The Oscillations and Stability of Rotating Fluids: The Two-Potential Formalism[a]

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INTRODUCTION

Rotation plays an essential role in determining the observable properties of a number of interesting astrophysical phenomena, for example, the accretion disks that we observe as compact x-ray sources and active galactic nuclei, and the rapidly rotating neutron stars that we observe as pulsars. This paper is concerned with the mathematical problem of describing the oscillations and determining the stability of these objects. Such oscillations may eventually be observed as temporal variations in the radiation emitted by these sources. And, the effects of such instabilities may eventually be observed either directly—by watching the dynamical evolution of unstable objects, such as too-rapidly rotating supernova cores—or indirectly—by measuring the physically allowed range of equilibrium states, such as the maximum angular velocities of neutron stars. The quantitative understanding of the oscillations and stability of rotating self-gravitating fluids must form the foundation for any deep understanding of those observations that measure the dynamics of these astrophysical phenomena.

A mathematical technique—the two-potential formalism—has been developed in recent years[1–3] that provides a relatively simple and elegant description of the oscillations of rotating self-gravitating fluids. In this approach the equations that describe these pulsations are reduced to an eigenvalue problem whose eigenfunctions are a pair of scalar potentials: one potential that describes the hydrodynamic perturbations and a second that describes the gravitational perturbations of the fluid. All other perturbation quantities, such as the fluid's velocity perturbation or the Lagrangian displacement, are determined as linear combinations of these potentials and their derivatives. This formalism was designed to describe the adiabatic oscillations of equilibrium fluid states that may have arbitrary differential rotation, and an arbitrary—possibly nonbarotropic—equation of state.[3] The first application of this formalism was to evaluate numerically the modes of rapidly rotating neutron stars.[4,5] The effects of gravitational radiation and viscosity on the stability of those modes have also been determined using these techniques.[6] The mathematical simplification

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that leads to the two-potential formalism has proved to be remarkably adaptable. It has been extended to include post-Newtonian effects on the oscillations of rotating stars.[7,8] It has also been adapted to describe the perturbations of a rotating fully general-relativistic fluid in terms of one scalar potential—to describe the hydrodynamic perturbations—and one tensor—to describe the perturbations in the space–time metric.[9] For nonrotating relativistic fluids this description reduces to an eigenvalue problem that again involves only a pair of scalar potentials.[10,11]

We believe that the two-potential formalism is a powerful tool that will have numerous additional astrophysical applications. In particular, we believe that these techniques will be useful in the study of astrophysical disks. One motivation for preparing this paper for this volume, then, is to provide a succinct review of this formalism in a representation that is appropriate for the study of disks. In particular, we present the formalism here for the case of the general adiabatic perturbations of differentially rotating fluid states that may have nonbarotropic equations of state. We also give explicit representations of the differential equations and the integral variational-principle expression for the frequencies of these modes in cylindrical coordinates. These new coordinate representations of the equations are probably the most suitable for the analysis of disks. The explicit coordinate representations of the equations presented here reveal an interesting fact that was not transparent in previous covariant discussions:[3] the variational-principle expression for the pulsation frequencies is valid even when those frequencies are complex. Thus, we think that the variational principle may be a useful tool for investigating the dynamical stability of rotating fluids.

THE TWO-POTENTIAL FORMALISM

The oscillations of a rotating self-gravitating fluid are described by the solutions of the linearized fluid equations:

$$\partial_t \delta\rho + v^a \nabla_a \delta\rho + \nabla_a(\rho \delta v^a) = 0, \tag{1}$$

$$\partial_t \delta v^a + v^b \nabla_b \delta v^a + \delta v^b \nabla_b v^a = -\frac{\nabla^a \delta p}{\rho} + \frac{\delta\rho \nabla^a p}{\rho^2} + \nabla^a \delta\Phi, \tag{2}$$

$$\nabla^a \nabla_a \delta\Phi = -4\pi G \delta\rho. \tag{3}$$

In these equations $\delta\rho$ represents the perturbation in the fluid density, δp the perturbed pressure, δv^a the perturbed fluid velocity, and $\delta\Phi$ the perturbed gravitational potential. Those quantities without the prefix δ represent the equilibrium values of those fields. The equilibrium state whose perturbations are being studied is assumed to be stationary and rotationally symmetric, and the equilibrium fluid velocity is assumed to be purely rotational. Time derivatives in these equations are denoted ∂_t, while ∇_a represents the three-dimensional spatial covariant derivative (i.e., just the partial derivatives $\partial/\partial x^a$ in Cartesian coordinates).

We are interested in the normal mode solutions to **(1)–(3)**, that is, those solutions having time dependence $e^{i\omega t}$, where the frequency ω is a constant. Since the equilibrium configurations are axisymmetric, the angular dependence of the sepa-

rated solutions has the form $e^{im\varphi}$, where φ is the angle that measures rotations about the symmetry axis of the star and m is an integer. The solutions to (1)–(3) having these properties are completely determined by two scalar potentials, δU and $\delta\Phi$, which have the form

$$\delta U = \delta U_0 e^{i\omega t + im\varphi}, \tag{4}$$

$$\delta\Phi = \delta\Phi_0 e^{i\omega t + im\varphi}, \tag{5}$$

where δU_0 and $\delta\Phi_0$ are independent of t and φ. All of the fluid perturbations are determined by these two potentials and their derivatives:

$$\delta p = \rho(\delta U + \delta\Phi), \tag{6}$$

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_s \delta p + i\frac{\rho^2}{\sigma}\delta v^a A_a, \tag{7}$$

$$\delta v^a = iQ^{ab}[\nabla_b \delta U + \rho(\delta U + \delta\Phi)A_b]. \tag{8}$$

Equation (6) is essentially the definition of δU, (7) is the condition that the perturbation be adiabatic, and (8) is equivalent to (2) for these perturbations. In these expressions $\sigma = \omega + m\Omega$ is the frequency of the mode as measured in a frame rotating with the (position = dependent) angular velocity of the fluid Ω. The thermodynamic derivative $(\partial\rho/\partial p)_s$ is to be computed at constant entropy per unit mass s. The vector A_a and the tensor Q^{ab} depend only on the equilibrium configuration and the frequency of the mode σ:

$$A_a = \frac{1}{\rho^2}\left[\nabla_a\rho - \left(\frac{\partial\rho}{\partial p}\right)_s \nabla_a p\right], \tag{9}$$

$$Q^{ab} = \frac{\lambda}{\sigma^3}\left[(\sigma^2 - A^c\nabla_c p)g^{ab} + 2i\hat{\varphi}^b\epsilon^{ac}\left(\sigma\Omega_c - \frac{\Omega^d\nabla_d p}{\sigma}A_c\right)\right.$$

$$\left. - 2\omega^a\Omega^b + \nabla^a p A^b - i\hat{\varphi}^a\epsilon^{bc}\left(\sigma\omega_c - \frac{\omega^d A_d}{\sigma}\nabla_c p\right)\right]. \tag{10}$$

The scalar λ, which is proportional to the determinant of Q^{ab}, is given by

$$\lambda = \sigma^4[\sigma^4 - \sigma^2 A^a\nabla_a p - 2\sigma^2\Omega^a\omega_a + 2A^a\omega_a\Omega^b\nabla_b p]^{-1}. \tag{11}$$

The vector $\omega^a = (\nabla \times v)^a = \epsilon^{abc}\nabla_b v_c$ is the vorticity of the fluid; $\Omega^a = \Omega z^a$ is the angular velocity vector; z^a is the unit vector that is parallel to the rotation axis, and $\hat{\varphi}^a$ is the unit vector in the φ direction; spatial indices are lowered and raised by the metric g_{ab} (i.e., the identity matrix in Cartesian coordinates) and its inverse g^{ab}, and finally $\epsilon^{ab} = \epsilon^{abc}\hat{\varphi}_c$ where ϵ^{abc} is the totally antisymmetric tensor whose components have the values ± 1 or 0 in Cartesian coordinates. The Lagrangian displacement ξ_a is also determined by the two potentials δU and $\delta\Phi$ via (8) and the formula:

$$\xi_a = -i\left(\frac{g_{ab}}{\sigma} - i\frac{v_a\nabla_b\Omega}{\sigma^2\Omega}\right)\delta v^b. \tag{12}$$

The two potentials δU and $\delta\Phi$ are determined in turn by the two second-order partial differential equations:

$$\nabla_a(\rho Q^{ab}\nabla_b\delta U) + \Psi_3\delta U = -\rho^2 Q^{ab}A_b\nabla_a\delta\Phi - \Psi_2\delta\Phi, \tag{13}$$

$$\nabla^a\nabla_a\delta\Phi + 4\pi G\Psi_1\delta\Phi = \frac{4\pi G\rho^2}{\sigma}A_aQ^{ab}\nabla_b\delta U - 4\pi G\Psi_1\delta U, \tag{14}$$

which are equivalent to (1) and (3) for these perturbations. The scalars Ψ_1, Ψ_2, and Ψ_3 depend only on the equilibrium fluid state and the frequency of the mode σ:

$$\Psi_1 = \rho\left(\frac{\partial\rho}{\partial p}\right)_s - \frac{\rho^3}{\sigma}A_aQ^{ab}A_b, \tag{15}$$

$$\Psi_2 = \sigma\Psi_1 + \nabla_a(\rho^2Q^{ab}A_b), \tag{16}$$

$$\Psi_3 = \Psi_2 + im\frac{\rho^2}{\varpi}(Q^{ab} - Q^{ba})\hat{\varphi}_aA_b, \tag{17}$$

where ϖ is the radial distance from the rotation axis. The solution of (13) and (14) for the functions δU and $\delta\Phi$ that satisfy the appropriate boundary conditions is an eigenvalue problem. One boundary condition is that the perturbed gravitational potential $\delta\Phi$ must vanish at infinity. The second boundary condition, that the Lagrangian perturbation in the pressure $\Delta p = \delta p + \xi^a\nabla_a p$ must vanish on the surface of the fluid, is automatically satisfied by the bounded solutions of these equations if the density ρ vanishes on the surface of the equilibrium configuration.

It is often helpful to have a variational principle for the pulsation equations that can be used as a tool for estimating the frequencies of the modes. Such a variational principle for (13) and (14) is given by

$$\begin{aligned} S = \frac{1}{2\pi}\int\Bigg\{&\rho(\sigma Q^{ab} - i\varpi Q^{ac}\nabla_c\Omega\hat{\varphi}^b)\nabla_a\left(\frac{\delta U^\dagger}{\sigma}\right)\nabla_b\left(\frac{\delta U}{\sigma}\right) - \frac{\Psi_3}{\sigma}\delta U^\dagger\delta U \\ &+ \frac{\rho^2}{\sigma}(\sigma Q^{ab} - i\varpi Q^{ac}\nabla_c\Omega\hat{\varphi}^b)\left[\delta\Phi^\dagger A_a\nabla_b\left(\frac{\delta U}{\sigma}\right) + \delta\Phi A_b\nabla_a\left(\frac{\delta U^\dagger}{\sigma}\right)\right] \\ &+ \frac{\nabla^a\delta\Phi^\dagger\nabla_a\delta\Phi}{4\pi G} - \Psi_1(\delta\Phi^\dagger\delta\Phi + \delta\Phi^\dagger\delta U + \delta\Phi\delta U^\dagger)\Bigg\}d^3x, \end{aligned} \tag{18}$$

where d^3x is the proper spatial volume element, and the adjoints $\delta U^\dagger$ and $\delta\Phi^\dagger$ are defined by

$$\delta U^\dagger = \delta U_0 e^{-i\omega t - im\varphi}, \tag{19}$$

$$\delta\Phi^\dagger = \delta\Phi_0 e^{-i\omega t - im\varphi}. \tag{20}$$

This expression (18) is a variational principle in the sense that arbitrary independent variations of S with respect to δU_0 and $\delta\Phi_0$ vanish if and only if (13) and (14) are satisfied. The proof of this is given by Ipser and Lindblom[3] for the case of real ω. A

proof is given in the last section of this paper for the case of complex ω. The frequencies of the modes are estimated by using the equation $S = 0$ to "define" ω as a function of δU_0, $\delta\Phi_0$, and m. For given δU_0, $\delta\Phi_0$, and m, S depends only on ω, and consequently the zeros of S determine ω. The stationary values of ω—with respect to infinitesimal variations in δU_0 and $\delta\Phi_0$—are eigenvalues therefore of **(13)** and **(14)**. In practice the eigenvalues ω of **(13)** and **(14)** may be estimated using **(18)** by first selecting a set of trial eigenfunctions $\delta U_0(\lambda_i)$ and $\delta\Phi_0(\lambda_i)$ that depend on N parameters λ_i. The optimal values of these parameters and the best estimate of ω are then obtained by solving $S = 0$ together with $\partial S/\partial\lambda_i = 0$ for each λ_i, keeping $\partial\omega/\partial\lambda_i = 0$ (the Ritz method). We note that this version of the variational principle involves parameterizing both trial potentials δU_0 and $\delta\Phi_0$. When **(14)** does not depend on ω, a more efficient variational principle exists[3] that involves parameterizing only the single potential δU_0. In this case, $\delta\Phi_0$ is considered to be the function of δU_0 that is obtained by solving **(14)**. However, in the case of fluids where $A_a \neq 0$ (like the hot material in an accretion disk), **(14)** does depend on ω, and so the more complicated variational principle given in **(18)** must be used.

Self-gravitational effects are often negligible in accretion disks. In this case (the Cowling approximation), the two-potential formalism simplifies considerably: The fluid perturbations are determined from the single potential δU by setting $\delta\Phi = 0$ in **(6)–(8)**. The potential δU is determined in turn by the single partial differential equation obtained by setting $\delta\Phi = 0$ in **(13)**. The frequencies of the modes in this approximation may be estimated with the aid of the variational principle obtained by setting $\delta\Phi = 0$ in **(18)**.

CYLINDRICAL COORDINATES

The covariant form of the two-potential equations, **(14)** and **(13)**, is simple and compact. The equations are most conveniently derived in this form,[3] and abstract manipulations of the equations are often easiest using this form as well. The covariant form of the equations also has the advantage that it can be transformed into any convenient choice of coordinates in a perfectly straightforward manner. However, before any explicit solution of the equations can be found it is inevitable that the equations must be expressed in some particular choice of coordinates. For the study of accretion disks, we feel that cylindrical coordinates are the most natural choice. Therefore, we present in this section the explicit representation of the two-potential equations in cylindrical coordinates.

Let z, ϖ, and φ denote the standard cylindrical coordinates: z measuring translations parallel to the rotation axis, ϖ the radial distance from the axis, and φ measuring rotations about the axis. In cylindrical coordinates, then, the equations that express $\delta\rho$ and the components of δv^a in terms of the two potentials δU_0 and $\delta\Phi_0$ are given, via **(4)**, **(5)**, **(7)**, and **(8)**, by

$$\delta\rho = \left\{\Psi_1(\delta U_0 + \delta\Phi_0) - \frac{2m\Omega\lambda\rho^2}{\varpi\sigma^3}A_\varpi\delta U_0 - \frac{\lambda\rho^2}{\sigma^2}A_\varpi\frac{\partial\delta U_0}{\partial\varpi}\right.$$

$$\left. - \frac{\lambda\rho^2}{\sigma^4}\left[\sigma^2 A_z - 2\Omega(A_z\omega_z + A_\varpi\omega_\varpi)\right]\frac{\partial\delta U_0}{\partial z}\right\} e^{i\omega t + im\varphi}, \qquad (21)$$

$$\delta v^z = i\frac{\lambda}{\sigma^3}\left\{\rho[\sigma^2 A_z - 2\Omega(A_z\omega_z + A_\varpi\omega_\varpi)](\delta U_0 + \delta\Phi_0) + \frac{2m\Omega}{\varpi\sigma}A_\varpi\partial_z p\delta U_0\right.$$
$$\left. + (\sigma^2 - A_\varpi\partial_\varpi p - 2\Omega\omega_z)\frac{\partial\delta U_0}{\partial z} + A_\varpi\partial_z p\frac{\partial\delta U_0}{\partial\varpi}\right\} e^{i\omega t + im\varphi}, \tag{22}$$

$$\delta v^\varpi = i\frac{\lambda}{\sigma^3}\left\{\rho\sigma^2 A_\varpi(\delta U_0 + \delta\Phi_0) + \frac{2m\Omega}{\varpi\sigma}(\sigma^2 - A_z\partial_z p)\delta U_0\right.$$
$$\left. + (\sigma^2 - A_z\partial_z p)\frac{\partial\delta U_0}{\partial\varpi} + A_\varpi\partial_z p\frac{\partial\delta U_0}{\partial z}\right\} e^{i\omega t + im\varphi}, \tag{23}$$

$$\delta v^\varphi = \frac{\lambda}{\varpi\sigma^4}\left\{\rho[\sigma^2(A_z\omega_\varpi - A_\varpi\omega_z) - 2\Omega\omega_\varpi(A_z\omega_z + A_\varpi\omega_\varpi)](\delta U_0 + \delta\Phi_0)\right.$$
$$- \frac{m\sigma}{\varpi}(\sigma^2 - A_z\partial_z p - A_\varpi\partial_\varpi p)\delta U_0 + \sigma^2\left(\omega_\varpi\frac{\partial\delta U_0}{\partial z} - \omega_z\frac{\partial\delta U_0}{\partial\varpi}\right)$$
$$\left. + (A_z\omega_z + A_\varpi\omega_\varpi)\left(\partial_z p\frac{\partial\delta U_0}{\partial\varpi} - \partial_\varpi p\frac{\partial\delta U_0}{\partial z}\right)\right\} e^{i\omega t + im\varphi}. \tag{24}$$

The components of A_a and ω_a used in these expressions are given by

$$\begin{pmatrix} A_z \\ A_\varpi \\ A_\varphi \end{pmatrix} = \frac{1}{\rho^2}\begin{pmatrix} \partial_z\rho \\ \partial_\varpi\rho \\ 0 \end{pmatrix} - \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial p}\right)_s\begin{pmatrix} \partial_z p \\ \partial_\varpi p \\ 0 \end{pmatrix}, \tag{25}$$

$$\begin{pmatrix} \omega_z \\ \omega_\varpi \\ \omega_\varphi \end{pmatrix} = \begin{pmatrix} 2\Omega + \varpi\partial_\varpi\Omega \\ -\varpi\partial_z\Omega \\ 0 \end{pmatrix}, \tag{26}$$

and the scalars λ and Ψ_1 are given by

$$\lambda = \sigma^4[\sigma^4 - \sigma^2(A_z\partial_z p + A_\varpi\partial_\varpi p + 2\Omega\omega_z) + 2\Omega(A_z\omega_z + A_\varpi\omega_\varpi)\partial_z p]^{-1}, \tag{27}$$

$$\Psi_1 = \rho\left(\frac{\partial\rho}{\partial p}\right)_s - \frac{\rho^3\lambda}{\sigma^2}(A_z^2 + A_\varpi^2) + \frac{2\rho^3\lambda\Omega}{\sigma^4}A_z(A_z\omega_z + A_\varpi\omega_\varpi). \tag{28}$$

In transcribing the covariant expressions for these quantities into cylindrical coordinates we have made repeated use of the identity $2\Omega\omega_\varpi = A_z\,\partial_\varpi p - A_\varpi\,\partial_z p$, which is satisfied by these equilibrium fluid configurations. The shorthand $\partial_z p = \partial p/\partial z$, $\partial_\varpi p = \partial p/\partial\varpi$, etc., is sometimes used to denote partial differentiation in these expressions. We note that the covariant and contravariant z- and ϖ-components of vectors are equal, $\delta v^z = \delta v_z$ and $\delta v^\varpi = \delta v_\varpi$, while the φ-components are related by $\delta v_\varphi = \varpi^2\delta v^\varphi$. The components of the Lagrangian displacement ξ^a are related to the components of

the perturbed velocity, via **(12)**, by

$$\begin{pmatrix} \xi^z \\ \xi^\varpi \\ \xi^\varphi \end{pmatrix} = -\frac{i}{\sigma^2}\begin{pmatrix} \sigma\delta v^z \\ \sigma\delta v^\varpi \\ \sigma\delta v^\varphi - i\delta v^z\partial_z\Omega - i\delta v^\varpi\partial_\varpi\Omega \end{pmatrix}. \tag{29}$$

The potentials δU_0 and $\delta\Phi_0$, which depend only on z and ϖ, are determined by the cylindrical coordinate representations of **(13)** and **(14)**:

$$\begin{aligned} &\frac{\partial}{\partial z}\left[\frac{\rho\lambda}{\sigma^2}(\sigma^2 - A_\varpi\partial_\varpi p - 2\Omega\omega_z)\frac{\partial}{\partial z}\left(\frac{\delta U_0}{\sigma}\right) + \frac{\rho\lambda}{\sigma^2}A_\varpi\partial_z p\,\frac{\partial}{\partial\varpi}\left(\frac{\delta U_0}{\sigma}\right)\right] \\ &+ \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left[\frac{\rho\lambda\varpi}{\sigma^2}(\sigma^2 - A_z\partial_z p)\frac{\partial}{\partial\varpi}\left(\frac{\delta U_0}{\sigma}\right) + \frac{\rho\lambda\varpi}{\sigma^2}A_\varpi\partial_z p\,\frac{\partial}{\partial z}\left(\frac{\delta U_0}{\sigma}\right)\right] \\ &+ \left[\sigma\Psi_3 - m\sigma\frac{\partial}{\partial z}\left(\frac{\rho\Lambda_z}{\sigma}\right) - \frac{m\sigma}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho\varpi\Lambda_\varpi}{\sigma}\right) - m^2\rho\Lambda_\varphi\right]\frac{\delta U_0}{\sigma} \\ &\quad = -\frac{\rho^2\lambda}{\sigma}\left\{\left[A_z - \frac{2\Omega}{\sigma^2}(A_z\omega_z + A_\varpi\omega_\varpi)\right]\frac{\partial\delta\Phi_0}{\partial z} + A_\varpi\frac{\partial\delta\Phi_0}{\partial\varpi}\right\} \\ &\qquad - \left[\Psi_2 + \frac{m\rho^2}{\sigma}(\Lambda_z A_z + \Lambda_\varpi A_\varpi)\right]\delta\Phi_0, \end{aligned} \tag{30}$$

$$\begin{aligned} &\frac{1}{4\pi G}\left[\frac{\partial^2\delta\Phi_0}{\partial z^2} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\delta\Phi_0}{\partial\varpi}\right) + \left(4\pi G\Psi_1 - \frac{m^2}{\varpi^2}\right)\delta\Phi_0\right] \\ &\quad = \frac{\rho^2\lambda}{\sigma}\left\{\left[A_z - \frac{2\Omega}{\sigma^2}(A_z\omega_z + A_\varpi\omega_\varpi)\right]\frac{\partial}{\partial z}\left(\frac{\delta U_0}{\sigma}\right) + A_\varpi\frac{\partial}{\partial\varpi}\left(\frac{\delta U_0}{\sigma}\right)\right\} \\ &\qquad - \left[\Psi_1 + \frac{m\rho^2}{\sigma^2}(\Lambda_z A_z + \Lambda_\varpi A_\varpi)\right]\delta U_0. \end{aligned} \tag{31}$$

The scalars Ψ_1, Ψ_2, and Ψ_3 that appear in these equations are given by **(28)** and

$$\Psi_2 = \sigma\Psi_1 + \frac{\partial}{\partial z}\left\{\frac{\rho^2\lambda}{\sigma}\left[A_z - \frac{2\Omega}{\sigma^2}(A_z\omega_z + A_\varpi\omega_\varpi)\right]\right\} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\rho^2\lambda\varpi}{\sigma}A_\varpi\right), \tag{32}$$

$$\begin{aligned} \Psi_3 = \Psi_2 &+ \frac{m\rho^2\lambda}{\sigma^4\varpi}(A_\varpi\partial_z p - A_z\partial_\varpi p)(A_z\omega_z + A_\varpi\omega_\varpi) \\ &+ \frac{m\rho^2\lambda}{\sigma^2\varpi}(A_z\omega_\varpi - A_\varpi\omega_z - 2\Omega A_\varpi), \end{aligned} \tag{33}$$

and the quantities Λ_z, Λ_ϖ, and Λ_φ are defined by

$$\begin{pmatrix} \Lambda_z \\ \Lambda_\varpi \\ \Lambda_\varphi \end{pmatrix} = \frac{\lambda}{\varpi\sigma^3}\begin{pmatrix} \sigma^2\omega_\varpi - (A_z\omega_z + A_\varpi\omega_\varpi)\partial_\varpi p \\ -\sigma^2\omega_z + (A_z\omega_z + A_\varpi\omega_\varpi)\partial_z p \\ \sigma(\sigma^2 - A_z\partial_z p - A_\varpi\partial_\varpi p)/\varpi \end{pmatrix}. \tag{34}$$

For real values of the frequency ω, **(30)** and **(31)** are real; thus the potentials δU_0 and $\delta\Phi_0$ may be taken (without loss of generality) to be real. For cases where the frequency is complex, however, these potentials will necessarily be complex as well. We point out that these coordinate representations of the equations are significantly more complicated than their covariant counterparts. This illustrates why the covariant expressions are in many cases a more convenient choice for abstract manipulations.

THE VARIATIONAL PRINCIPLE

As we remarked earlier, a variational principle is often a useful tool for estimating the eigenvalues of partial differential equations. In order to facilitate the use of **(18)** for the study of the pulsations and stability of accretion disks, we present here its transcription into cylindrical coordinates:

$$
\begin{aligned}
S = \int\int \Bigg\{ & \frac{\rho\lambda}{\sigma^2}(\sigma^2 - A_\varpi \partial_\varpi p - 2\Omega\omega_z)\left[\frac{\partial}{\partial z}\left(\frac{\delta U_0}{\sigma}\right)\right]^2 \\
& + \frac{\rho\lambda}{\sigma^2}(\sigma^2 - A_z \partial_z p)\left[\frac{\partial}{\partial \varpi}\left(\frac{\delta U_0}{\sigma}\right)\right]^2 + \frac{2\rho\lambda}{\sigma^2} A_\varpi \partial_z p \frac{\partial}{\partial z}\left(\frac{\delta U_0}{\sigma}\right)\frac{\partial}{\partial \varpi}\left(\frac{\delta U_0}{\sigma}\right) \\
& - \left[\sigma\Psi_3 - m\sigma\frac{\partial}{\partial z}\left(\frac{\rho\Lambda_z}{\sigma}\right) - \frac{m\sigma}{\varpi}\frac{\partial}{\partial \varpi}\left(\frac{\rho\varpi\Lambda_\varpi}{\sigma}\right) - m^2\rho\Lambda_\varphi\right]\left(\frac{\delta U_0}{\sigma}\right)^2 \\
& + \frac{2\rho^2\lambda}{\sigma}\left[A_z - \frac{2\Omega}{\sigma^2}(A_z\omega_z + A_\varpi\omega_\varpi)\right]\delta\Phi_0\frac{\partial}{\partial z}\left(\frac{\delta U_0}{\sigma}\right) \\
& + \frac{2\rho^2\lambda}{\sigma} A_\varpi \delta\Phi_0 \frac{\partial}{\partial \varpi}\left(\frac{\delta U_0}{\sigma}\right) - 2\left[\Psi_1 + \frac{m\rho^2}{\sigma^2}(\Lambda_z A_z + \Lambda_\varpi A_\varpi)\right]\delta U_0 \delta\Phi_0 \\
& + \frac{1}{4\pi G}\left[\left(\frac{\partial\delta\Phi_0}{\partial z}\right)^2 + \left(\frac{\partial\delta\Phi_0}{\partial \varpi}\right)^2\right] + \left[\frac{m^2}{4\pi G\varpi^2} - \Psi_1\right](\delta\Phi_0)^2\Bigg\}\varpi\, d\varpi\, dz, \qquad (35)
\end{aligned}
$$

where the integral is to be carried out over the $\varpi \geq 0$ half-plane. The various functions that appear in **(35)**, for example, λ, Ψ_1, Λ_z, etc., are expressed in terms of cylindrical coordinates in the previous section. It is straightforward (but somewhat tedious) to verify that the variation of this S with respect to δU_0 vanishes if and only if **(30)** is satisfied, and its variation with respect to $\delta\Phi_0$ vanishes if and only if **(31)** is satisfied. It is also straightforward to verify that $S = 0$ when **(30)** and **(31)** are satisfied. The calculations needed to verify these relationships do not depend on the frequency ω being real. It follows, then, that this variational principle is valid even for modes having complex frequencies. It should be suitable therefore for the study of the dynamical instabilities of rotating stars and accretion disks. The derivation of the variational principle presented here generalizes the work of Ipser and Lindblom[3] by showing that it applies to modes having complex frequencies. The original argument was based on a covariant analysis of the equations and used the fact that the tensor $\sigma Q^{ab} - i\varpi Q^{ac}\nabla_c\Omega\hat{\varphi}^b$ was Hermitian. This tensor is *not* Hermitian unless the

frequency ω is real, however. Thus, the proof given by Ipser and Lindblom[3] fails for complex ω. However, as we have shown here, the vanishing of the variations of the functional S is equivalent to the pulsation equations even for complex ω. Thus the variational principle *is valid* even in this more general case. This is one example, then, when the analysis of the coordinate representations of the equations lead to an important insight that was not apparent in the covariant analysis.

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