. Choosing the parameter $\gamma_0 > 0$ causes the constraint $\mathcal{C}_a$ to be exponentially suppressed via Eq. (6). Choosing the parameter $\gamma_1 = -1$ makes the new system linearly degenerate, so shocks do not form from smooth initial data.[7] Choosing the parameter $\gamma_2 > 0$ in this new system causes the constraint $\mathcal{C}_{iab}$ to be exponentially suppressed,[8] because the modified Eq. (9) implies an evolution equation for $\mathcal{C}_{iab}$ having the form $\partial_t \mathcal{C}_{iab} - N^k \partial_k \mathcal{C}_{iab} \simeq -\gamma_2 N \mathcal{C}_{iab}$.

Boundary conditions for hyperbolic evolution systems are applied to the characteristic fields of those systems. The characteristic fields for the new GH evolution system, Eqs. (7)–(9), are given by

$$u^{\hat{0}}_{ab} = \psi_{ab}, \tag{12}$$

$$u^{\hat{1}\pm}_{ab} = \Pi_{ab} \pm n^i \Phi_{iab} - \gamma_2 \psi_{ab}, \tag{13}$$

$$u^{\hat{2}}_{iab} = P_i{}^k \Phi_{kab}, \tag{14}$$

where n_i is the outgoing unit normal at a point on the boundary, and $P_i{}^k = \delta_i{}^k - n_i n^k$. The characteristic fields $u^{\hat{0}}_{ab}$ have coordinate characteristic speed $-(1+\gamma_1) n_k N^k$, the fields $u^{\hat{1}\pm}_{ab}$ have speed $-n_k N^k \pm N$, and the fields $u^{\hat{2}}_{iab}$ have speed $-n_k N^k$. Characteristic fields with negative characteristic speeds propagate into the computational domain, so boundary conditions must be imposed on each characteristic field that has a negative characteristic speed. The simplest boundary condition that enforces the physical idea of no incoming waves sets each incoming characteristic speed to zero at the boundary. A similar condition, which we often find useful, freezes each incoming characteristic field to its initial value. We have also derived a set of rather more complicated constraint preserving and physical boundary conditions for this system (see Ref. [6]).

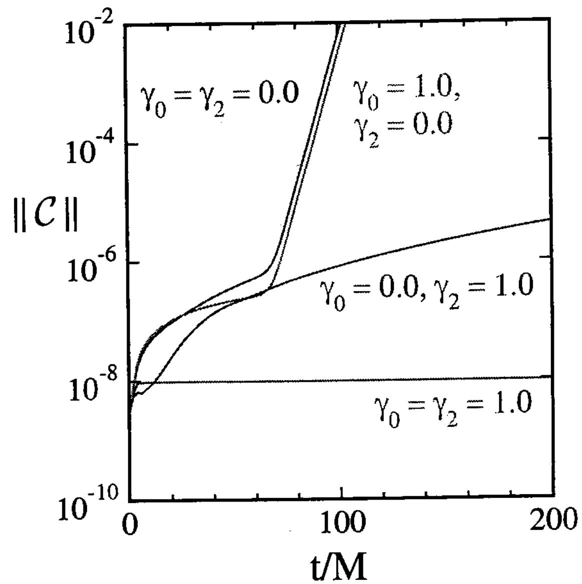

Fig. 1. Evolution of Schwarzschild initial data using different values of the constraint damping parameters γ_0 and γ_2.

The well-posedness of the initial-boundary value problem can be analyzed using the Fourier-Laplace technique[9] for the complicated physical and constraint preserving boundary conditions that we use. We have analyzed the well-posedness of this system for high-frequency perturbations of any spacetime in any GH gauge. Applying the Fourier-Laplace technique to this case yields a necessary (but not sufficient) condition for well-posedness, the so-called determinant condition;[9] failure to satisfy this condition would mean the system admits exponentially growing solutions with arbitrarily large growth rates. We have verified that this determinant condition is satisfied for the GH system using the combined set of physical and constraint preserving boundary conditions that we use.

We tested this new evolution system by evolving initial data for a Schwarzschild black hole. In these evolutions we "freeze" the values of the incoming characteristic fields on the boundaries. We performed these numerical evolutions using spectral methods as described in Ref. [10] for a range of numerical resolutions specified by N_r (the highest order radial basis function) and L_{max} (the highest order spherical harmonic). Figure 1 shows the time dependence of the constraint norm $||C||$ for several val-

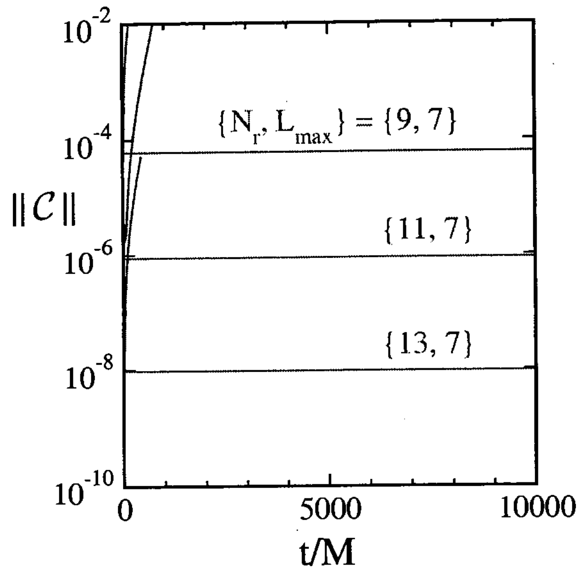

Fig. 2. Evolution of Schwarzschild initial data with $\gamma_0 = \gamma_2 = 1$ show stability and convergence for several numerical resolutions.

ues of the constraint damping parameters γ_0 and γ_2. These tests show that without constraint damping the extended evolution system is extremely unstable. But Figure 2 illustrates that with constraint damping, $\gamma_0 = \gamma_2 = 1$, the evolutions of the Schwarzschild spacetime are completely stable up to $t = 10,000M$ (and forever, we presume). These tests illustrate that both the γ_0 and the γ_2 constraint damping terms are essential.

We also tested our new constraint-preserving boundary conditions by evolving a black hole perturbed by an incoming gravitational wave (GW) pulse. We perturb Schwarzschild initial data by injecting a GW pulse through the outer boundary of the computational domain with time profile $f(t) = \mathcal{A}\, e^{-(t-t_p)^2/w^2}$ and $\mathcal{A} = 10^{-3}$, $t_p = 60M$, and $w = 10M$. Figure 3 shows the evolution of $||\mathcal{C}||$ using constraint-preserving boundary conditions (dashed curves) and simple freezing boundary conditions (solid curves). These results illustrate that the new boundary conditions are effective in preventing the influx of constraint violations. Figure 4 illustrates the time dependence of the Weyl tensor component $|\Psi_4|$ averaged over the outer boundary of the computational domain. The dashed curve (using constraint-preserving boundary conditions) shows black-hole quasi-normal

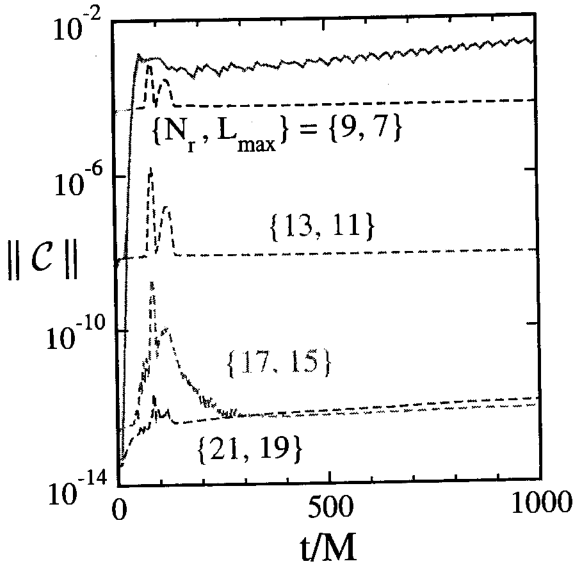

Fig. 3. Evolution of perturbed Schwarzschild spacetime. Solid curves use boundary conditions that freeze all the incoming characteristic fields, while dashed curves use constraint preserving and physical boundary conditions.

oscillations with the correct complex frequency, while the solid curve (using freezing boundary conditions) is completely unphysical. These results show that proper constraint preserving boundary conditions are essential if accurate gravitational waveforms are needed.

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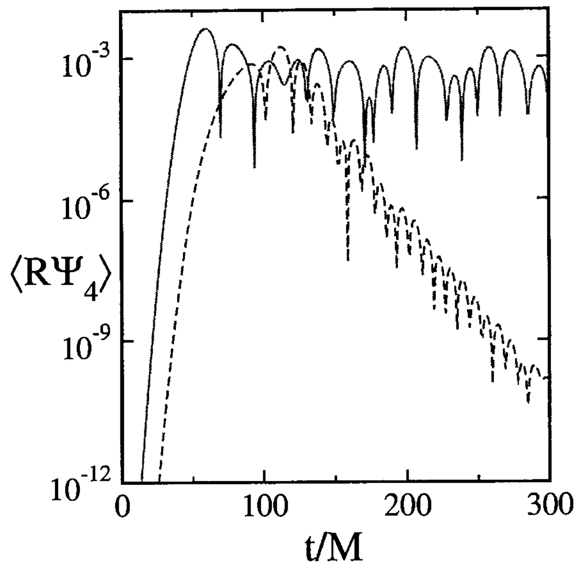

Fig. 4. Evolution of the Weyl curvature component $|\Psi_4|$ in a perturbed Schwarzschild spacetime. Solid curves use boundary conditions that freeze all the incoming characteristic fields, while dashed curves use constraint preserving and physical boundary conditions.

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