Neutron Star Pulsations and Instabilities

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Abstract

Gravitational radiation (GR) drives an instability in certain modes of rotating stars. This instability is strong enough in the case of the r-modes to cause their amplitudes to grow on a timescale of tens of seconds in rapidly rotating neutron stars. GR emitted by these modes removes angular momentum from the star at a rate which would spin it down to a relatively small angular velocity within about one year, if the dimensionless amplitude of the mode grows to order unity. A pedagogical level discussion is given here on the mechanism of GR instability in rotating stars, on the relevant properties of the r-modes, and on our present understanding of the dissipation mechanisms that tend to suppress this instability in neutron stars. The astrophysical implications of this GR driven instability are discussed for young neutron stars, and for older systems such as low mass x-ray binaries. Recent work on the non-linear evolution of the r-modes is also presented.

1 Introduction

The non-radial pulsations of stars couple to gravitational radiation (GR) in general relativity theory [1, 2], and the GR produced by these oscillations carries away energy and angular momentum from the star. In non-rotating stars the effect of these GR losses is dissipative, and the pulsations of the star are damped. Chandrasekhar first noted [3, 4] that in rotating stars the situation can be quite different: the emission of GR causes the amplitudes of certain modes to grow. The mechanism that drives this GR instability is fairly easy to understand: Modes that propagate in the direction opposite the star's rotation (as seen in the co-rotating frame of the fluid) have *negative* angular momentum, because these modes lower the total angular momentum of the star. In a rotating star some of these counter rotating modes are dragged forward and appear to an inertial observer to propagate in the same direction as the star's rotation. Such modes, as illustrated in Fig. 1, emit *positive* angular momentum GR since the density and momentum perturbations appear to an observer at infinity to be rotating in the same direction as the star. The angular momentum removed by GR lowers the (already negative) angular momentum of such a mode, and therefore the amplitude of the mode grows.

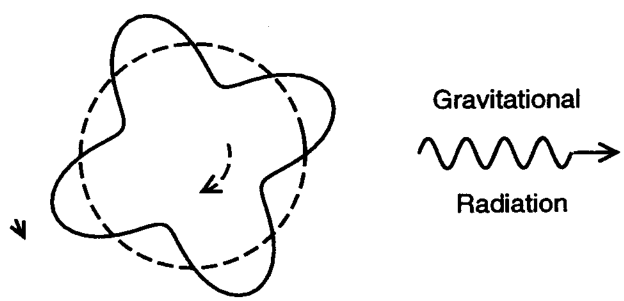

Figure 1: A counter-rotating mode (solid curve) that is dragged forward by the rotation of the background star (dashed curve) is driven unstable by the emission of gravitational radiation.

This GR driven instability was first studied extensively by Friedman and Schutz [5, 6] for the fundamental (f-) modes of rotating stars. They demonstrated that the GR instability has the remarkable property that it makes *every* rotating perfect fluid star unstable in general relativity theory. This discovery sparked an interest in the possibility that GR might play a significant role in the evolution of real neutron stars. Does the GR instability determine the maximum spin rate of pulsars? Is the GR emitted by an unstable rapidly rotating neutron star detectable? Unfortunately the generic nature of this destabilizing process does not guarantee that it plays any role at all in real neutron stars. Internal dissipation (*e.g.*, viscosity) within a star tends to damp the pulsations that are driven unstable by GR. If the internal dissipation is sufficiently strong, then the GR instability can even be completely suppressed [7, 8]. Detailed calculations of the effects of GR and internal dissipation on the f-modes of rotating stars revealed that the GR instability is effective only in very rapidly rotating stars [9, 10, 11, 12]. Stars with angular velocities smaller than some critical value, $\Omega < \Omega_c$ are stable, while those rotating more rapidly, $\Omega > \Omega_c$, are subject to the GR instability. This critical angular velocity, Ω_c, is depicted in Fig. 2 for realistic neutron-star models. The strength of the internal dissipation processes in neutron stars is temperature dependent, and consequently the critical angular velocity Ω_c is temperature dependent as well. Figure 2 illustrates that the GR instability is completely suppressed in the f-modes except when the temperature of the neutron star lies in the range, $10^7 < T < 10^{10}$K. Further, the internal dissipation is so strong that the f-modes are never unstable unless the angular velocity of the star exceeds $0.91\Omega_{\rm max}$. Thus the GR instability in the f-modes cannot significantly reduce the spin of a neutron star below the maximum, and substantial amounts of GR cannot be emitted by this process.

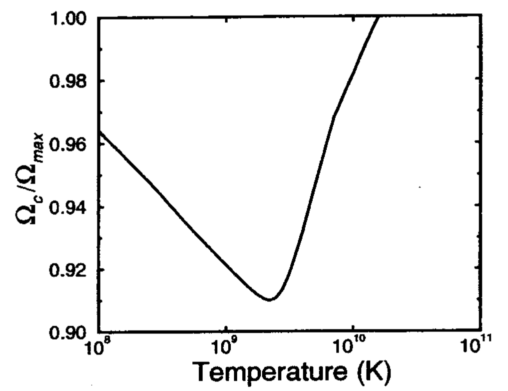

Figure 2: Temperature dependence of the critical angular velocity Ω_c in rotating neutron stars: an f-mode is driven unstable by gravitational radiation when the star's angular velocity exceeds Ω_c.

This pessimistic view of the GR instability began to change when Andersson [13] and Friedman and Morsink [14] showed that the r-modes were also subject to the GR instability. Indeed they showed that *all* the r-modes are driven unstable by GR in *all* rotating perfect fluid stars. Subsequent calculations by Lindblom, Owen and Morsink [15] showed that the GR instability in the r-modes was also strong enough to overcome the most common internal dissipation processes in neutron-star matter, even in relatively slowly rotating stars. Thus the GR instability in the r-modes appears capable of significantly reducing the angular momenta of rotating neutron stars, and the GR emitted during such spin-down events may well be detectable by LIGO [16]. The remainder of this paper discusses recent developments related to the GR instability of the r-modes. Section 2 discusses the basic properties of the r-modes and their GR instability. Section 3 considers the astrophysical scenarios in which the r-mode GR instability may play an important role. Section 4 discusses recent work on the non-linear hydrodynamic evolution of the r-modes. And finally in Section 5 a set of important but presently unresolved issues is briefly discussed.

2 Gravitational Radiation Instability in the r-Modes

The r-modes (also called rotation dominated modes, inertial modes, or Rossby waves) are oscillations of rotating stars whose restoring force is the Coriolis force [17]. These modes are primarily velocity perturbations, which for slowly rotating barotropic stars have the simple analytical form

$$\delta\vec{v} = \alpha R\Omega \left(\frac{r}{R}\right)^m \vec{Y}^B_{mm} e^{i\omega t} + \mathcal{O}(\Omega^3), \tag{1}$$

where α is the dimensionless amplitude of the mode; R and Ω are the radius and angular velocity of the equilibrium star; $\vec{Y}^B_{lm} = \hat{r} \times r\vec{\nabla} Y_{lm}/\sqrt{l(l+1)}$ is the magnetic-type vector spherical harmonic; and ω is the frequency of the mode. The associated density perturbation, $\delta\rho = \mathcal{O}(\Omega^2)$, vanishes at lowest order. Because the Coriolis force dominates, the frequencies of the r-modes are independent of the equation of state and are proportional to the angular velocity of the star (at lowest order),

$$\omega = -\frac{(m-1)(m+2)}{m+1}\Omega + \mathcal{O}(\Omega^3). \tag{2}$$

The velocity field of the r-mode, Eq. (1), is everywhere orthogonal to the radial direction $\hat{r}$, and has an angular structure determined by Y_{mm}. Figure 3 gives equatorial and polar views of this velocity field for the $m = 2$ r-mode, which plays the dominant role in the GR instability. Figure 4 shows another view of the same field in standard polar coordinates (θ, φ). The four circulation zones propagate through the fluid with angular velocity $-\frac{1}{3}\Omega$, toward the left in Fig. 4. The fluid elements respond by moving on paths determined by the Lagrangian displacement, $\vec{\xi} = -i\delta\vec{v}/(\omega+m\Omega)$. To first order these are ellipses, with θ-dependent eccentricities, as illustrated on the left side of Fig. 4.

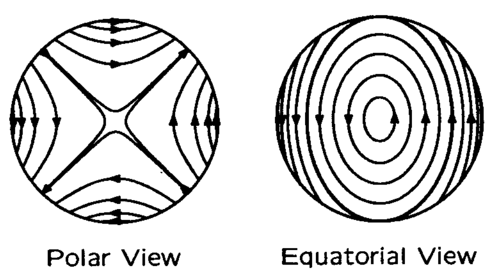

Figure 3: Polar and equatorial views of the flow pattern of the $m = 2$ r-mode. This velocity field propagates through the fluid with angular velocity $\frac{2}{3}\Omega$ relative to the inertial frame, and $-\frac{1}{3}\Omega$ relative to the fluid.

The effects of dissipation (*i.e.*, viscosity and GR) on the evolution of the r-mode are most easily studied by considering $\tilde{E}$, the energy of the perturbation (as measured in the co-rotating frame of the fluid). To lowest order in Ω, $\tilde{E}$ is given by

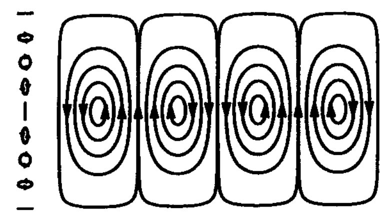

Figure 4: Polar coordinate (θ, φ) representation of the flow pattern of the $m = 2$ r-mode. The pattern moves past the individual fluid elements which move on small elliptical paths as illustrated on the left.

$$\tilde{E} = \frac{1}{2}\int \rho\, \delta\vec{v}^* \cdot \delta\vec{v}\, d^3x + \mathcal{O}(\Omega^4). \tag{3}$$

This energy is conserved in the absence of dissipation, and more generally satisfies [15]

$$\begin{aligned} \frac{d\tilde{E}}{dt} &= -\omega(\omega + m\Omega) \sum_{l\geq m} N_l\, \omega^{2l} \left[|\delta D_{lm}|^2 + \frac{4l|\delta J_{lm}|^2}{c^2(l+1)} \right] \\ &\quad - \int \left(2\eta\, \delta\sigma^*_{ab}\delta\sigma^{ab} + \zeta\, \delta\sigma^*\delta\sigma \right) d^3x, \end{aligned} \tag{4}$$

where $N_l = 4\pi G(l+1)(l+2)\{c^{2l+1}l(l-1)[(2l+1)!!]^2\}^{-1}$ are positive constants; and δD_{lm} and δJ_{lm} are the mass and current multipole moments of the perturbation,

$$\delta D_{lm} = \int \delta\rho\, r^l Y^*_{lm} d^3x, \tag{5}$$

$$\delta J_{lm} = \int r^l \left(\rho\, \delta\vec{v} + \delta\rho\, \vec{v} \right) \cdot \vec{Y}^{B*}_{lm}\, d^3x. \tag{6}$$

The second term on the right side of Eq. (4) represents the dissipation due to the shear and bulk viscosity of the fluid: η and ζ are the viscosity coefficients, and $\delta\sigma^{ab}$ and $\delta\sigma$ are the shear and expansion of the perturbed fluid respectively. These viscosity terms in Eq. (4) always decrease the energy $\tilde{E}$ and so tend to damp the r-modes. The first term on the right side of Eq. (4) represents the effect of GR on the perturbation. The sign of this term is determined by the sign of $\omega(\omega + m\Omega)$, the product of the frequencies in the inertial and rotating frame. This product,

$$\omega(\omega + m\Omega) = -\frac{2(m-1)(m+2)}{(m+1)^2}\Omega^2 < 0, \tag{7}$$

is negative for the r-modes, thus GR tends to drive the r-modes toward instability. Further this destabilizing force is *generic* [13, 14]: GR drives all the r-modes in all rotating stars (*i.e.*, for all values of m and Ω) towards instability.

To evaluate the relative strengths of the destabilizing GR force and the dissipative viscous forces, it is convenient to define the combined dissipative timescale $1/\tau$,

$$\frac{1}{\tau} = -\frac{1}{2\tilde{E}}\frac{d\tilde{E}}{dt} = -\frac{1}{\tau_{GR}} + \frac{1}{\tau_V}, \tag{8}$$

which is just the imaginary part of the frequency of the mode. The integrals on the right sides of Eqs. (3)–(6) are easily performed to determine the GR and the viscous contributions to $1/\tau$ respectively. Using Newtonian stellar models based on fairly realistic neutron-star matter these timescales are [15, 18]:

$$\frac{1}{\tau_{GR}} = \frac{1}{3.3\text{s}}\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right)^3, \tag{9}$$

$$\frac{1}{\tau_V} = \frac{1}{3\times 10^8\text{s}}\left(\frac{10^9\text{K}}{T}\right)^2 + \frac{1}{2\times 10^{11}\text{s}}\left(\frac{T}{10^9\text{K}}\right)^6\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right). \tag{10}$$

For small angular velocities, $\Omega \ll \sqrt{\pi G\bar{\rho}}$, the GR timescale is very large so viscous dissipation always dominates, $1/\tau_{GR} \ll 1/\tau_V$. Thus neutron stars are always stable in this limit. Conversely, when Ω is sufficiently large the GR timescale is shorter than the viscous timescale and the neutron star is unstable. The critical angular velocity Ω_c,

$$\frac{1}{\tau(\Omega_c)} = 0, \tag{11}$$

marks the boundary between stability and instability. Since the viscosities are temperature dependent in neutron-star matter, so too is Ω_c. The solid curve in Fig. 5 illustrates the temperature dependence of Ω_c for the r-modes. The minimum of this curve occurs at $\min\Omega_c = 0.045\Omega_{\max}$. For comparison Fig. 5 also illustrates Ω_c for the GR instability in the f-modes. It is obvious that the r-modes are driven unstable by GR over a far wider range of angular velocities than the f-modes. Thus the GR instability in the r-modes may play an interesting role in limiting the angular velocities of neutron stars, and the GR emitted during a spin-down event may well be detectable.

3 Astrophysical Implications

Two astrophysical scenarios have been proposed in which the GR instability of the r-modes might play an interesting role in the evolution of real neutron stars. These are illustrated by the two evolution curves, A and B, in Fig. 6. In scenario A a rapidly rotating neutron star is formed with a very high temperature ($T \geq 10^{11}$K) as the result of the gravitational collapse of the neutron-star progenitor [15]. In this scenario the star cools within a few seconds to a point that lies above the r-mode

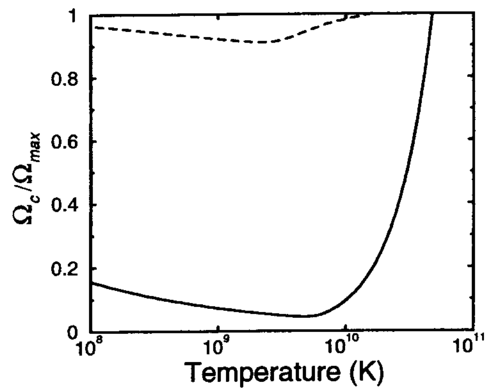

Figure 5: Temperature dependence of the critical angular velocity Ω_c for rotating neutron stars. Solid curve gives Ω_c for the instability in the $m = 2$ r-mode, while for comparison the dashed curve gives Ω_c for the f-modes.

instability curve (the dashed curve in Fig. 6). The amplitude of the r-mode then grows exponentially (with a timescale of about 40 s for a very rapidly rotating star), and becomes large within a few minutes. If the dimensionless r-mode amplitude α saturates (by some unknown process) with a value of order unity, it would take about 1 y for the star to spin down to a point where stability is re-gained [16]. In this scenario a star could lose up to 95% of its angular momentum, and up to about 99% of its rotational kinetic energy by emitting GR. This scenario provides a natural explanation for the lack of rapidly rotating neutron stars in young supernova remnants. The GR emitted in this scenario might be detectable for neutron stars as far away as the Virgo cluster [16].

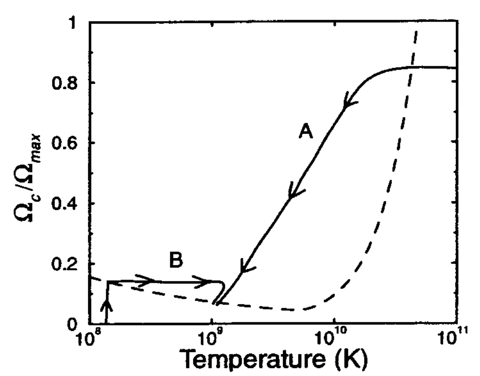

Figure 6: Rotating neutron stars may become unstable to the r-mode instability in two ways: *a)* hot young rapidly rotating stars may cool along path A, become unstable, and finally spin down to a small angular velocity; or *b)* old, cold, slowly rotating stars may be spun up by accretion along path B, becoming unstable, then heated by the growing r-mode, and finally spun down to a smaller angular velocity.

In scenario B an old, cold, slowly rotating neutron star is spun up by accreting high angular momentum material from a companion star [19, 20]. Once the neutron star's angular velocity reaches the critical value Ω_c, the amplitude of the unstable r-mode grows exponentially. It was once thought that in this situation the amplitude of the unstable mode would grow until the rate of angular momentum lost to GR just balances the amount gained from accretion [21]. However Levin [22] has shown that viscous dissipation in the growing r-mode rapidly increases the temperature of the low specific-heat neutron-star matter. This moves the star along the horizontal section of the evolution curve B in Fig. 6. At some point the r-mode amplitude saturates (by some unknown mechanism) and thermal equilibrium is established between viscous heating and neutrino cooling. The star then spins down by emitting GR until stability is regained. It has been suggested that this scenario provides the explanation for the relatively narrow range of rotation periods observed for the neutron stars in low mass x-ray binaries (LMXBs) [23].

These scenarios are just rough sketches and considerable work has been (and continues to be) done to fill in the details and see whether they represent realistic astrophysical possibilities. In the case of scenario B for example, it is clear that the sketch given above is too simple. The core temperatures of neutron stars in accreting systems like the LMXBs are expected to be in the range 10^8-10^9K [24, 25]. Simple shear viscosity gives rise to $\Omega_c \leq 0.16\Omega_{\max}$ in this temperature range, as seen in Fig. 5. This upper limit (of about 160 Hz) on the angular velocities of accreting systems is in conflict with the observed 300 Hz spin frequencies of the neutron stars in LMXBs, and the 600 Hz frequencies of pulsars that are believed to have been spun up in LMXB-like systems. Thus some additional dissipation mechanism must act to suppress the r-mode instability in these accreting systems. It was suggested [15] that additional dissipative effects associated with the superfluid transition in the neutron-star matter at about 10^9K might effectively suppress the r-mode instability. However, calculations have shown that the dominant superfluid-dissipation mechanism (mutual friction) is generally not effective in suppressing the r-mode instability [26]. Bildsten and Ushomirsky [27] have suggested that viscous dissipation in the boundary layer between the liquid core and the solid crust of a neutron star might provide the needed stability. At present this appears to be the most likely possibility.

At the interface between a viscous fluid and a solid (*e.g.,* the crust of a neutron star) the fluid velocity must match the velocity of the solid. Therefore viscosity significantly modifies the velocity field of an r-mode, at least in the neighborhood of the crust-core boundary. The solution of the viscous fluid equations in this boundary region [27, 28, 29] shows that the r-mode velocity field is modified significantly only in a thin layer with scale-height d,

$$d = \sqrt{\frac{\eta}{2\rho\Omega}} \approx 0.5\text{cm}\left(\frac{10^9\text{K}}{T}\right)\left(\frac{\sqrt{\pi G\bar{\rho}}}{\Omega}\right)^{1/2}. \tag{12}$$

The magnitude of the shear of the fluid in this boundary layer is approximately $|\delta\sigma^{ab}| \approx |\vec{\nabla}\delta\vec{v}| \approx |\delta\vec{v}|/d$, which is larger by the factor $R/d \approx 10^6$ than the shear of the in-viscid r-mode velocity field. The formation of a rigid crust therefore increases the total dissipation due to shear viscosity by approximately the factor R/d. The viscous timescale for the $m = 2$ r-mode (using a typical neutron-star model) then becomes [29],

$$\tau_V = \begin{pmatrix} 230\,\mathrm{s}, & T < 10^9\mathrm{K} \\ 530\,\mathrm{s}, & T > 10^9\mathrm{K} \end{pmatrix} \left(\frac{T}{10^9\mathrm{K}}\right) \left(\frac{\sqrt{\pi G\bar{\rho}}}{\Omega}\right)^{1/2}. \tag{13}$$

Figure 7 illustrates the critical angular velocity Ω_c for the r-mode GR instability including the effects of this boundary-layer dissipation. The solid curves are based on neutron-star models from a number of realistic equations of state. Figure 7 illustrates that dissipation in the boundary layer significantly increases the stability of the r-modes. This suggests that rapidly rotating neutron stars, such as the 1.6 ms pulsars, are consistent with a spin-up process that operates in the $10^8 - 10^9$ K temperature range. And this suggests that the clustering of spin frequencies in the LMXBs may not be due to the GR instability in the r-modes. However, additional work is needed to understand fully whether scenario B operates in real neutron stars or not. In particular the effects of a semi-rigid crust (which tend to reduce the boundary layer dissipation) have not been included in Fig. 7 [30], nor have the effects of the neutron star's magnetic field [31, 32], nor have other possible effects of the superfluid core (*e.g.*, the [unlikely?] possibility that the core vortices are pinned to the crust, or a possible dissipative interaction between neutron vortices and magnetic flux tubes).

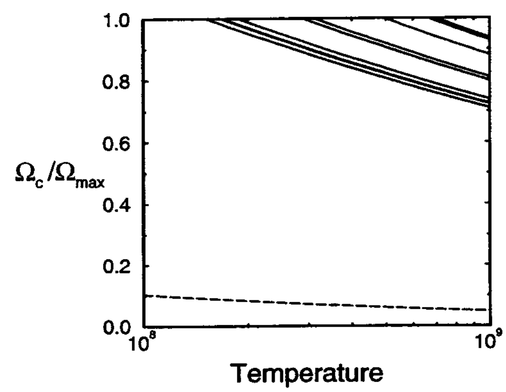

Figure 7: Solid curves represent Ω_c for neutron star models (from a variety of realistic equations of state) with rigid crust. Dashed curve represents the stability curve for a neutron star without crust.

At present scenario A seems more likely to play an interesting role in the astrophysics of neutron stars. But a number of important questions remain unresolved about this mechanism as well. In particular it is not yet fully understood what role the formation of a solid crust, non-linear hydrodynamic effects, or the influence of a magnetic field might play. Let us consider first the role of a solid crust. When a neutron star is about 30 s old and the temperature falls to about 10^{10}K, a solid crust begins to form initially at densities of about $\rho_c \approx 1.5 \times 10^{14}$gm/cm^3 [29]. Figure 8 shows Ω_c (the dot-dashed curve) including the boundary-layer dissipation from a rigid crust. Rapidly rotating neutron stars may become unstable and spin down, as illustrated by evolution curve A in Fig. 8. However, stars with small initial angular velocities will cool into the stable region before a significant r-mode amplitude develops and spin-down occurs. The exact location of the dividing line between stars that can spin down and those that cannot is difficult to estimate. If a solid monolithic crust forms when the temperature first drops to the melting temperature, then only stars rotating faster than about $0.5\Omega_{\rm max}$ will become unstable and spin down. However, if the formation of a monolithic crust is substantially delayed by differential rotation, pulsations, or some re-heating mechanism in the nascent neutron star, then more slowly rotating stars may develop substantial r-mode amplitudes and spin down as well.

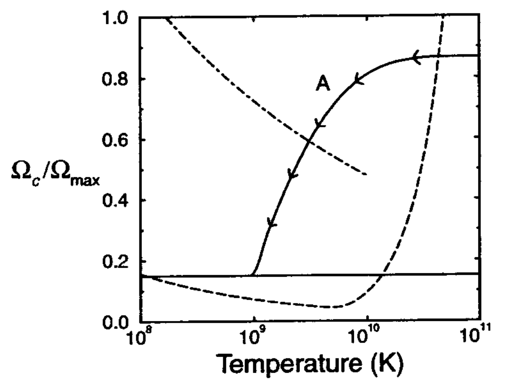

Figure 8: Spin-down of a rapidly rotating neutron star is not impeded by the formation of a crust if the amplitude of the unstable r-mode exceeds α_c before the crust forms. Dashed curve is the stability curve without crust, dash-dot curve is the stability curve with crust.

How far can a neutron star spin down in scenario A? Will the spin-down be halted at the (dot-dashed) stability curve in Fig. 8, or will the star spin down beyond this as indicated in evolution curve A? If a solid crust is present and an r-mode is excited, then viscous dissipation in the boundary layer will heat the fluid adjacent to the crust. If the r-mode amplitude exceeds a critical value, α_c, then the energy dissipation in the boundary layer will re-melt the crust. A fairly simple calculation that balances dissipative heating with thermal conduction and neutrino emission [29] gives the following expression for α_c,

$$\alpha_c = 2.8 \times 10^{-3} f(\theta) \left(\frac{T_m}{10^{10}\mathrm{K}}\right)^{5/2} \left(\frac{\sqrt{\pi G\bar\rho}}{\Omega}\right)^{5/4}, \tag{14}$$

where $1 \le f(\theta) < 2$ (except for a small region near the rotation axis of the star). A solid crust would be re-melted by the r-mode if its amplitude exceeds α_c. Paradoxically, if no solid crust formed then the temperature of the neutron star would quickly drop below the melting temperature by neutrino emission. It has been suggested [29] that instead of a solid crust, an ice-flow (similar to the pack ice that forms on the fringes of the arctic ocean) will form. Dissipation in this ice-flow will keep the temperature at just the melting temperature of the ice. Should the temperature fall below this, the chunks would become larger; the dissipation from collisions between chunks would rise and the temperature would increase. Conversely should the temperature rise above the melting temperature, the chunks would partially melt; dissipation within the ice-flow would fall and the temperature would drop. In summary: if the amplitude of the r-mode grows beyond the value α_c, then a rigid crust will not form at all and the (dot-dashed) stability curve in Fig. 8 has no relevance. Instead the star will spin down to a point (illustrated by the horizontal solid curve in Fig. 8) where the energy in the r-mode is no longer sufficient to melt the crust.

Non-linear hydrodynamic effects may also limit the class of stars which can be spun down by GR emission. For example, above some r-mode amplitude the dissipation at the crust-core boundary is dominated by turbulent viscosity. Wu, Matzner, and Arras [33] have shown that non-linearities in the energy dissipation from this mechanism prevent the GR instability from increasing the amplitude of the r-mode beyond the saturation value $\alpha_{\text{sat}} \approx 0.015(\Omega/\sqrt{\pi G\bar{\rho}})^5$. In rapidly rotating stars, $\Omega > 0.87\Omega_{\text{max}}$, this saturation amplitude is larger than the critical value, α_c, needed to melt the crust [29]. But in more slowly rotating stars the mode saturates before the critical amplitude is reached. It is hard to estimate the exact value of the angular velocity below which non-linear saturation from this effect will halt the growth of the r-mode. A semi-rigid crust tends to reduce the viscous coupling between the core and crust [30], and this raises the turbulence-limited saturation amplitude [33]. In this case the r-mode amplitudes will be sufficient to re-melt the crust in more slowly rotating stars than the $0.87\Omega_{\text{max}}$ limit derived for a rigid crust. Further, any delay in the formation of a monolithic crust (*e.g.*, because of differential rotation or pulsations in the nascent neutron star) may also allow the amplitude of the r-mode to grow beyond the critical value α_c before a crust actually forms.

4 Non-Linear Evolution

Other non-linear hydrodynamic effects—such as mode-mode coupling—might also limit the growth of the r-mode amplitude. If these effects are sufficiently strong, then the nascent neutron star might cool so quickly that the star is unable to lose much angular momentum to GR emission before the instability is suppressed (*e.g.*, as discussed in Sec. 3). And even if not completely suppressed, the GR emitted by any spin-down event may not be detectable by LIGO if the amplitude of the mode is limited to a small value by some non-linear process. Several calculations have been (or are presently being) done to investigate the effects of non-linear hydrodynamic evolution on the development of the r-modes. Stergioulas and Font have shown that large amplitude non-linear r-modes evolve without significant dispersion in rapidly rotating fully relativistic stellar models [34]. And Morsink has found that the amplitude of the $m = 2$ r-mode is limited by non-linear coupling to other r-modes only when the dimensionless amplitude is much larger than unity [35]. Here I will discuss in some detail a three-dimensional numerical simulation by Lindblom, Tohline, and Vallisneri [36] of the non-linear growth and evolution of an r-mode driven unstable by the GR reaction force. This more general simulation is consistent with the previous results: the dimensionless amplitude of the r-mode grows to a maximum value $\alpha_{\max} \approx 3.4$ before non-linear effects (shock waves) damp the mode.

In this simulation a neutron star is modeled as a fluid that obeys the Newtonian hydrodynamic equations,

$$\partial_t \rho + \vec{\nabla} \cdot \left(\rho \vec{v}\right) = 0, \tag{15}$$

$$\rho\left(\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v}\right) = -\vec{\nabla} p - \rho \vec{\nabla} \Phi + \rho \vec{F}_{GR}, \tag{16}$$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{17}$$

Here ρ and p are the density and pressure of the fluid, $\vec{v}$ is the fluid velocity, Φ the Newtonian gravitational potential, and $\vec{F}_{GR}$ is the GR reaction force. For the case of the r-modes, the GR reaction force is dominated by the contribution from the J_{22} current multipole, and is given by the expression [36],

$$F^x_{GR} - iF^y_{GR} = -\kappa i (x + iy) \left[3v^z J^{(5)}_{22} + z J^{(6)}_{22}\right], \tag{18}$$

$$F^z_{GR} = -\kappa \mathrm{Im} \left\{ (x + iy)^2 \left[3 \frac{v^x + iv^y}{x + iy} J^{(5)}_{22} + J^{(6)}_{22}\right]\right\}, \tag{19}$$

where $J^{(n)}_{22}$ represents the n^{th} time derivative of the current multipole,

$$J_{22} = \int \rho r^2 \, \vec{v} \cdot \vec{Y}^{B*}_{22} d^3x. \tag{20}$$

The coupling constant κ has the value $32\sqrt{\pi}G/(45\sqrt{5}c^7)$ for the post-Newtonian limit of general relativity theory. The idea is to evolve a rotating neutron star having a small amplitude r-mode perturbation using the full non-linear hydrodynamics of Eqs. (15)–(20). How large will the amplitude of the r-mode grow? What non-linear hydrodynamic process will halt the growth?

Unfortunately it is essentially impossible to solve the evolution Eqs. (15)–(20) numerically as written. There are two basic problems. First, the timescale for the GR reaction force to act is longer than the dynamical timescale of the problem by at least a factor of 10^4. Thus a rapidly rotating neutron star would have to be evolved through about 10^4 complete revolutions before the amplitude of the r-mode doubles. It simply is not possible (with presently available computer resources) to evolve the hydrodynamic equations numerically for a long enough time to study the effects of the non-linear growth of the r-modes. Second, the GR reaction force in Eqs. (18)–(19) depends on the sixth time derivative of J_{22}. It is not possible to compute this many derivatives accurately using any of the traditional numerical techniques. We resolve the first problem by artificially increasing the value of the coupling constant κ in the GR reaction force. In the simulation presented here the value of κ was taken to be about 4500 times larger than the correct physical value. In our simulation the ratio of the GR growth time to the r-mode period is 12.6, where it should be 5.64×10^4 for a real neutron star with the same mass and angular velocity. We resolve the second problem by noting that in the linear perturbation regime the time dependence of J_{22} is exactly sinusoidal and the derivatives are easily computed: $J_{22}^{(n)} = (i\omega)^n J_{22}$. We find (see below) that the evolution of J_{22} is quite sinusoidal even in the non-linear regime. Thus we use the expressions $J_{22}^{(6)} = -\omega^6 J_{22}$, and $J_{22}^{(5)} = \omega^4 J_{22}^{(1)}$ to evaluate the needed time derivatives. And (using the trick of Finn and Evans [37]) we evaluate $J_{22}^{(1)}$ after eliminating time derivatives from the integrand with the evolution equations:

$$J_{22}^{(1)} = \int \rho \left[\vec{v} \cdot \vec{\nabla}\left(r^2 \vec{Y}_{22}^{B*}\right) \cdot \vec{v} - r^2 \vec{\nabla}\Phi \cdot \vec{Y}_{22}^{B*}\right] d^3x. \tag{21}$$

Finally, we must have a way of evaluating numerically the frequency of the r-mode as the star evolves. We tried several alternatives, but found the expression

$$\omega = -\frac{|J_{22}^{(1)}|}{|J_{22}|}, \tag{22}$$

to be the most stable numerically.

We studied the non-linear growth of an r-mode by solving Eqs. (15)–(20) numerically for a rapidly rotating stellar model represented on a $64 \times 128 \times 128$ cylindrical grid. We constructed initial data for this simulation by building first a rigidly rotating equilibrium stellar model using the polytropic equation of state $p = K\rho^2$. The model discussed here was very rapidly rotating with initial angular velocity $\Omega_0 = 0.635\sqrt{\pi G \bar{\rho}_0} \approx 0.95\Omega_{\rm max}$, where $\bar{\rho}_0$ is the average density of the non-rotating star with the same mass. At the beginning of our simulation we take the fluid density to be that of this equilibrium stellar model, while the fluid velocity is taken to include the rigid rotation of the equilibrium model plus a small amplitude r-mode perturbation:

$$\vec{v} = \Omega_0 \vec{\varphi} + \alpha_0 R_0 \Omega_0 \left(\frac{r}{R_0}\right)^2 \mathrm{Re}(\vec{Y}_{22}^{B}). \tag{23}$$

In our simulation we take $\alpha_0 = 0.1$. Figure 9 shows the numerically determined evolution of $\mathrm{Re}(J_{22})$, and illustrates the fact that this evolution is essentially sinusoidal with a (relatively) slowly varying amplitude and frequency. Thus the approximations used to compute ω and $J_{22}^{(n)}$ are in fact quite good in this situation.

We monitor the evolution of the r-mode into the non-linear regime by defining the generalized amplitude:

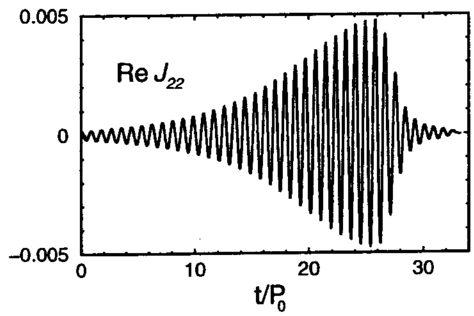

Figure 9: Evolution of the real part of the current quadrupole moment, Re(J_{22}). Time is given here in units of the initial rotation period of the star P_0.

$$\alpha = \frac{2R_0|J_{22}|}{\Omega_0 \int \rho_0 r^4 d^3x}, \tag{24}$$

where R_0, Ω_0, and ρ_0 represent the radius, angular velocity and density of the initial model. This amplitude is normalized so that it agrees with the standard dimensionless r-mode amplitude [15] in the limit of slow rotation and small amplitudes. This definition of α is (up to an overall constant factor) just the magnitude of J_{22}, the only non-vanishing multipole moment for the $m = 2$ r-mode in slowly rotating stars. Figure 10 illustrates the evolution of this α in our simulation. We see that at first α grows exponentially as predicted by linear perturbation theory. Then some non-linear process halts the growth at $t \approx 26P_0$ when $\alpha_{\max} \approx 3.4$. After reaching this maximum the mode is damped on a timescale that is approximately equal to the rotation period.

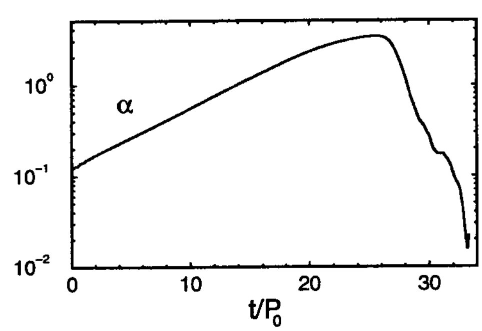

Figure 10: Evolution of the dimensionless amplitude of the r-mode α.

What non-linear process halts the growth and then quickly damps out the r-mode? Figure 11 illustrates the evolution of the total mass M (dashed curve), total angular momentum J (solid curve), and the total kinetic energy T (dot-dashed curve) of the star. The constancy of the mass demonstrates that the damping of the r-mode is not caused by the ejection of matter from the numerical grid in this simulation. Another possibility is that non-linear coupling between modes causes the energy in the r-mode to be transferred to other modes once its amplitude becomes sufficiently large. However, this is not the process taking place in this simulation. The evolution of the total angular momentum J depicted in Fig. 11 agrees (within a few percent) with the predicted loss into GR:

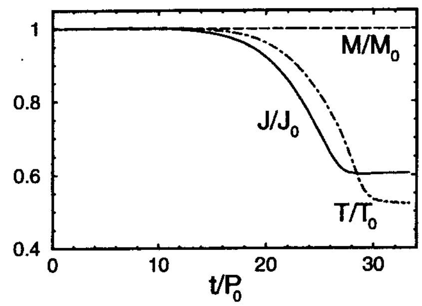

Figure 11: Evolution of the total angular momentum J/J_0 (solid curve), mass M/M_0 (dashed), and kinetic energy T/T_0 (dot-dashed) of the star.

$$\frac{dE}{dt} = \frac{|\omega|}{2}\frac{dJ}{dt} = -\frac{128\pi}{225}\frac{G}{c^7}\kappa\omega^6|J_{22}|^2. \tag{25}$$

Note that the evolution of the total kinetic energy T does not track the evolution of J very well. In particular T continues to decrease significantly even after J becomes essentially constant. If the r-mode were being damped by transferring its energy to other modes, then the total kinetic energy would be essentially conserved in this process. The kinetic energy, along with the total energy of the star, should be conserved then once the losses into GR become negligable. But that is not what is happening in this simulation. Instead some purely hydrodynamic process continues to decrease T even after the GR losses become negligable. This hydrodynamic damping process turns out to be the breaking of waves and the formation of shocks near the surface of the star. Figure 12 illustrates the breaking of these surface waves.

This simulation suggests that non-linear hydrodynamic processes do not prevent the GR instability from driving the dimensionless amplitude of the r-mode to values of order unity. Two caveats prevent us from making this statement more definitive. First, this simulation treated the neutron-star matter as barotropic. Thus our simulation does not determine whether or not there is non-linear coupling between the $m = 2$ r-mode and g-modes that could prevent the growth of the r-mode amplitude. Second, the GR driving force in this simulation is larger than the physical GR force by a factor of about 4500. Thus it is possible that non-linear hydrodynamic coupling does occur but on timescales that are much longer than the basic hydrodynamic timescale of the problem. Such couplings would not have any significant effect in this simulation, but might in a real neutron star significantly limit the growth of the r-mode.

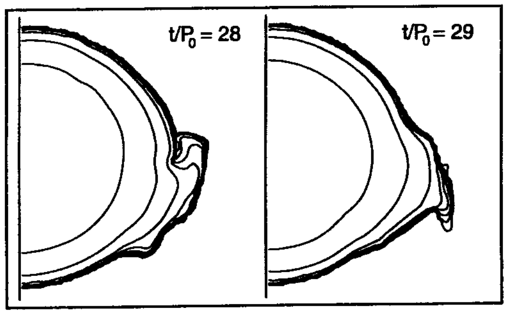

Figure 12: Density contours (at $10^{-n/2}\rho_{\max}$ with $n = 1, 2, ...$) in selected meridional planes at times $t = 28P_0$ and $29P_0$ illustrate the breaking of surface waves. Shocks at the leading edges of these waves appear to be the primary mechanism that suppresses the r-mode.

5 Questions

At the present time it appears that the GR instability in the r-modes may be strong enough to overcome the numerous dissipative processes that act to suppress it real neutron stars. This instability may determine the maximum spin rates of newly formed neutron stars and perhaps the range of spin rates for the neutron stars in LMXBs as well. But there remain a number of important questions that have yet to be resolved. It is not yet known what role magnetic fields play in the evolution of the r-modes. Will magnetic fields suppress the instability, limit its growth, or merely change the values of the frequency and growth times? Is the formation of a solid crust delayed long enough by differential rotation or pulsations after the birth of a neutron star to allow the r-mode instability to act? Does a non-linear coupling to the g-modes limit the growth of the GR instability? Do semi-rigid crust effects move the critical angular velocity to small enough values that the GR instability acts in the LMXBs? Or, do superfluid effects (*e.g.*, pinning of the core vortices or vortex-fluxtube dissipation) suppress the r-mode instability completely in these stars?

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