Post-Newtonian Effects on the Oscillations of Rotating Stars[a]

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INTRODUCTION

It is well known that gravitational radiation tends to make all rotating stars unstable.[1,2] Viscosity, however, tends to counteract this instability so that only sufficiently rapidly rotating stars are in fact unstable.[3,4] In order to determine which stars are stable, therefore, a detailed calculation of the pulsations of rapidly rotating stars must be carried out, which includes the effects of viscosity and gravitational radiation. Such calculations are very difficult. The problem of finding solutions to the pulsation equations for rapidly rotating relativistic stellar models has never been seriously attempted, let alone solved. Various approximate calculations have been completed, however. For example, the equations that describe the pulsations of rapidly rotating Newtonian stars have been solved, including the effects of viscosity and gravitational radiation.[5-8] These calculations are unrealistic due to their neglect of relativistic effects in the equations for the structure and pulsations of the stars, and due to their use of very idealized equations of state for the stellar matter. More realistic calculations have also been carried out using the full relativistic equations and using more realistic equations of state,[9,10] but these calculations are limited to nonrotating stars.

Although idealized, these calculations do give some approximate understanding of the gravitational radiation instability in rotating neutron stars. The shear viscosity of neutron-star matter scales with temperature like T^{-2}. Therefore, in sufficiently cold neutron stars, the viscosity is so large that it completely suppresses the gravitational radiation instability in all rotating stars. The approximate calculations just described indicate that this complete suppression occurs when $T \leq 10^7 K$. In hotter stars the instability may occur, but only in stars rotating faster than about 90 percent of the maximum equilibrium angular velocity. In the very hottest stars, $T \geq 5 \times 10^{10} K$, the bulk viscosity (which scales with temperature like T^6) becomes very large and completely suppresses the instability in all rotating stars.

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In an attempt to improve our understanding of these instabilities, we report here on another approximate calculation of the pulsation frequencies of rapidly rotating stars. In typical realistic neutron-star models the gravitational field is fairly weak in the sense that GM/c^2R is considerably less than one. Under these circumstances the structure of the star and its gravitational field can be reasonably well approximated in a post-Newtonian expansion of general relativity. Thus, we review (and extend) in the next two sections the formalism developed by Cutler[11] for computing the structure and pulsations of rotating stars in the post-Newtonian approximation. In the fourth section we present numerical results for the frequencies of the $l = m$ f-modes of rapidly rotating neutron-star models. We illustrate the accuracy of this method by comparing the post-Newtonian frequencies computed here with those computed for nonrotating stars using the full general-relativistic equations. We show that the post-Newtonian frequencies agree with the exact ones to within a few percent. We then compute, in the post-Newtonian approximation, the critical angular velocities where the frequencies of the $l = m$ f-modes vanish. These are the angular velocities where these modes would become unstable to the gravitational-radiation-induced instability in the absence of viscosity. We find that post-Newtonian effects lower, by up to 8 percent, the ratios of these critical angular velocities to $\sqrt{\pi G\bar{\rho}_0}$, where $\bar{\rho}_0$ is the average density of the star. Thus, post-Newtonian effects tend to make the gravitational radiation instability more important.

THE POST-NEWTONIAN APPROXIMATION

The dynamics of the material that makes up a general-relativistic stellar model is constrained (if not completely determined) by Einstein's equation,

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}. \tag{1}$$

For our purposes it is sufficient to approximate the stress energy tensor of the stellar matter as that of a perfect fluid,

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}, \tag{2}$$

where ϵ is the energy density, p the pressure, and u^α the four-velocity of the stellar fluid. The space–time metric is denoted $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. We limit our attention here to cases where the pressure is determined completely by the energy density of the fluid: $p = p(\epsilon)$. Under these conditions the dynamics of the stellar matter is determined completely by (1) and (2). Even under these idealized assumptions, however, it is very difficult to find solutions to these equations except under the simplest time-independent equilibrium conditions.

For most realistic stellar models the gravitational fields are relatively weak and the fluid velocities are only a small fraction of the speed of light. Under these conditions it is possible to approximate the solutions to (1) and (2) in a "post-Newtonian" expansion. Following Chandrasekhar[12] (and for the notation used here, Cutler[11]) we expand the space–time metric, and the fluid variables as formal series in inverse powers of the speed of light $1/c^n$:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\left[1 + \frac{2}{c^2}\Phi + \frac{2}{c^4}(\Phi^2 + \Psi) + O(c^{-6})\right]c^2 dt^2$$
$$+ \left[\frac{2}{c^3}A_a + O(c^{-5})\right]cdt\,dx^a + \left[\delta_{ab}\left(1 - \frac{2}{c^2}\Phi\right) + O(c^{-4})\right]dx^a dx^b, \quad (3)$$

$$\epsilon = \rho c^2 + (\sigma + 2\rho\Phi - \rho v^2) + O(c^{-2}), \quad (4)$$

$$p = p(\rho) + \frac{1}{c^2}\frac{dp}{d\rho}(\sigma + 2\rho\Phi - \rho v^2) + O(c^{-4}), \quad (5)$$

$$cu^t = 1 + \frac{1}{2c^2}(v^2 - 2\Phi)$$
$$+ \frac{1}{2c^4}[\Phi^2 - 3\Phi v^2 - 2\Psi - \tfrac{1}{4}v^4 + 2v^a(A_a + w_a)] + O(c^{-6}), \quad (6)$$

$$u^a = \frac{1}{c}v^a + \frac{1}{c^3}w^a + O(c^{-5}). \quad (7)$$

A particular choice of coordinates (t, x^a) has been made in order to cast the components of these quantities in this form.[11] The preceding equations serve as definitions of the Newtonian fields, ρ, v^a, and Φ, and of the post-Newtonian fields σ, w^a, Ψ, and A_a, which represent the next order "corrections" to the Newtonian quantities. Spatial indices are raised and lowered with the Euclidean metric δ_{ab} (i.e., the identity matrix in Cartesian coordinates) and its inverse δ^{ab}; and v^2 denotes $v^a v_a$.

The equations that determine the Newtonian fields ρ, v^a, and Φ, and their post-Newtonian counterparts σ, w^a, Ψ, and A^a, are obtained by expanding Einstein's equation **(1)** and the associated conservation law, $\nabla_\alpha T^{\alpha\beta} = 0$, as formal power series in $1/c^n$. Following Gunnarsen,[13] the coefficient of each power of $1/c$ in these expansions is set separately to zero. The coefficients of the lowest order terms in this expansion give the standard Newtonian equations for ρ, v^a, and Φ:

$$\partial_t\rho + D_a(\rho v^a) = 0, \quad (8)$$

$$\rho(\partial_t v^a + v^b D_b v^a) + D^a p + \rho D^a\Phi = 0, \quad (9)$$

$$D^a D_a \Phi = 4\pi G\rho, \quad (10)$$

where ∂_t represents the partial derivative with respect to t, and D_a is the covariant derivative associated with the spatial metric δ_{ab} (i.e., just the partial derivatives $\partial/\partial x^a$ in Cartesian coordinates). The next-order terms in this expansion give the post-Newtonian equations for the fields σ, w^a, Ψ, and A^a. The resulting equations are rather complicated (see Cutler[11]). Since these equations are not explicitly needed here in their general form, we do not reproduce them. Suffice it to say that the post-Newtonian fields σ, w^a, Ψ, and A^a are determined by solving the post-Newtonian analogs of **(8)**–**(10)**. Finally, the Newtonian and post-Newtonian fields are combined as prescribed in (3)–(7) to produce a solution of Einstein's equation (1) in the post-Newtonian approximation.

POST-NEWTONIAN STELLAR OSCILLATIONS

Here we are interested in analyzing the pulsations of rapidly rotating stars in the post-Newtonian approximation. Following the formalism described in the previous section, this analysis begins with the solution of the Newtonian equations for stellar pulsations. Fortunately, the solutions to (**8**)–(**10**) that correspond to small pulsations of rapidly rotating Newtonian stars are well understood. A useful formalism for finding these Newtonian pulsations has been developed in a series of papers by Ipser and Lindblom.[6–8] We briefly review that work here.

Newtonian stellar models are the time-independent and axisymmetric solutions of (**8**)–(**10**). We limit our attention here to rigidly rotating stars, that is, those in which the velocity is taken to be a constant multiple of the rotational Killing field: $v^a = \Omega\phi^a$. Under these conditions (**8**)–(**10**) reduce to

$$C = \frac{1}{2}v^2 - \Phi - \int_0^p \frac{d\bar{p}}{\rho(\bar{p})}, \tag{11}$$

$$D^aD_a\Phi = 4\pi G\rho, \tag{12}$$

where C is a constant. Numerical techniques for solving these equations for sequences of rotating stellar models of given mass were developed by Ipser and Lindblom[6,7] on the basis of the work of James.[14]

Once a Newtonian stellar model has been found, its pulsations may be studied by solving the linearized time-dependent equations for the small departures from equilibrium of each of the fluid fields $\delta\rho$, δv^a, and $\delta\Phi$. We will consider the solutions for these quantities that correspond to normal modes, that is, those having time and angular dependence $e^{-i\omega t+im\phi}$. The constant ω is the frequency of the mode, and m is an integer. Ipser and Managan[15,16] showed that such perturbations are determined completely by two scalar potentials δU and $\delta\Phi$. The remaining fluid perturbations are determined from these by the equations

$$\delta\rho = \rho\frac{d\rho}{dp}(\delta U - \delta\Phi), \tag{13}$$

$$\delta v^a = iQ^{ab}D_b\delta U, \tag{14}$$

where the tensor Q^{ab} is given by

$$Q^{ab} = \frac{1}{\omega - m\Omega}\left[-\lambda\delta^{ab} + (\lambda - 1)z^az^b - \frac{2i\lambda D^av^b}{\omega - m\Omega}\right]. \tag{15}$$

In this expression z^a is the unit vector that is parallel to the rotation axis and

$$\lambda = \frac{(\omega - m\Omega)^2}{(\omega - m\Omega)^2 - 4\Omega^2}. \tag{16}$$

The potentials δU and $\delta\Phi$ are determined by two second-order equations that are consequences of the linearized (**8**) and (**10**):

$$D_a(\rho Q^{ab}D_b\delta U) + (\omega - m\Omega)\rho\frac{d\rho}{dp}(\delta\Phi - \delta U) = 0, \tag{17}$$

$$D^aD_a\delta\Phi + 4\pi G\rho\frac{d\rho}{dp}(\delta\Phi - \delta U) = 0. \tag{18}$$

The methods needed to solve these equations numerically for rotating stellar models have been developed by Ipser and Lindblom.[6,7] The particular form of the pulsation equations presented here applies only to the barotropic perturbations $[\delta\rho = (d\rho/dp)\delta p]$ of rigidly rotating stars. The equations for the general adiabatic pulsations of arbitrary differentially rotating stars are qualitatively similar to (**13**)–(**18**), only they are somewhat more complicated.[17]

Having found a solution of the Newtonian equations (**8**)–(**10**) that corresponds to the pulsation of a rotating star, the next step in our analysis is to find the corresponding solution of the post-Newtonian equations. As in the Newtonian case, the post-Newtonian equations are solved in two stages. First, the time-independent fields σ, w^a, Ψ, and A^a that describe the post-Newtonian corrections to the structure of the equilibrium star are found. And then second, the time-dependent fields $\delta\sigma$, δw^a, $\delta\Psi$, and δA^a that describe the small amplitude pulsations about this equilibrium configuration are found.

The equations that determine the post-Newtonian corrections to a time-independent rigidly rotating stellar model are:

$$w^a = \frac{1}{2}\left(v^2 - 2\Phi + 2c^2\frac{\Delta\Omega}{\Omega}\right)v^a, \tag{19}$$

$$D^bD_bA^a = 16\pi G\rho v^a, \tag{20}$$

$$D^aD_a\Psi = 4\pi G(\sigma + \rho v^2 + 3p), \tag{21}$$

$$c^2\Delta C = \frac{1}{4}v^4 - \Psi - 2\Phi v^2 + v^aA_a + v^2c^2\frac{\Delta\Omega}{\Omega} - \frac{1}{\rho}\frac{dp}{d\rho}(\sigma - \rho v^2 + 2\rho\Phi) + \int_0^p\frac{\bar{p}\,d\bar{p}}{[\rho(\bar{p})]^2}, \tag{22}$$

where ΔC is a constant. The Newtonian quantities that appear in these equations are a time-independent and rigidly rotating solution to (**8**)–(**10**). In particular the velocity is a constant multiple of ϕ^a, the rotational Killing field: $v^a = \Omega\phi^a$. The quantity $\Delta\Omega$ represents the post-Newtonian correction to the angular velocity of the rotating star.[d] Thus, the full post-Newtonian angular velocity is $\Omega + \Delta\Omega$. We note that the post-Newtonian equations (**19**)–(**22**), as well as those that appear throughout the remainder of this paper, are (slightly) more general than those derived by

[d]Throughout this paper we use the notation ΔQ to denote the post-Newtonian change in a quantity Q.

Cutler,[11] in that we allow here for the possibility that the Newtonian and post-Newtonian angular velocities are not equal (i.e., $\Delta\Omega \neq 0$).

Equations (**19**)–(**22**) are easily solved using the same techniques that were developed to find the structure of a rotating Newtonian star.[7,11] In particular, the first step is to choose the post-Newtonian correction to the angular velocity of the star $\Delta\Omega$. There are two free parameters in selecting which (rigidly rotating) post-Newtonian stellar model to associate with a given Newtonian model; the choice of $\Delta\Omega$ is one of them. Having made a suitable choice for $\Delta\Omega$, (**19**) determines the post-Newtonian correction to the fluid velocity w^a. The second step is to solve (**20**) for the post-Newtonian gravitomagnetic field A^a. This involves the numerical solution of Poisson's equation for the various components of A^a (with the boundary condition $A^a \to 0$ as $r \to \infty$, where r is the spherical radial coordinate) using well-known numerical techniques. The third step is to determine the post-Newtonian gravitational potential Ψ by eliminating σ from (**21**) and (**22**). The resulting equation has the form

$$D^aD_a(\Psi + c^2\Delta C) + 4\pi G\rho\frac{d\rho}{dp}(\Psi + c^2\Delta C) = 4\pi G(2\rho v^2 + 3p - 2\rho\Phi)$$

$$+ 4\pi G\rho\frac{d\rho}{dp}\left(\frac{1}{4}v^4 - 2\Phi v^2 + v^aA_a + v^2c^2\frac{\Delta\Omega}{\Omega} + \int_0^p\frac{\bar{p}\,d\bar{p}}{[\rho(\bar{p})]^2}\right). \quad (23)$$

This is an elliptic equation for $\Psi + c^2\Delta C$ whose right side depends only on the Newtonian and the previously determined post-Newtonian fields. A boundary condition for this potential must be specified in order that this equation have a unique solution. This boundary condition fixes the post-Newtonian correction to the mass of the stellar model ΔM. This is the remaining free parameter needed to specify which post-Newtonian model is associated with a given Newtonian model. It is possible, for example, to select the post-Newtonian star with the same gravitational mass as the Newtonian model. This is accomplished by imposing the boundary condition $r\Psi \to 0$ as $r \to \infty$. Alternative boundary conditions can be imposed by adding to this Ψ some appropriate solution of the homogeneous version of (**23**). Once the appropriate boundary condition has been selected, (**23**) can be solved numerically using standard techniques.[7] Finally, having solved for $\Psi + c^2\Delta C$, the post-Newtonian field σ is fixed by (**22**).

The time-dependent fields $\delta\sigma$, δw^a, $\delta\Psi$, and δA^a that describe the post-Newtonian corrections to the pulsations of a star are determined from the linearized post-Newtonian equations [i.e., the time-dependent generalizations of (**19**)–(**22**)]. Cutler[11] has shown that the solutions to these equations having time dependence $e^{-i(\omega+\Delta\omega)t}$ and angular dependence $e^{im\phi}$ are determined completely by two scalars δW and $\delta\Psi$ and the vector δA^a. The other post-Newtonian fields are determined from these by equations that are analogous to the Newtonian expressions (**13**) and (**14**):

$$\delta\sigma = \rho\frac{d\rho}{dp}(\delta W - \delta\Psi), \quad (24)$$

$$\delta w^a = iQ^{ab}[D_b\delta W - ic^2\Delta\omega\delta v_b - i(\omega - m\Omega)\delta A_b - D_b(v^c\delta A_c) - \delta B_b]. \quad (25)$$

In these expressions Q^{ab} is the tensor defined in (15), $\Delta\omega$ is the post-Newtonian correction to the frequency of the mode, and δB_a is a rather complicated quantity that depends only on the previously determined Newtonian and stationary post-Newtonian fields:

$$\delta B_a = \frac{1}{\rho} D_a\left\{\left[(v^2 - 2\Phi)\frac{d}{d\rho}\left(\rho\frac{dp}{d\rho}\right) - \sigma\frac{d^2p}{d\rho^2}\right]\delta\rho - 2\rho\frac{dp}{d\rho}(\delta\Phi - v^b\delta v_b)\right\} - \frac{\delta\rho}{\rho}D_a\Psi$$
$$- \left(2\Phi + v^2 + \frac{p+\sigma}{\rho}\right)D_a\delta\Phi - \left(\Phi + \frac{1}{2}v^2 - \frac{p+\sigma}{\rho} - c^2\frac{\Delta\Omega}{\Omega}\right)D_a\delta U$$
$$+ iv_a\left\{\frac{\omega}{\rho}\frac{dp}{d\rho}\delta\rho - (2\omega - 3m\Omega)\delta\Phi - [(\omega - m\Omega)v_b + 4iD_b\Phi]\delta v^b\right\}$$
$$- D_a\Phi\left[2v^b\delta v_b + 2\delta\Phi + \left(v^2 + 2\Phi + \frac{dp}{d\rho}\right)\frac{\delta\rho}{\rho}\right] + D_a(v^bA_b)\frac{\delta\rho}{\rho} - i\omega c^2\frac{\Delta\Omega}{\Omega}\delta v_a$$
$$- v^bD_av_b\left[2v_c\delta v^c + \left(2\Phi - \frac{dp}{d\rho} - 2c^2\frac{\Delta\Omega}{\Omega}\right)\frac{\delta\rho}{\rho}\right] + (D_aA_b - D_bA_a)\delta v^b. \tag{26}$$

The equation that determines the vector δA^a is

$$D^bD_b\delta A_a = 16\pi G(v_a\delta\rho + \rho\delta v_a) - i\omega D_a\delta\Phi. \tag{27}$$

This is simply Poisson's equation with a source that depends only on the previously determined Newtonian fields. It can be solved numerically using standard techniques. The equations for the two scalar potentials δW and $\delta\Psi$ are inhomogeneous generalizations of the Newtonian pulsation equations (17) and (18):

$$D^a(\rho Q^{ab}D_b\delta W) + (\omega - m\Omega)\rho\frac{d\rho}{dp}(\delta\Psi - \delta W) =$$
$$\Delta\omega\, c^2[\delta\rho + iD_a(\rho Q^{ab}\delta v_b)] + \delta X_W, \tag{28}$$

$$D^aD_a\delta\Psi + 4\pi G\rho\frac{d\rho}{dp}(\delta\Psi - \delta W) = 4\pi G\delta X_\Psi. \tag{29}$$

The linear operators on the left sides of these equations are precisely the same as those in (17) and (18). The right sides depend on the post-Newtonian correction to the frequency $\Delta\omega$, and the previously determined fields. In particular, the quantities δX_W and δX_Ψ depend only on previously determined fields:

$$\delta X_W = -\rho(\omega + m\Omega)\delta\Phi + D_a\{\rho Q^{ab}[\delta B_b + i(\omega - m\Omega)\delta A_b + D_b(v^c\delta A_c)]\}$$
$$+ m\Omega\left[\rho v^a\delta v_a - \left(\frac{dp}{d\rho} + c^2\frac{\Delta\Omega}{\Omega}\right)\delta\rho\right] + iD_a\left[\left(\sigma + p + \rho\Phi - \frac{1}{2}\rho v^2\right)\delta v^a\right], \tag{30}$$

$$\delta X_\Psi = \left(v^2 + 3\frac{dp}{d\rho}\right)\delta\rho + 2\rho v_a\delta v^a. \tag{31}$$

Now, the homogeneous versions of (28) and (29) admit a nontrivial solution, namely

δU and $\delta\Phi$. It follows that the inhomogeneous equations will admit no solution at all unless the sources on the right side are orthogonal (in an appropriate sense, see Cutler[11]) to this homogeneous solution. This constraint determines the post-Newtonian change in the frequency:

$$\Delta\omega\, c^2 \int (\rho\delta v_a^* \delta v^a + \delta\rho\delta U^*)\, d^3x = \int [(\omega - m\Omega)\delta X_\Psi \delta\Phi^* - \delta X_W \delta U^*]\, d^3x + i\int \sigma\delta\Phi^* \delta v^a n_a d^2x. \quad (32)$$

The volume integrals are to be performed over the interior of the stellar model where $\rho > 0$, while the surface integral is to be performed over the boundary of this region. The outward directed unit normal to this surface is denoted n_a. The surface integral that appears in this expression is needed to enforce the boundary condition $\delta\Psi \to 0$ as $r \to \infty$. It arises because $\delta\Psi$ is not continuously differentiable at the star's surface unless $\sigma = 0$ there. The post-Newtonian correction to the frequency $\Delta\omega$ is completely determined by evaluating these integrals. It can be shown that $\Delta\omega$ is real whenever ω is real. Once $\Delta\omega$ has been determined, (**28**) and (**29**) can be solved for the potentials δW and $\delta\Psi$. Note that it is not necessary to determine these potentials if only the frequency of the mode is desired.

NUMERICAL RESULTS

We have used the methods described in the previous section to evaluate numerically the post-Newtonian corrections to the pulsation frequencies of rapidly rotating neutron stars. For this study we have selected a simple polytropic equation of state, $p = 10^5\rho^2$ (in cgs units), whose parameters were chosen to reproduce approximately the macroscopic properties of more realistic neutron-star models.

This analysis begins with the construction of an appropriate sequence of equilibrium rotating stellar models. Even for nonrotating stars, there is freedom to select which post-Newtonian model to associate with a given Newtonian model. After a certain amount of numerical experimentation, we found it to be convenient to associate nonrotating models having the same M/R ratios, where M is the gravitational mass and R is the radius of the star (see also Balbinski *et al.*[18]). This constraint is imposed by adding to any given solution to (**23**) a sufficient amount of the homogeneous solution so that $\Delta R/R = \Delta M/M$. In this study we consider post-Newtonian models having gravitational mass $M + \Delta M = 1.400\, M_\odot$. The values of M and R for this model are listed in TABLE 1, along with the corresponding values for the Newtonian and general-relativistic stellar models based on the same equation of state and having the same M/R ratio. We note that relativistic effects are fairly important in these stellar models: $\Delta M/M \approx -0.2$. We also note that the post-Newtonian parameters agree with the general-relativistic values to within about $(\Delta M/M)^2 \approx 0.04$. This is the expected magnitude of the second-order post-Newtonian corrections.

A sequence of rotating Newtonian stellar models is constructed numerically using the methods developed by Ipser and Lindblom.[6,7] Each model in this sequence has the same mass, $M = 1.736\, M_\odot$, as the nonrotating model previously selected. Associated with this sequence of Newtonian models is a sequence of rotating post-Newtonian models, each with mass $M + \Delta M = 1.400\ M_\odot$. Having selected the mass of the post-Newtonian models, there is still the freedom to choose the post-Newtonian correction to the angular velocity for each model in the sequence. It might appear natural to allow the post-Newtonian model to have the same angular velocity as its Newtonian counterpart: $\Delta\Omega = 0$. Numerical experimentation shows, however, that this is in fact a poor choice. The reason is that post-Newtonian gravity is stronger than Newtonian gravity. Thus, a post-Newtonian star will be less distorted in shape by its rotation than its Newtonian counterpart rotating at the same angular velocity. In fact the $1.400\ M_\odot$ post-Newtonian star that rotates at the maximum angular velocity of our sequence of $1.736\ M_\odot$ Newtonian models is not particularly rapidly rotating. It is more appropriate, therefore, to associate models whose angular velocities are related in some more dynamically meaningful way. Various studies[7,19,20] have shown that sequences of rotating stellar models all terminate when the ratio $\Omega/\Omega_0 \approx 0.6$, where $\Omega_0^2 = \pi G\bar{\rho}_0 \equiv 3GM/4R^3$ (with M the mass and R the radius of the nonrotating star in the sequence). This result applies to both Newtonian and general-relativistic stellar models, and is essentially independent of the equation of

TABLE 1. NONROTATING STELLAR PARAMETERS*

	Q_N	$Q_N + \Delta Q_{PN}$	Q_{GR}
$M/M_\odot$	1.736	1.400	1.352
R(km)	15.343	12.374	11.959
$\Omega_0(s^{-1})$	6917	8256	8871
$-\omega_2/\Omega_0$	1.416	1.423	1.374
$-\omega_3/\Omega_0$	1.960	1.854	1.801
$-\omega_4/\Omega_0$	2.351	2.177	2.124
$-\omega_5/\Omega_0$	2.667	2.448	2.393
$-\omega_6/\Omega_0$	2.939	2.684	2.629

*The Newtonian (Q_N), post-Newtonian ($Q_N + \Delta Q_{PN}$), and general-relativistic (Q_{GR}) value is given for each quantity.

state of the stellar material. This ratio is, therefore, a dynamically meaningful measure of the star's angular velocity. Thus, we choose the angular velocity of the post-Newtonian stellar model so that this ratio is the same as its Newtonian counterpart: $\Delta(\Omega/\Omega_0) = 0$. This condition is equivalent to $\Delta\Omega/\Omega = \Delta\Omega_0/\Omega_0$.

A sequence of stellar models (Newtonian or general relativistic) terminates when the angular velocity of the star is equal to the angular velocity of a test particle that orbits at the star's surface in the equatorial plane. For Newtonian stellar models this orbital or "Keplerian" angular velocity is given by the expression

$$\Omega_K^2 = \frac{1}{r}\frac{d\Phi}{dr}, \tag{33}$$

where the quantity on the right side is to be evaluated in the equatorial plane at the surface of the star. The sequence of equilibrium stellar models terminates when $\Omega = \Omega_K$. For the equation of state used in this study, this maximum angular velocity occurs at $\Omega_K/\Omega_0 = 0.635$. By solving the geodesic equation for the metric in (3), the post-Newtonian correction to the Keplerian angular velocity is found to be

$$\Delta\Omega_K = \frac{1}{2rc^2\Omega_K}$$

$$\left[2\Phi\frac{d\Phi}{dr} + \frac{d\Psi}{dr} - \frac{\Omega_K}{\Omega}\frac{d(v^a A_a)}{dr} + \Omega_K^2\frac{d(r^2\Phi)}{dr} + c^2\left(\frac{d^2\Phi}{dr^2} - \Omega_K^2\right)\Delta r\right], \quad (34)$$

where the quantities on the right side are evaluated in the equatorial plane at the surface of the star, and Δr represents the post-Newtonian change in the radial coordinate of the surface of the star at the equator. The sequence of post-Newtonian stellar models considered in this study terminates at $\Omega_K/\Omega_0 + \Delta(\Omega_K/\Omega_0) = 0.634$. This near equality between the Newtonian and post-Newtonian values of Ω_K/Ω_0 shows that it is appropriate to associate the Newtonian and post-Newtonian models having the same Ω/Ω_0 ratio.

Having constructed an appropriate sequence of rotating post-Newtonian stellar models, it is fairly straightforward to determine the post-Newtonian corrections to the frequencies of the modes $\Delta\omega$. The first step is to solve (27) for the potential δA^a. This can be accomplished using fairly standard numerical techniques. The second, and final, step is to evaluate the integrals in (32). The details of the numerical methods involved in these steps will be published in a forthcoming paper on this work.

We have used the techniques just described to investigate the properties of the $l = m$ f-modes of rapidly rotating neutron stars. These are the modes that are most subject to the gravitational-radiation-driven secular instability.[8,11,12] The frequencies of these modes for the nonrotating stellar models previously described are given in TABLE 1 for $2 \leq m \leq 6$. In this paper we are concerned with the modes that propagate in the direction opposite to the star's rotation (i.e., the $l = m$ modes), so $\omega_m(0)$ is negative. For comparison, the frequencies of these modes for the corresponding general-relativistic stellar models are also given in TABLE 1. We expect that the post-Newtonian approximation will give the frequencies of rotating stars to a similar level of accuracy. It is convenient to describe the frequencies of the modes of rotating stars in terms of the dimensionless function[5,7]

$$\alpha_m(\Omega) = \frac{\omega_m(\Omega) - m\Omega}{\omega_m(0)}. \quad (35)$$

The post-Newtonian correction to this function, $\Delta\alpha_m(\Omega)$, is related to the post-Newtonian change in the frequency of the mode, $\Delta\omega_m(\Omega)$, by

$$\Delta\alpha_m(\Omega) = \frac{1}{\omega_m(0)}[\Delta\omega_m(\Omega) - m\Delta\Omega - \alpha_m(\Omega)\Delta\omega_m(0)]. \quad (36)$$

The Newtonian functions $\alpha_m(\Omega)$ and their post-Newtonian counterparts $\alpha_m(\Omega) + \Delta\alpha_m(\Omega)$ are depicted in FIGURE 1. We note that the post-Newtonian functions are smaller than their Newtonian counterparts by as much as 10 percent.

The $l = m$ f-modes are unstable to the emission of gravitational radiation when the angular velocity of the star exceeds the angular velocity where the frequency of the mode passes through zero. Thus it is of great interest to determine the values of

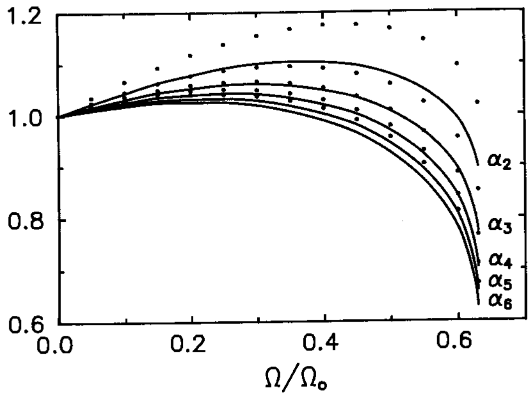

FIGURE 1. The angular-velocity dependence of the frequencies of the $l = m$ f-modes for $2 \leq m \leq 6$. These frequencies are displayed as the Newtonian (*dotted curves*) and post-Newtonian (*solid curves*) values of the functions $\alpha_m(\Omega)$ $[\omega_m(\Omega) = -m\Omega]/\omega_m(0)$. *Key:* · Newtonian; —— post-Newtonian.

these critical angular velocities. The Newtonian equation for the critical angular velocities $\omega_m(\Omega_c) = 0$ can be transformed into the form

$$\frac{\Omega_c}{\Omega_0} = -\frac{\omega_m(0)}{m\Omega_0}\alpha_m(\Omega_c). \tag{37}$$

This equation is easily solved numerically, since the function α_m as defined in (35) is slowly varying. The post-Newtonian condition for a critical angular velocity is $\omega_m + \Delta\omega_m = 0$. This may be transformed into an equation for $\Delta\Omega_c$ using the expressions for α_m and $\Delta\alpha_m$ in (35) and (36):

$$\Delta\frac{\Omega_c}{\Omega_0} = -\frac{\alpha_m(\Omega_c)\Delta[\omega_m(0)/\Omega_0] + [\omega_m(0)/\Omega_0]\,\Delta\alpha_m(\Omega_c) + [\Omega_c\omega_m(0)/\Omega_0^2][d\alpha_m(\Omega_c)/d\Omega]\,\Delta\Omega_0}{m + \omega_m(0)\,[d\alpha_m(\Omega_c)/d\Omega]}. \tag{38}$$

We find no critical angular velocity (Newtonian or post-Newtonian) having $\Omega < \Omega_K$ for the $l = m = 2$ mode. The values of the critical angular velocities are given in TABLE 2 for the $3 \leq m \leq 6$ modes. The ratios of the post-Newtonian critical angular velocities to Ω_0 are smaller than their Newtonian counterparts by up to about 8 percent. (Since the post-Newtonian Ω_0 is larger than its Newtonian counterpart by about 20 percent, the actual critical angular velocity of the post-Newtonian star is larger, however.) Thus, post-Newtonian hydrodynamic effects tend to lower the maximum value of this ratio in rotating neutron stars by up to about 8 percent. In order to determine the actual upper limit on the angular velocity, however, a more complicated calculation that includes the effects of gravitational radiation and viscous dissipation on the modes would have to be carried out.[8]

The angular velocities of rotating stellar models are often parameterized by the dimensionless quantity $\tau = -K/W$, the ratio of the rotational kinetic energy of the star to its gravitational potential energy. We use the general-relativistic definitions of these quantities given by Friedman, Ipser, and Parker:[19]

$$K = \frac{\Omega}{2c} \int (\epsilon + p) u^\alpha \phi_\alpha u^\beta \, dS_\beta, \tag{39}$$

$$W = \int \{[2(\epsilon + p) u^\alpha u^\beta + (\epsilon - p) g^{\alpha\beta}] t_\alpha - \epsilon u^\beta\} \, dS_\beta - K, \tag{40}$$

where t_α is the globally timelike Killing field, ϕ_α is the rotational Killing field, and the integrals are performed over a t = constant hypersurface with volume element dS_β. Using the expansions in (3)–(7) for the various quantities that appear in these integrals, expressions may be obtained for K and W in the Newtonian and the post-Newtonian approximations. The first-order terms in these expansions are the

TABLE 2. CRITICAL ANGULAR VELOCITIES

$l = m$	$\frac{\Omega_c}{\Omega_0}$	$\frac{\Omega_c}{\Omega_0} + \Delta\frac{\Omega_c}{\Omega_0}$	τ_C	$\tau_C + \Delta\tau_C$
3	0.610	0.583	0.0798	0.0615
4	0.560	0.522	0.0582	0.0439
5	0.515	0.475	0.0453	0.0340
6	0.477	0.436	0.0368	0.0277

standard Newtonian expressions for K and W:[21]

$$K = \frac{1}{2} \int \rho v^2 \, d^3x, \tag{41}$$

$$W = \frac{1}{2} \int \rho \Phi \, d^3x. \tag{42}$$

The post-Newtonian corrections to these quantities are given by the second-order terms in the expansions of (39) and (40):

$$\Delta K = \frac{1}{2c^2} \int \left[\left(\sigma + p - 4\rho\Phi + 2\rho c^2 \frac{\Delta\Omega}{\Omega} \right) v^2 + \rho v^a A_a \right] d^3x, \tag{43}$$

$$\Delta W = -\Delta K + \frac{1}{2c^2} \int \left[\rho \left(-13v^2\Phi - \Phi^2 + \frac{1}{4}v^4 + 2\Psi + 6v^2c^2\frac{\Delta\Omega}{\Omega} + 2v^a A_a \right) \right.$$
$$\left. + 4p(v^2 - 3\Phi) + \sigma(3v^2 + 2\Phi) + 6\frac{dp}{d\rho}(\sigma + 2\rho\Phi - \rho v^2) \right] d^3x. \tag{44}$$

We have evaluated the ratio $\tau = -K/W$ and its post-Newtonian correction $\Delta\tau = K\Delta W/W^2 - \Delta K/W$ for the stars rotating at the critical angular velocities of the $l = m$

f-modes. These values are given in TABLE 2. We note that the values of $\tau_c + \Delta\tau_c$ are reduced from their Newtonian counterparts by almost 20 percent. This large discrepancy is due to the fact that $\tau = \tau(\Omega)$ is a rather nonlinear function of Ω. We have also evaluated τ for the stellar model rotating at the maximum angular velocity of the sequence. We find $\tau \leq 0.1026$ for the Newtonian sequence of rotating stars, and $\tau + \Delta\tau \leq 0.1035$ for the post-Newtonian sequence.

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