Lee Lindblom
Department of Physics, Montana State University, Bozeman, MT 59717, USA

Instabilities in Rotating Neutron Stars

The maximum angular velocity of rotating neutron stars—and hence the minimum pulsation period of pulsars—is determined by the instabilities to which these objects are subject. This paper reviews the properties of the gravitational-radiation driven instability that is presently believed to limit the rotation of neutron stars. Numerical models of these instabilities are described along with estimates of the maximum angular velocities of rotating neutron stars.

1. INTRODUCTION

Numerous pulsars have been observed with millisecond pulsation periods, and more are being discovered each year. At present the shortest of these periods is 1.56 ms in pulsar PSR1937+21,[1] followed by 1.61 ms in PSR1957+20.[2] It is of considerable theoretical interest to understand what physical mechanism limits these periods, and to determine quantitatively what those limits are. The standard model of a pulsar is a neutron star whose pulsation period is determined by the star's rotation. This paper explores the instabilities that limit the rotation rates of neutron stars and hence, in the standard model, the pulsation periods of pulsars.

The angular velocities of rotating stars are limited by the nature of the equilibrium equations as well as by the existence of unstable solutions to the dynamical equations. The equilibrium states of neutron stars are expected to be rigidly rotating within a few years of their birth. The viscosity of neutron-star matter damps out differential rotation on a time scale of about[3] $\tau \approx 10^7 T_9^2$ s, where T_9 is the temperature in units of 10^9K. Since the interior temperature is expected to drop to $T_9 \approx 1$ within a few years after its birth,[4] a neutron star is expected to be rigidly rotating after this time. Rigidly rotating stellar models can exist only if the angular velocity of the star does not exceed the "Keplerian" angular velocity of the equatorial circular orbit that coincides with the star's surface. Various studies indicate that this equilibrium limit on the angular velocity of a neutron star is given approximately by $\Omega < \Omega_{\max} \approx 0.6\sqrt{\pi G \bar{\rho}_o}$, where $\bar{\rho}_o$ is the average density of the non-rotating star of the same mass.[5,6] This limit appears to be rather insensitive to the equation of state of the stellar matter and applies to both Newtonian and to general-relativistic stellar models.

The angular velocities of rotating neutron stars may be limited further by the existence of unstable solutions to the dynamical equations. It appears that a gravitational-radiation driven instability is probably responsible for determining the maximum angular velocity of a neutron star. This instability was discovered by Chandrasekhar,[7] and was shown to be generic (i.e., it tends to make *all* rotating stars unstable) by Friedman and Schutz.[8,9] Therefore if no other physical mechanism operates, the maximum angular velocity of a neutron star is zero.

The action of the gravitational-radiation instability is not difficult to understand. Consider a neutron star that rotates with angular velocity Ω, and a small perturbation of this star having time dependence $e^{-i\omega t}$ and angular dependence $e^{im\phi}$, with m an integer. This perturbation propagates with angular velocity ω/m in the direction opposite the star's rotation (assuming $m > 0$) in sufficiently slowly rotating stars. (Note that with this sign convention, $\omega < 0$ for these perturbations.) Figure 1 depicts this situation schematically for $m = 4$. These perturbations create time-dependent mass-multipoles, which causes the star to emit gravitational radiation having negative angular momentum. The perturbation itself has negative angular momentum—since it propagates in the direction opposite the star's rotation—and so the gravitational radiation reduces the perturbation's amplitude in order to conserve angular momentum. Thus gravitational radiation damps out the perturbation. In sufficiently rapidly rotating stars these perturbations are forced to move in the opposite direction. The waves are in effect dragged along by the fluid in the star. In this case the star emits gravitational radiation having positive angular momentum. Since the angular momentum in the perturbation is negative—it still propagates against the rotational flow of the star—the perturbation's amplitude must grow in order to conserve angular momentum. Thus, any counter-rotating perturbation will become unstable when the star rotates rapidly enough to force it to corotate with the star.

This gravitational-radiation instability is quite generic. These perturbations are rather superficial and propagate much like waves on the surface of the ocean. Thus, the speed of the wave relative to the matter is rather independent of the rotation

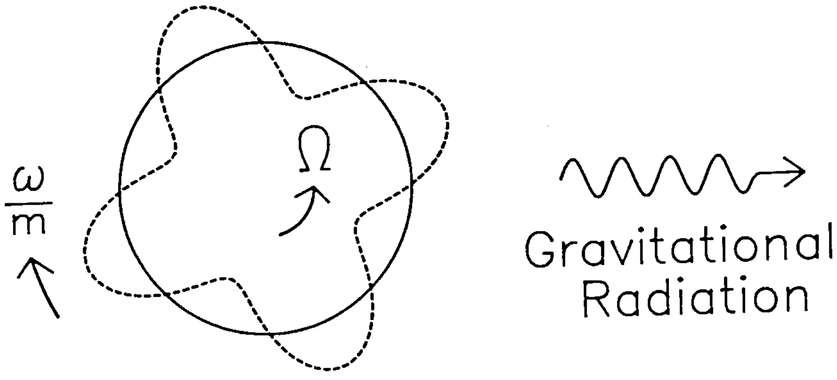

FIGURE 1 Representation of an $m = 4$ perturbation of a rotating neutron star

of the star. To a fairly good approximation, then, the angular-velocity dependence of the frequency of a perturbation is given by $\omega(\Omega) \approx \omega(0) + m\Omega$, where $\omega(0)$ is the frequency when the star is not rotating. These perturbations will reverse direction at the angular velocity where the frequency passes through zero, that is, when $\Omega \approx -\omega(0)/m$. It is easy to see why the instability is generic from this formula. The frequencies of the modes of nonrotating stars increase with m roughly as $\sqrt{m}$. Thus the angular velocity where a perturbation becomes unstable varies with m approximately as $\Omega \propto 1/\sqrt{m}$. An unstable perturbation can be found in any rotating star, therefore, simply by choosing m sufficiently large.

We know of course that all rotating stars are *not* unstable. The argument outlined above merely shows that some other physical mechanism must act to prevent gravitational radiation from driving these perturbations unstable. One such mechanism is internal dissipation in the stellar matter.[10] Viscosity and thermal conductivity quickly damp out any large gradients in the velocity or thermal perturbations. Those perturbations with angular dependence $e^{im\phi}$ have gradients that increase as m increases. Thus, the time scales for the internal dissipation mechanisms to damp out a perturbation tend to decrease as m increases. In contrast, the time scale for gravitational radiation to drive a perturbation unstable becomes very long as m gets large. This is because the radiation couples more weakly to the higher mass-multipole moments. Thus, for sufficiently large m, viscosity will suppress the gravitational-radiation instability.[11] As a consequence, sufficiently slowly rotating stars are stable.

While the presence of dissipation ensures the stability of some rotating stars, it also complicates considerably the analysis needed to determine which stars are actually stable. If the viscosity is large enough, for example, the gravitational-radiation instability can be suppressed in *all* rotating stars.[10] If the viscosity is very small, however, only the most slowly rotating stars will be stable. In order to determine which stars are stable, then, a detailed analysis of their perturbations must be carried out which includes the influences of gravitational radiation and viscosity. The remainder of this paper outlines the techniques that have been developed during the past several years to carry out this analysis, and some of the numerical results of that work are described.

2. THE THEORY OF STELLAR PULSATIONS

Consider the perturbations of a rotating star with time dependence $e^{-i\omega t}$ and angular dependence $e^{im\phi}$. (Since the analysis of such perturbations has only been completed to date in the context of Newtonian physics, the discussion here will be limited to that case.) All of the properties of such a perturbation are determined by two scalar potentials $\delta\Phi$ and δU.[12,13] The potential $\delta\Phi$ represents the perturbed Newtonian gravitational field, while δU is a potential related to the density perturbation $\delta\rho$ of the star by

$$\delta\rho = \rho\frac{d\rho}{dp}\Big(\delta U - \delta\Phi\Big), \tag{1}$$

where ρ and p are the density and pressure. Quantities not preceded by δ are equilibrium quantities. (While a more general formalism exists,[14] the equations given here apply only to barotropic perturbations, $\delta p = [dp/d\rho]\delta\rho$, of rigidly rotating stars.) The velocity of this perturbation δv^a is also determined by the potential δU. In neutron stars the dissipative forces (both viscosity and gravitational radiation) are weak in the sense that the dissipative time scales are much longer than the pulsation period $1/\omega$. Thus the dissipative effects may be ignored in the first approximation. In this case the perturbed Euler equation has a particularly simple form which determines δv^a in terms of δU:

$$\delta v^a = iQ^{ab}\nabla_b\delta U. \tag{2}$$

In this equation the tensor Q^{ab} is given by

$$Q^{ab} = -\frac{1}{\hat\omega}z^a z^b - \frac{\hat\omega}{\hat\omega^2 - 4\Omega^2}\Big(g^{ab} - z^a z^b + \frac{2i}{\hat\omega}\nabla^a v^b\Big), \tag{3}$$

where v^a is the velocity and Ω is the angular velocity of the unperturbed star, z^a is the unit vector parallel to the rotation axis, and $\hat\omega = \omega - m\Omega$. The Euclidean metric g_{ab} (i.e., the identity matrix in Cartesian coordinates) and its inverse g^{ab} are used to raise and lower tensor indices. The covariant derivative ∇_a associated with g_{ab} is just the partial derivative $\partial/\partial x^a$ in Cartesian coordinates.

The two scalar potentials δU and $\delta\Phi$ are determined by the perturbed mass-conservation and gravitational-potential equations. These form a system of second-order (in most cases elliptic) equations for the two potentials:

$$\nabla_a\left(\rho Q^{ab}\nabla_b\delta U\right)+\hat{\omega}\rho\frac{d\rho}{dp}\left(\delta\Phi-\delta U\right)=0, \tag{4}$$

$$\nabla^a\nabla_a\delta\Phi+4\pi G\rho\frac{d\rho}{dp}\left(\delta\Phi-\delta U\right)=0, \tag{5}$$

where G is Newton's gravitation constant. The boundary condition $\delta\Phi\to 0$ must be imposed in the limit $r\to\infty$, where r is the spherical radial coordinate. The frequency of the perturbation ω plays the role of an eigenvalue in these equations. Although the most difficult step in the problem, it is reasonably straightforward to solve these equations numerically for the frequency ω and the eigenfunctions $\delta\Phi$ and δU even in rapidly rotating stars. The needed techniques are described in detail elsewhere[6,15] and will not be reviewed here.

It is easy to evaluate the effects of (weak) dissipation on the pulsation of a star once the frequency ω and the potentials δU and $\delta\Phi$ have been determined by the non-dissipative equations as outlined above. To this end, it is useful to introduce the following "energy" associated with the pulsations:

$$E(t)=\tfrac{1}{2}\int\left[\rho\delta v^a\delta v_a^*+\tfrac{1}{2}\left(\delta\rho\delta U^*+\delta\rho^*\delta U\right)\right]d^3x, \tag{6}$$

where * represents complex conjugation. This energy is conserved, $dE/dt=0$, in the absence of dissipation. In general its time derivative is determined by the equations for the evolution of a viscous fluid coupled to gravitational radiation:

$$\frac{dE}{dt}=-\int\left(2\eta\delta\sigma^{ab}\delta\sigma_{ab}^*+\zeta\delta\sigma\delta\sigma^*\right)d^3x-\hat{\omega}\sum_{l=l_{\min}}^{\infty}N_l\,\omega^{2l+1}\delta D_l^m\delta D_l^{*m}. \tag{7}$$

In this expression $l_{\min}$ is the larger of 2 or $|m|$. The functions ζ and η are the bulk- and shear-viscosity coefficients, while $\delta\sigma^{ab}$ and $\delta\sigma$ are the shear and expansion of the perturbed fluid motion:

$$\delta\sigma^{ab}=\tfrac{1}{2}\left(\nabla^a\delta v^b+\nabla^b\delta v^a-\tfrac{2}{3}g^{ab}\nabla_c\delta v^c\right), \tag{8}$$

$$\delta\sigma=\nabla_a\delta v^a. \tag{9}$$

The gravitational-radiation energy loss is determined by the multipole moment

$$\delta D_l^m=\int\delta\rho\, r^l Y_l^{*m}d^3x \tag{10}$$

and the coupling constant N_l (with c the speed of light),

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \tag{11}$$

When dissipation is present in the star, it is convenient to represent the time dependence of a perturbation in the form $e^{-i\omega t - t/\tau}$, where ω and τ are real. The energy $E(t)$ defined in Eq. (6) is a real function that is quadratic in the perturbation variables. Thus, its time derivative is given by

$$\frac{dE}{dt} = -\frac{2E}{\tau}. \tag{12}$$

This formula can be used to evaluate $1/\tau$. The integrals in Eqs. (6) and (7) that determine E and dE/dt may be evaluated to lowest order (in the strength of the dissipative forces) by using nondissipative values of the frequency ω and the potentials δU and $\delta\Phi$. Once evaluated, these integrals determine $1/\tau$ via Eq. (12). It is convenient to decompose the imaginary part of the frequency into contributions from each of the dissipative forces: $1/\tau = 1/\tau_\zeta + 1/\tau_\eta + 1/\tau_{GR}$. These individual damping times are defined—using Eqs. (7) and (12)—by the integrals

$$\frac{1}{\tau_\zeta} = \frac{1}{2E} \int \zeta \delta\sigma \delta\sigma^* d^3x, \tag{13}$$

$$\frac{1}{\tau_\eta} = \frac{1}{E} \int \eta \delta\sigma^{ab} \delta\sigma^*_{ab} d^3x, \tag{14}$$

$$\frac{1}{\tau_{GR}} = \frac{\hat{\omega}}{2E} \sum_{l=l_{\min}}^{\infty} N_l \omega^{2l+1} \delta D_l^m \delta D_l^{*m}. \tag{15}$$

Consider a sequence of rotating stars—parameterized by the angular velocity Ω—of fixed mass and equation of state. A perturbation of one of these stars is stable whenever the imaginary part of the frequency of that perturbation $1/\tau(\Omega)$ is positive. Stars whose angular velocities do not exceed the smallest root of the equation $1/\tau(\Omega_c) = 0$ are stable (assuming that the nonrotating star in this sequence is stable). The problem of determining the maximum angular velocity of a neutron star has been reduced, therefore, to finding the values of the critical angular velocities Ω_c that are the roots of the equation

$$0 = \frac{1}{\tau(\Omega_c)} = \frac{1}{\tau_\zeta(\Omega_c)} + \frac{1}{\tau_\eta(\Omega_c)} + \frac{1}{\tau_{GR}(\Omega_c)}. \tag{16}$$

The integrals in Eqs. (6) and (13)–(15) are easily evaluated once the nondissipative pulsation problem has been solved. Then Ω_c is determined from Eq. (16) for each solution to the perturbation equations. The smallest of these critical angular velocities is, therefore, the maximum angular velocity of a stable neutron star.

3. A NUMERICAL EXAMPLE

In this section the techniques for analyzing the pulsations and stability of rotating neutron stars are illustrated with a numerical example. For simplicity, attention is limited here to neutron stars based on the idealized polytropic equations of state: $p = \kappa\rho^{1+1/n}$. The index n determines the "stiffness" of the equation of state. Realistic equations of state for neutron-star matter have $n \approx 1$, and so this discussion will focus on this value. Most of the results presented here are independent of the parameter κ in the polytropic equation of state. For numerical purposes its value is chosen to make the physical size of these models comparable to those based on more realistic equations of state.

The first task in analyzing the stability of rotating stellar models is to solve Eqs. (4) and (5) for the frequencies ω and the eigenfunctions δU and $\delta\Phi$ that describe the pulsations of a star in the absence of dissipation. As the discussion in section 1 indicates, the modes of primary interest here are those which propagate in the direction opposite the star's rotation. These are the modes which may become unstable via the emission of gravitational radiation in sufficiently rapidly rotating stars. The modes that are the most susceptible to this instability are those that reduce to the $l = m$ f-modes in nonrotating stars.[16] Table 1 gives the frequencies of these modes for $2 \leq l = m \leq 6$ in nonrotating $n = 1$ polytropes.[6] The frequencies are given here in units of $\Omega_o = \sqrt{\pi G\bar{\rho}_o}$, where $\bar{\rho}_o$ is the average density of these nonrotating stars. The ratios ω/Ω_o are independent of the parameter κ that appears in the polytropic equation of state. The angular velocity dependence of these frequencies is most conveniently expressed in terms of the dimensionless functions $\alpha(\Omega)$ defined by

$$\alpha(\Omega) = \frac{\omega(\Omega) - m\Omega}{\omega(0)}. \tag{17}$$

These functions are displayed in Figure 2 for the $l = m$ f-modes of $n = 1$ polytropes.[6] The $\alpha(\Omega)$ are very slowly varying with $\alpha \approx 1$ over the entire range of angular velocities. This fact justifies the argument given in section 1 that the frequency of these modes is given approximately by $\omega(\Omega) \approx \omega(0) + m\Omega$. Also displayed in Figure 2 are the post-Newtonian versions of these functions.[17] These were computed for a reasonably relativistic ($GM/c^2R \approx 0.2$) sequence of $n = 1$ polytropes with post-Newtonian mass $M = 1.4M_\odot$. Thus, Figure 2 illustrates the errors that result from the neglect of general-relativistic effects.

Before the dissipation time scales τ_ζ and τ_η can be determined, expressions for the bulk- and shear-viscosity coefficients ζ and η must be given.[3] Bulk viscosity arises in neutron-star matter because the pressure and density perturbations become slightly out of phase due to the long time scale needed for the weak interactions to reestablish local thermodynamic equilibrium. Sawyer[18] calculates the bulk viscosity of neutron-star matter to be $\zeta = 6.0 \times 10^{-59}\rho^2\omega^{-2}T^6$ in cgs units. Shear viscosity in neutron-star matter is primarily the result of neutron-neutron scattering (when the temperature exceeds the superfluid-transition temperature). Flowers and Itoh[19] calculate this form of shear viscosity to be approximately $\eta = 347\rho^{9/4}T^{-2}$.

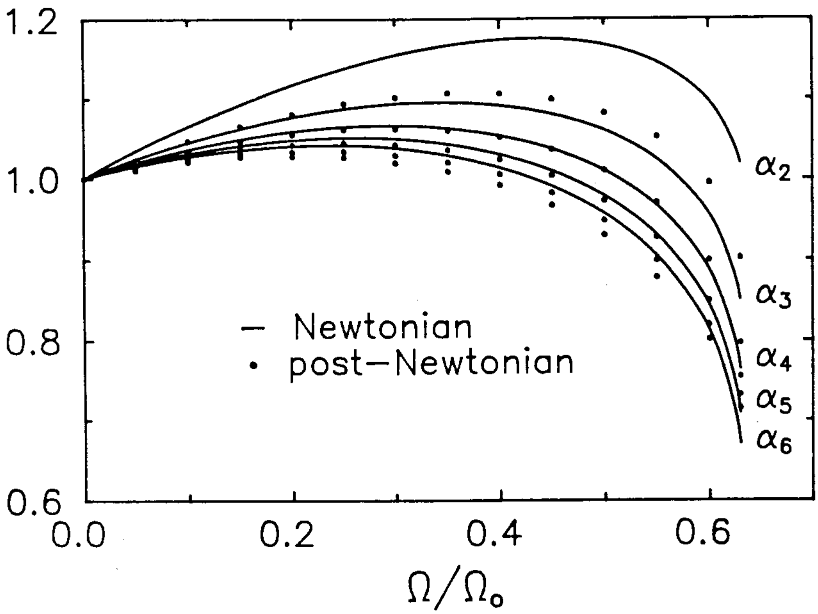

FIGURE 2 Frequencies of the $l = m$ f-modes are represented in terms of the functions $\alpha_m(\Omega)$ defined in Eq. (17)

Given these formulas for ζ and η, the frequency ω, and the eigenfunctions δU and $\delta\Phi$, it is straightforward to perform the numerical integrals needed to evaluate the expressions in Eqs. (13)–(15) for τ_ζ, τ_η, and τ_{GR}. These damping times are given in Table 1 for the nonrotating stellar models described in this study.[20] The viscous damping times τ_ζ and τ_η are given for neutron stars having the temperature $T = 10^9$K. These damping times scale simply with temperature: τ_ζ as T^{-6} and τ_η as T^2.

TABLE 1 Frequencies and Damping Times of the $l = m$ f-modes of Nonrotating $n = 1$ Polytropes. These quantities are given in units of $\Omega_o = \sqrt{\pi G \bar{\rho}_o}$, where $\bar{\rho}_o$ is the average density.

$l = m$	$-\omega/\Omega_o$	$\tau_{GR}\Omega_o$	$\tau_\eta\Omega_o$	$\tau_\zeta\Omega_o$
2	1.415	2.43×10^2	6.00×10^{11}	1.85×10^{17}
3	1.959	1.06×10^4	6.15×10^{11}	2.66×10^{17}
4	2.350	5.21×10^5	7.00×10^{11}	4.90×10^{17}
5	2.667	2.95×10^7	7.98×10^{11}	9.14×10^{17}
6	2.939	1.92×10^9	8.99×10^{11}	1.65×10^{18}

The angular-velocity dependence of the damping times are most conveniently expressed as dimensionless functions:

$$\gamma(\Omega) = \frac{\omega(\Omega)}{\omega(0)}\left[\frac{\tau_\eta(0)}{\tau_{GR}(0)}\frac{\tau_{GR}(\Omega)}{\tau_\eta(\Omega)}\right]^{1/(2l+1)}, \tag{18}$$

$$\epsilon(\Omega) = \frac{\tau_\zeta(0)}{\tau_\eta(0)}\frac{\tau_\eta(\Omega)}{\tau_\zeta(\Omega)}. \tag{19}$$

These functions are independent of the temperature of the neutron-star matter and the parameter κ that appears in the polytropic equation of state. Figures 3 and 4 illustrate these functions for the $l = m$ f-modes of $n = 1$ polytropes.[20] These functions are very slowly varying except for the very highest angular velocities. The effects of bulk viscosity are suppressed in spherical stars because the nonradial pulsations have very little expansion $\delta\sigma$ associated with them. In very rapidly rotating stars, however, spherical symmetry is broken and the pulsations are no longer constrained to have small $\delta\sigma$. Figure 4 illustrates that as a consequence the τ_ζ are much shorter in rapidly rotating stars.

Having determined the angular-velocity dependence of the damping times $\tau_\zeta(\Omega)$, $\tau_\eta(\Omega)$, and $\tau_{GR}(\Omega)$, Eq. (16) can be solved for the critical angular velocities Ω_c where the perturbation becomes unstable. The numerical determination of Ω_c is made easier by transforming Eq. (16) into the form

$$\Omega_c = \frac{\omega(0)}{m}\left\{\alpha(\Omega_c) + \gamma(\Omega_c)\left[\frac{\tau_{GR}(0)}{\tau_\eta(0)} + \frac{\tau_{GR}(0)}{\tau_\zeta(0)}\epsilon(\Omega_c)\right]^{1/(2l+1)}\right\}. \tag{20}$$

This equation is easy to solve numerically because $\alpha(\Omega_c) \approx \gamma(\Omega_c) \approx \epsilon(\Omega_c) \approx 1$. Eq. (20) must be evaluated separately for each solution of the perturbation equations. The smallest of these Ω_c for a given sequence of stellar models is the maximum

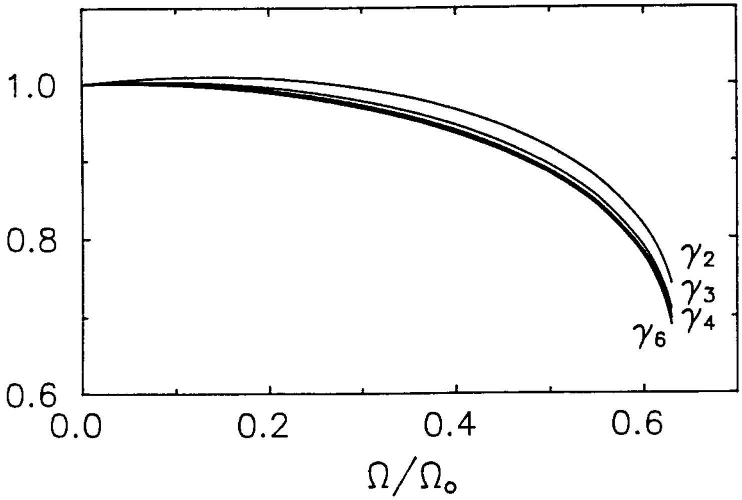

FIGURE 3 Angular-velocity dependence of the gravitational-radiation damping times as represented by the functions $\gamma_m(\Omega)$ defined in Eq. (3.2).

angular velocity with which a stable star may rotate. Since the viscosity of neutron-star matter depends on the temperature of the star, so too will these critical angular velocities. Figure 5 illustrates the smallest Ω_c for a range of neutron-star temperatures. These Ω_c are displayed as ratios, with $\Omega_{\max}$ the angular velocity above which mass shedding occurs. For the $n = 1$ polytropes considered here $\Omega_{\max} = 0.639\Omega_o$. Figure 5 shows that the gravitational-radiation instability is completely suppressed in neutron stars except for those with temperatures in the range 10^7 to about 10^{10}K. Shear viscosity suppresses the instability for lower temperatures while bulk viscosity suppresses it for higher temperatures. The analysis described here has not taken into account the superfluid nature of neutron-star matter at temperatures below about 10^9K. A preliminary investigation[21] indicates that dissipation in the superfluid state due to electron-vortex scattering completely suppresses the gravitational-radiation instability in all neutron stars cooler than the superfluid-transition temperature, $T \approx 10^9$K.

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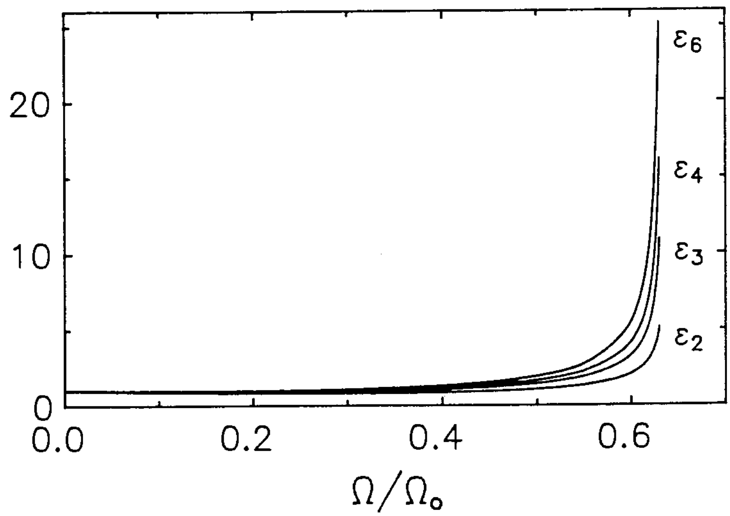

FIGURE 4 Angular-velocity dependence of the viscous damping times of the $l = m$ f-modes as represented by the functions $\epsilon_m(\Omega)$ defined in Eq. (19).

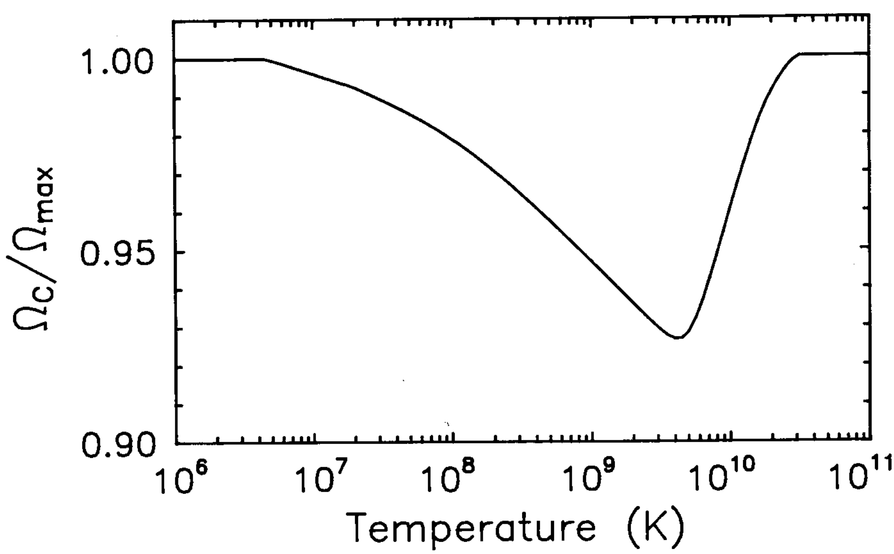

FIGURE 5 Critical angular velocities for rotating $n = 1$ polytropes.

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