STELLAR STABILITY ACCORDING TO NEWTONIAN THEORY AND GENERAL RELATIVITY

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This paper reviews the theory of the stability of stellar models in both the Newtonian theory and general relativity. The emphasis here is on recent work on the stability of rapidly rotating stellar models, and the effects of dissipation on stability.

1 Introduction

Observations are beginning to provide a wealth of information about stars in which relativistic effects play an important role. Measurements of the pulsation periods of pulsars show that neutron stars can have rotation periods as short as 1.56ms.[1] Measurements of the orbital elements of pulsars in binary systems now give accurate determinations of the masses of about ten neutron stars.[2] To understand the meaning of these (and other related) observations the appropriate theoretical tools must be developed for analyzing the structures and stability of relativistic stellar models. The techniques for constructing and analyzing equilibrium stellar models from a given equation of state are now well understood.[3,4] So it is relatively easy now to compare the observable macroscopic properties (e.g. masses, angular velocities, etc.) of these models with the observations. The inverse problem of determining the poorly known equation of state from the observable properties of relativistic stars is only beginning to be understood.[5]

In addition to a thorough understanding of equilibrium stellar models, the stability of these models must also be understood in order to interpret the observations. Stability theory is required, for example, to determine the ranges of masses and angular velocities present in stable (and thus physically possible) stars. This paper reviews the theory of the stability of stellar models in both the Newtonian theory and general relativity. The emphasis here is on recent work on the stability of rapidly rotating stellar models, and the effects of dissipation on stability.

Our understanding of the stability of rotating stars was (until quite recently) based entirely on the analysis of the stability of the uniform density rigidly rotating stellar models: the Maclaurin spheroids. It has been known for over a century that rapidly rotating Maclaurin spheroids were subject to an instability driven by viscosity.[6] This instability causes a rapidly rotating Maclaurin spheroid to evolve into a rigidly rotating but non-axisymmetric configuration such as a Jacobi ellipsoid.[7,8] This type of instability is referred to as *secular* since it is driven by dissipative forces in the star. The Maclaurin spheroids are also subject to a second type of secular instability that is driven by gravitational radiation reaction.[9,10] This instability causes the Maclaurin spheroid to evolve into a stationary but non-axisymmetric configuration such as a Dedekind ellipsoid.[11] Maclaurin spheroids with very large angular momenta (about 1.7 times that required to trigger the viscous secular instability) are also subject to a *dynamical* instability that is driven by purely hydrodynamical forces.[12]

During the past two decades new mathematical techniques have been developed which make it possible to study and analyze the stability of more realistic stellar models than the simple Maclaurin spheroids. These recent developments will be described and reviewed in this paper. Section 2 begins with a brief discussion of the criteria that have been developed for evaluating the stability of non-rotating stellar models. Section 3 describes techniques for evaluating the stability of rotating stars based on global energy functionals. Section 4 presents the analytical techniques needed to analyze the normal modes of rotating stars in the Newtonian theory and general relativity. Section 5 completes the discussion of the normal modes by showing how dissipation effects their stability.

2 Non-Rotating Stars

The theory of the stability of non-rotating, spherical, stellar models in general relativity theory is now rather well understood. Criteria have been found which allow the stability of these stars to be evaluated without solving explicitly the dynamical perturbation equations. While this has not been an active research area for some time, the results are rather interesting and not generally well known. This section is devoted to a brief summary of this work.

The stability of static spherical stars in general relativity theory to spherically symmetric perturbations can be determined by examining the mass-radius curve for equilibrium stellar models. Consider the one-parameter family of stellar models that is constructed from a particular equation of state: $\rho = \rho(p)$. The central pressure, p_c, of these models is a convenient choice for the parameter that distinguishes members of the family. Now consider $M(p_c)$ and $R(p_c)$, the total mass and total radius of the model with central pressure p_c, as determined by solving the equilibrium structure equations.[3] Figure 1 illustrates the curve $[M(p_c), R(p_c)]$ with parameter p_c, which represents the masses and radii of the stellar models constructed for one particular "realistic" neutron star equation of state. The points on this curve where the mass is an extremum (labeled by the letters A, B, C, ...) are places where the stability changes in one of the spherical modes.[13] The stellar models represented by the thickened portion of the curve are unstable.

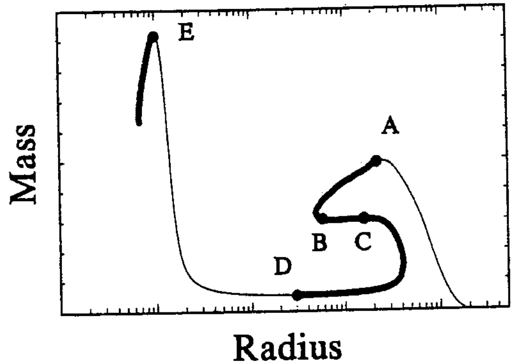

Figure 1: Total Mass-Radius Curve for Neutron Stars.

The dynamics of the spherically symmetric modes of these stars is determined by a second-order equation of Sturm-Liouville type with eigenvalue ω^2, where ω is the frequency of the mode.[14,15] The eigenvalues of this equation are real, hence the frequencies are either real and the mode is stable, or purely imaginary and the mode is unstable. The frequencies of the modes change continuously along a smooth one-parameter family of stellar models.[a]The stability of a mode, and hence the stability of the stellar model, can change therefore only at points along the sequence where a zero frequency modes exists.

At an extremum of the mass-radius curve, the transformation that takes the extreme model into an infinitesimally nearby model is a solution to the time-independent linearized structure equations. This solution leaves the mass of the model fixed, and is a solution of the oscillation equation with zero frequency. Thus, the extrema of the mass-radius curve are models where the stability of some mode is changing. For realistic neutron star equations of state,

[a]The smoothness conditions required of the equation of state to guarantee the continuity of these eigenvalues have not been worked out to date.

the lowest density models are known to be stable.[13] Thus the first extremum, A in Figure 1, is a point where the models first become unstable. The thickened portion of the curve represents the higher density models which are unstable. The second extremum, B in Figure 1, represents another point where some mode is changing stability. In the zero-frequency mode at this point, however, the outer surface of the star gets larger as the central density of the star increases. Therefore the function that describes the radial displacement of each fluid element (which is proportional to the eigenfunction of this mode) must have an odd number of nodes, unlike the mode changing stability at A. The eigenfunction associated with the smallest eigenvalue of a Sturm-Liouville systems has no nodes, and the modes with successively larger eigenvalues have eigenfunctions with successively larger numbers of nodes.[16] This implies that the mode changing stability at B is not the fundamental mode, and the stellar models beyond point B must have two unstable modes.[b] Similar arguments reveal that the stellar models remain unstable up to the point D where stability is regained. The models between D and E are stable neutron stars, while those beyond point E are unstable.

The stability of stars to non-radial perturbations is determined by the quantity S,

$$S(r) = \frac{dp}{dr} - \left(\frac{\partial p}{\partial \rho}\right)_s \frac{d\rho}{dr}, \tag{1}$$

where r is the radial spherical coordinate, p is the pressure of the fluid, ρ is the total energy density (including rest mass), and s is the entropy per particle of the stellar fluid. When S is positive the adiabatic exchange of fluid masses at different "elevations" within the star requires the addition of energy to the system.[17] When S is negative in some region, however, the energy of the configuration can be lowered by re-arranging the fluid. In this region, consequently, the stellar fluid is unstable to convection. It has been shown that the condition $S > 0$ everywhere within the star is the necessary and sufficient condition for the stability of the non-radial modes of Newtonian stellar models.[18,19,20] In general relativity theory it has also been shown that the non-radial outgoing quasi-normal modes are stable if $S > 0$ throughout the star.[21] The proof that this is also a necessary condition for stability of the non-radial modes in general relativity theory has not been completed to date.

[b]The standard theorems apply only to non-singular Sturm-Liouville systems. The radial pulsation equation is singular at $r = 0$ and to date the relevant properties of the eigenvalues and eigenfunctions of this equation have not been established rigorously to my knowledge.

3 Rotating Stars

The conditions that determine when rotating stars are unstable are not as well understood as in the non-rotating case. However, a few general results are known. The stability of uniformly rotating stars with respect to axisymmetric perturbations can be determined by a technique that is analogous to the mass-radius diagram analysis described above. Consider a one-parameter family of uniformly rotating equilibrium stellar models of fixed angular momentum, all based on a particular equation of state $\rho = \rho(p)$. It has been shown that the region of stable stars in this family is bounded by points where the mass is an extremum.[22] As in the non-rotating case, the extrema of the mass signal the onset of instabilities to axisymmetric perturbations in rotating stars. At these points there exists a time independent solution of the perturbation equations which takes one uniformly rotating solution into a nearby one. This solution transfers angular momentum among the fluid elements in order to preserve uniform rotation. Hence the instability that sets in at this point will develop on a time scale set by the dissipation process (e.g. viscosity) that facilitates the angular momentum redistribution in the fluid. The proof of this result is based on a very general stability theorem[23] which does not rely on the properties (e.g. existence or completeness) of the normal modes.

The stability of rotating stars with respect to non-axisymmetric perturbations is a very interesting and difficult subject, and this has been the focus of most of the research effort in this area in recent years. Simple local stability conditions analogous to eq. (1) have not been found and probably do not exist for rotating stars. A few global conditions have been found, however, and these have been extremely useful in understanding a number of interesting instabilities in rotating stars. These global conditions determine the stability of rotating stars from the properties of certain non-local functionals of the perturbations. Perhaps the most important example of such a functional is the energy E. This functional can be expressed as a Hermitian quadratic form in the perturbation fields integrated over the volume of the star. For perturbations that satisfy dissipation-free (e.g. no viscosity) evolution equations the energy E is conserved for all perturbations. Thus E is not a useful tool for diagnosing the presence of dynamical instabilities. When the effects of dissipation are considered, however, the energy functional E evolves with time; and in some circumstances it decreases monotonically for all fluid perturbations. Under these conditions E can be used to diagnose secular instabilities. If E is positive for all possible perturbations then the star is stable. The evolution equations in this case may only change E by decreasing its value toward its lower bound, zero. This ensures that the perturbation remains bounded (at least in an $\mathcal{L}^2$ sense). If the energy E were negative for some perturbation, however, then E would have no lower bound. The evolution equations would cause a perturbation with negative E to decrease without bound and the star would be unstable.

The emission of gravitational radiation by a stellar perturbation causes the energy functional E to decrease. Thus E can be used to test the secular stability of rotating stars with respect to the emission of gravitational radiation. When the functional E for rotating stars is examined in detail a remarkable fact emerges: every rotating star is unstable to the emission of gravitational radiation.[24,25] That is, there exists some perturbation in every rotating star for which E is negative. An example of such a negative energy perturbation can be found by considering the perturbations of the star that correspond to "waves" which propagate around the star in the direction opposite its rotation. If these waves propagate slowly enough then the underlying rotation of the star drags them along so they appear from infinity to propagate in the same direction as the star's rotation. Such waves have negative E and are unstable. The physical nature of this instability can be visualized as follows. These waves emit positive angular momentum gravitational radiation since they are seen from infinity to propagate in the same direction as the star's rotation. The waves themselves, however, carry negative angular momentum since they propagate (relative to the fluid in the star) in the direction opposite this rotation. Thus angular momentum can only be conserved for these perturbations by increasing the amplitude of the perturbation in order to decrease its angular momentum as it emits gravitational radiation.

A closely related functional $\tilde{E}$, which represents the energy of a perturbation as measured in the co-rotating reference frame of the star, has also been useful for diagnosing instabilities in rotating stars. For Newtonian stellar models this functional has an extremely simple form:

$$\tilde{E} = \frac{1}{2}\int \left(\rho\,\delta v_a^* \delta v^a + \frac{\delta\rho^*\delta p}{\rho} - \delta\rho^*\delta\Phi\right) d^3x, \tag{2}$$

where ρ is the mass density, and δv^a, $\delta\rho$, δp, and $\delta\Phi$ are the perturbations in the fluid velocity, density, pressure, and gravitational potential respectively. An analogous functional is also known in the general relativistic case.[26] $\tilde{E}$ is conserved for fluid perturbations that satisfy dissipation-free evolution equations, hence it is not a useful diagnostic of dynamical instabilities. Internal fluid dissipation causes $\tilde{E}$ to decrease with time. Thus, $\tilde{E}$ can be used to diagnose secular instabilities that are driven by viscous forces in rotating stars. The study of this functional has revealed that thermal conductivity and bulk viscosity can cause the same type of secular instability as shear viscosity in rotating stars.[27]

The use of these energy functionals to diagnose instabilities is based on the expectation that any negative energy perturbation will grow without bound and thus represent an instability. While this is believed to be the case for each of the energy functionals discussed above, the careful mathematical analysis needed to establish this has only been completed to date for the Newtonian $\tilde{E}$ in a star having viscosity and thermal conductivity but no interaction with gravitational radiation. In this case it has been shown that $\tilde{E}$ is strictly decreasing with time unless $\tilde{E}$ vanishes.[28] This shows that a necessary condition for stability is that $\tilde{E} \geq 0$ for all fluid perturbations.

The effects of gravitational radiation cause the functional E to decrease with time while viscous effects cause $\tilde{E}$ to decrease. Unfortunately, neither functional is decreasing for every perturbation when both viscous and gravitational radiation effects are considered simultaneously. Thus in general neither functional (nor any known combination of them) can be used to diagnose these secular instabilities except in special cases. For very slowly rotating stars the waves with negative E that are subject to the gravitational radiation driven secular instability have very short wavelengths. These waves couple only weakly to gravitational radiation but very strongly to viscosity. Under these conditions it has been shown that the functional $\tilde{E}$ is a decreasing function of time while E is not.[26] Thus, $\tilde{E}$ may be used to evaluate the secular stability of these perturbations while E may not. This analysis reveals that any amount of viscosity suppresses the gravitational radiation driven secular instability in sufficiently slowly rotating stars.

4 Normal Modes

The analysis of the energy functional stability criteria discussed in Section 3 has revealed that gravitational radiation tends to make all rotating stars unstable, while viscous forces tend to suppress this instability. Unfortunately there is no known functional that always decreases with time when all of the relevant dissipative forces are present together. Thus no generally applicable test for the stability of rotating stars is presently available at all. The study of the stability of rotating stars has been directed therefore toward the study of the normal modes of rotating stars: solutions of the perturbation equations having time dependence $e^{i\omega t}$. This analysis provides a sufficient test for instability: the instability of one mode proves that the star is unstable.[c] Even the analysis of the normal modes of rotating stars turns out to be a rather difficult and interesting

[c] Lacking a proof of the completeness of the normal modes, however, stability of all normal modes does not prove that the star is stable.

subject however. Considerable progress has been made in transforming this problem into a more tractable form in recent years. The analysis that leads to this simplification is simple and elegant, and so it is presented here in some detail for the simplest case of Newtonian stellar models.

In real stars the effects of dissipation are rather weak in that dissipative effects occur on time scales that are much longer than the dynamical time scale. Under these conditions it is possible to ignore the effects of dissipation as a first approximation. In this section the discussion is confined therefore to the simpler problem of the dissipation-free modes of rotating stars. The techniques for evaluating the effects of dissipation are discussed in Section 5.

The equations that govern the perturbations of a dissipation-free self-gravitating Newtonian fluid are given by

$$\partial_t \delta\rho + v^a \nabla_a \delta\rho + \nabla_a(\rho\delta v^a) = 0, \tag{3}$$

$$\partial_t \delta v^a + v^b \nabla_b \delta v^a + \delta v^b \nabla_b v^a = -\nabla^a \left(\frac{\delta p}{\rho} - \delta\Phi\right), \tag{4}$$

and

$$\nabla^a \nabla_a \delta\Phi = -4\pi G \delta\rho, \tag{5}$$

where any quantity preceded by δ represents the (Eulerian) perturbation of that quantity, while those not preceded by δ represent equilibrium values. In these equations ρ, p, Φ, and v^a represent the mass density, pressure, gravitational potential, and the fluid velocity. This system of equations is completed by specifying the thermodynamic relationship between the perturbed pressure and density. For simplicity here the equation of state is taken to be barotropic so that

$$\delta p = \frac{dp}{d\rho}\delta\rho. \tag{6}$$

The unperturbed equilibrium stellar model is assumed here to be rigidly rotating, i.e. $v^a = \Omega\varphi^a$ where Ω is the (constant) angular velocity and φ^a is the vector field representing rotations about the z^a axis.

The equations (3)–(6) that describe the perturbations of rotating stars constitute a complicated sixth-order system for the five independent components of the perturbation fields $(\delta\rho, \delta v^a, \delta\Phi)$. The solutions to these equations are known analytically only for the perturbations of uniform density stars[29] and have only been directly solved numerically for more realistic models quite recently.[30] Rather than attempt to solve these equations directly, two different approaches have been devised to reduce the complexity of the equations by analytical means. The first approach introduces a potential ξ^a, the Lagrangian displacement, for the velocity perturbation:

$$\delta v^a = \partial_t \xi^a + v^b \nabla_b \xi^a - \xi^b \nabla_b v^a. \tag{7}$$

Using this potential the perturbed continuity equation (3) can be solved analytically: $\delta\rho = -\nabla_a(\rho\xi^a)$. This substitution reduces the number of independent perturbation fields to four, $(\xi^a, \delta\Phi)$, and reduces the equations that must be solved to the system (4)–(6). One nice feature of this representation of the equations is the existence of a Lagrangian from which the equations in this form may be derived.[31] Unfortunately this representation also increases the order of the system of differential equations from sixth to eighth. For the purposes of actually solving the equations, this transformation does not offer much simplification. The equations have only been solved in this form (to my knowledge) numerically for the special case of axisymmetric normal modes.[32]

A second analytical transformation has been found that does significantly simplify the perturbation equations.[33] This transformation is limited to perturbations which are normal modes with angular dependence $e^{im\varphi}$, where φ is measured about the rotation axis of the star. For this case eq. (4) reduces to

$$\Big[i(\omega + m\Omega)\delta_{ab} + 2\nabla_b v_a\Big]\delta v^b = -\nabla_a\left(\frac{\delta p}{\rho} - \delta\Phi\right), \tag{8}$$

where δ_{ab} represents the three-dimensional Euclidean metric. This equation is *algebraic* in the velocity perturbation δv^a and can be solved analytically:

$$\delta v^a = iQ^{ab}\nabla_b \delta U, \tag{9}$$

where δU is defined as

$$\delta U = \frac{\delta p}{\rho} - \delta\Phi \tag{10}$$

and Q^{ab} is the tensor

$$Q^{ab} = \frac{1}{(\omega + m\Omega)^2 - 4\Omega^2}\left[(\omega + m\Omega)\delta^{ab} - \frac{4\Omega^2}{\omega + m\Omega} z^a z^b - 2i\nabla^a v^b\right]. \tag{11}$$

Using equation (9) to replace δv^a in the remaining perturbation equations reduces the system to a pair of second-order equations for the scalar potentials δU and $\delta\Phi$:

$$\nabla_a\left(\rho Q^{ab}\nabla_b\delta U\right) = -(\omega + m\Omega)\rho\frac{d\rho}{dp}(\delta U + \delta\Phi), \tag{12}$$

$$\nabla^a\nabla_a\delta\Phi = -4\pi G\rho\frac{d\rho}{dp}(\delta U + \delta\Phi). \tag{13}$$

This transformation has reduced the equations for the modes of rotating stars to this relatively simple fourth-order system for the two scalar potentials (δU, $\delta\Phi$). These equations constitute a reasonably standard eigenvalue problem with eigenvalue ω. The tensor Q^{ab} in eq. (12) is positive definite if $(\omega + m\Omega)^2 > 4\Omega^2$, so the equation is elliptic for sufficiently slowly rotating stars. These equations can be solved for the eigenfunctions δU and $\delta\Phi$ and the eigenvalue ω using fairly standard numerical techniques.[34,35] Figure 2 illustrates a typical eigenfunction δU for an $m = 3$ mode of a rapidly rotating Newtonian stellar model. Figure 3 illustrates the angular velocity dependence of the eigenvalue ω for two different sets of modes.[35,36] The frequencies in Figure 3 are displayed in terms of the dimensionless function $\alpha_m(\Omega)$,

$$\alpha_m(\Omega) = \frac{\omega(\Omega) + m\Omega}{\omega(0)}, \tag{14}$$

which is normalized so that $\alpha_m(0) = 1$ for non-rotating stars.

Once the eigenfunctions δU and $\delta\Phi$ are determined, then every other physical property of the stellar oscillation may be determined from them. Equation (9) gives the velocity perturbation δv^a in terms of δU, while the density perturbation $\delta\rho$ is given by

$$\delta\rho = \rho\frac{d\rho}{dp}(\delta U + \delta\Phi), \tag{15}$$

and the Lagrangian displacement ξ^a by

$$\xi^a = \frac{Q^{ab}\nabla_b\delta U}{\omega + m\Omega}. \tag{16}$$

The particular version of the equations presented here, eqs. (12)–(13), is for the special case of barotropic perturbations of rigidly rotating stellar models. This approach can also be used to reduce the equations for the general adiabatic perturbations of differentially rotating stellar models without any restriction (e.g. barotropic) on the equation of state.[37] The equations in the more general case remain, like eqs. (12)–(13), a fourth-order system for the two functions δU and $\delta\Phi$.

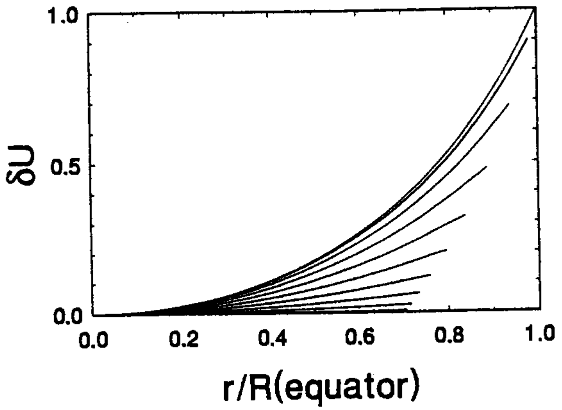

Figure 2: The eigenfunction δU for the $l = m = 3$ mode of a rapidly rotating Newtonian stellar model. Each curve represents the radial dependence of the eigenfunction along one angular spoke.

The problem of evaluating the modes of rapidly rotating stars has been rendered considerably simpler by the transformation that leads to eqs. (12)–(13). Nevertheless, there are some interesting questions that remain unresolved. The tensor Q^{ab} that appears in eq. (12) is positive definite whenever $(\omega + m\Omega)^2 > 4\Omega^2$. In this case eq. (12) is elliptic and can be solved using standard numerical techniques.[35] This condition is always satisfied in non-rotating stars; however, in more rapidly rotating models it may be violated. When this condition is violated eq. (12) becomes hyperbolic yet the physical solutions must still satisfy Dirchlet boundary conditions. Little appears to be known about hyperbolic eigenvalue problems of this kind. Numerical techniques based on a variational principle have been devised which give solutions to the equations even in this case however.[36,38] The change in signature of this equation does not appear to be connected to the onset of a physical instability. The physical significance of this change and the meaning of the characteristic surfaces that appear in eq. (12) are presently unknown.

In neutron stars the gravitational fields are rather strong and general relativistic effects significantly influence the structures and the dynamics. Thus it is of considerable interest to extend the analysis of the modes of rotating stars into the domain of general relativity theory. Unfortunately, this problem is extremely difficult. The chief obstacle is the coupling of these modes to

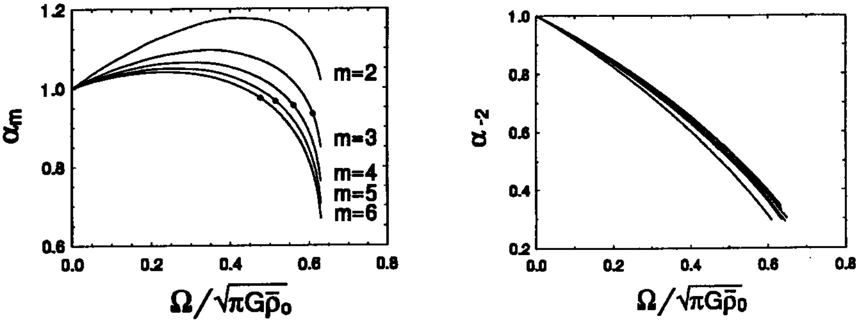

Figure 3: The angular velocity dependence of the frequencies of the modes of rapidly rotating Newtonian stellar models. The graph on the left gives the frequencies of the $l = m$ modes for $n = 1$ polytropic stellar models. The graph on the right gives the frequencies of the $l = -m = 2$ modes for stellar models constructed from thirteen realistic equations of state.

gravitational radiation. In general relativity theory a star may oscillate at any frequency at all! If gravitational radiation of a given frequency were directed toward a star, then the star would oscillate at that frequency. The definition of normal modes for general relativistic stars must be refined therefore to include as an additional boundary condition that there be no incoming gravitational radiation. These solutions are referred to as the quasi-normal modes. This boundary condition is difficult to enforce because it must be done far away from the star in the wave zone of the gravitational radiation. This is reasonably easy to deal with in the case of non-rotating stars where the spacetime outside the unperturbed star is simply the Schwarzschild geometry.[39,40,41] In rotating stars, however, the spacetimes outside the stars are only known numerically and only on rather small numerical grids. A practical method for imposing the outgoing radiation boundary condition on such spacetimes has not yet been devised. Fortunately there is a middle ground. The post-Newtonian approximation to general relativity provides a reasonably accurate description of the spacetimes associated with neutron stars. At the lowest orders the dynamics in the post-Newtonian approximation does not couple to gravitational radiation. Thus the problems associated with the outgoing radiation boundary condition does not arise in a post-Newtonian description of the modes of rotating stars.

It is reasonably straightforward to extend the Newtonian analysis of the modes of rotating stars to the post-Newtonian theory.[42,43] The oscillations of post-Newtonian stars are determined completely by the post-Newtonian corrections to the mode functions δU and $\delta\Phi$. These post-Newtonian eigenfunctions are determined by solving a pair of second-order equations having the same differential structures as eqs. (12)–(13) plus inhomogeneous terms that depend on δU and $\delta\Phi$ (and on the Newtonian and post-Newtonian structures of the equilibrium star). The post-Newtonian corrections to the frequency of a mode can be determined from the integrability condition for these pulsation equations, without solving the post-Newtonian pulsation equations at all! There exists an explicit formula for the post-Newtonian frequency that depends on δU and $\delta\Phi$ as well as the Newtonian and post-Newtonian structures of the star.[43] As is typical of post-Newtonian analyses, this formula is extremely complicated (and unenlightening). However, it is straightforward to evaluate the needed integrals numerically and so determine the frequencies of the modes in this approximation. Figure 4 compares the frequencies of several modes of non-rotating stars computed in this post-Newtonian approximation with the exact general relativistic values. The post-Newtonian approximation for the frequencies of $1.4M_\odot$ neutron stars agree with the exact general relativistic frequencies to within about 4%. In comparison, the Newtonian frequencies agree with the exact values only to within about 12% for these same neutron stars. Figure 5 illustrates the angular velocity dependencies of the frequencies of the modes of rotating stars in both the Newtonian and post-Newtonian approximations for stars with $GM/c^2R = 0.2$. The post-Newtonian frequencies for these modes differ from the Newtonian values by about 10%.

The analysis of the modes of rotating stars in full general relativity theory is far less complete. But, the general equations for these modes have been derived and a certain amount of analysis has been done with them. The general relativistic version of the Lagrangian displacement has been used to transform the equations into a simpler and more canonical form.[44] These equations have been very useful for analyzing the effects of general relativity on the secular instabilities of rotating stars.[25,26] These equations have never been solved (even numerically), however, except in the case of non-rotating stars.[39,40] The general relativistic version of the transformation that leads to eq. (9) has also been found. For modes with angular dependence $e^{im\varphi}$ the perturbed conservation laws, $\delta(\nabla_a T^{ab}) = 0$, can be solved analytically for the perturbed four velocity δu^a in terms of a scalar potential δU, defined by

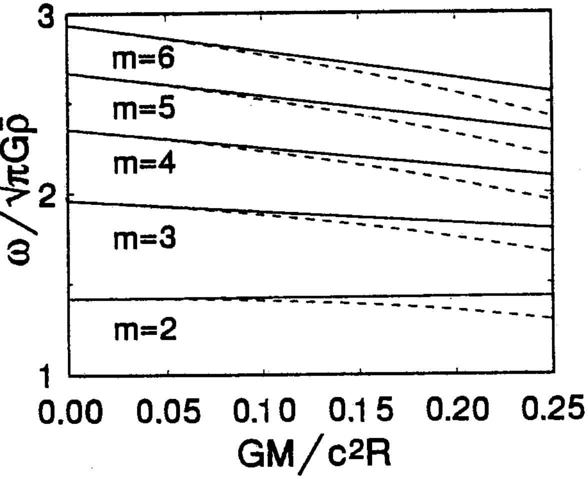

Figure 4: The post-Newtonian values for the frequencies of the modes of non-rotating stars (solid lines) are compared with the exact general relativistic values (dashed lines) for stars with different values of GM/c^2R.

$$\delta U = \frac{\delta p}{\rho + p}, \tag{17}$$

and the perturbed metric tensor δg_{ab}.[45] The resulting equation for δu^a,

$$\delta u^a = iQ^{ab}\nabla_b \delta U + \delta F^a(\delta g_{cd}), \tag{18}$$

is the relativistic analog of eq. (9). The vector δF^a that appears in eq. (18) depends on the metric perturbation δg_{ab} and the functions that describe the unperturbed star. The tensor Q^{ab} depends on the geometry of the unperturbed star and the frequency of the mode ω. This Q^{ab} is simply the relativistic generalization of eq. (11). There is also a general relativistic analog of eq. (12) which is derived by replacing the four-velocity perturbations in the energy conservation law using eq. (18). The resulting equation has the form

$$\nabla_a[(\rho + p)Q^{ab}\nabla_b \delta U] - Q^{ab}\nabla_a p \nabla_b \delta U + \Psi \delta U = \delta F(\delta g_{ab}) \tag{19}$$

where Ψ depends on the frequency of the mode and the unperturbed structure of the star, and δF depends on δg_{ab}. This equation is particularly useful when

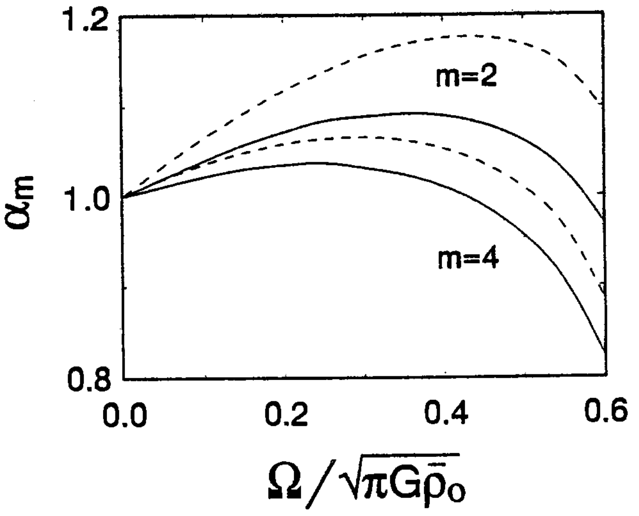

Figure 5: The angular velocity dependence of the frequencies of the modes of rotating stars. The post-Newtonian values for the frequencies (solid lines) are compared with Newtonian values (dashed lines) for stars with different angular velocities.

the dynamics of a mode is driven primarily by hydrodynamic rather than gravitational forces. Such is the case for the higher-order modes of stars,[46] as well as the modes of objects like accretion disks where self gravitational effects are not important. Under these circumstances the metric perturbations may be ignored and the complete dynamics of the general relativistic mode is determined by eq. (19) with $\delta F = 0$. This equation is no harder to solve in the relativistic case than it is for Newtonian stellar models. The equation in this form has been used to determine the modes of relativistic accretion disks.[47]

5 Dissipative Effects

Dissipation plays an important role in the stability of rotating stars. The general arguments outlined in Section 3 show that gravitational radiation tends to make all rotating stars unstable[24,25] while internal fluid dissipation processes (e.g. viscosity) tend to suppress this instability and make sufficiently slowly rotating stars stable.[26,48] In this section the techniques are described which have been used to evaluate the effects of dissipation on the stability of the normal modes of rotating stars. The principal tool that is used in this analysis is the equation that determines the evolution of the energy of the perturbation due to dissipative effects. For example, the evolution of $\tilde{E}$ defined in eq. (2) can be evaluated using the equations for a dissipative Newtonian fluid including the effects of gravitational radiation reaction forces:[49]

$$\begin{aligned} \frac{d\tilde{E}}{dt} &= -\int\left[2\eta\delta\sigma^{ab}\delta\sigma^*_{ab} + \zeta\delta\sigma\delta\sigma^*\right]d^3x \\ &\qquad -(\omega + m\Omega)\sum_l N_l\omega^{2l+1}\delta D_{lm}\delta D^*_{lm}. \end{aligned} \tag{20}$$

The thermodynamic functions η and ζ that appear on the right side of eq. (20) represent the viscosities of the fluid. The viscous forces in a fluid are driven by the shear $\delta\sigma_{ab}$ and the expansion $\delta\sigma$ of the perturbation:

$$\delta\sigma_{ab} = \frac{1}{2}\left(\nabla_a\delta v_b + \nabla_b\delta v_a - \frac{2}{3}\delta_{ab}\nabla_c\delta v^c\right), \tag{21}$$

$$\delta\sigma = \nabla_c\delta v^c. \tag{22}$$

The gravitational radiation reaction force couples to the evolution of the fluid via the mass multipole moments of the perturbation δD_{lm},

$$\delta D_{lm} = \int \delta\rho\, r^l Y^*_{lm} d^3x, \tag{23}$$

with coupling constant N_l:

$$N_l = \frac{4\pi G}{c^{2l+1}}\frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \tag{24}$$

Now consider the normal modes of a rotating star that is subject to dissipative effects. Assume that the time dependence of the mode is $e^{i\omega t - t/\tau}$, where ω is the real part of the frequency and $1/\tau$ is the imaginary part. A mode is stable if $1/\tau$ is positive and unstable if negative. Thus the problem of evaluating the stability of a mode is reduced to determining the sign of the imaginary part of its frequency. Equation (20) provides a means of evaluating this quantity. The functional $\tilde{E}$ is real and quadratic in the perturbations, so its time dependence is $e^{-2t/\tau}$. It follows that the imaginary part of the frequency is given by

$$\frac{1}{\tau} = -\frac{1}{2\tilde{E}}\frac{d\tilde{E}}{dt}. \tag{25}$$

The right side of eq. (25) is, using eqs. (2) and (20), a functional of the eigenfunction of the mode. This is an exact identity which is not however particularly useful. If the exact dissipative eigenfunctions of the star were known, then the frequency of the mode could easily be evaluated in a number of ways. Equation (25) is nevertheless an extremely useful tool for evaluating $1/\tau$ approximately. Dissipation is a relatively weak force in stars: gravitational radiation and internal fluid dissipative processes effect the evolution of the fluid in a star on time scales that are much longer than the dynamical time scale. Thus the presence of dissipation has a relatively small effect on the evolution of the fluid in a star, and so the exact eigenfunctions of a mode (including the effects of dissipation) differ only slightly from the more easily evaluated eigenfunctions based on dissipation-free hydrodynamics. Thus, the functional on the right hand side of eq. (25) has essentially the same value whether evaluated using the exact or the dissipation-free eigenfunctions. This functional is straightforward to evaluate approximately, therefore, using the dissipation-free eigenfunctions as determined in Section 4. This approximation is expected to give values for the imaginary part of the frequency that have fractional errors of order $\tau\omega$, the ratio of the dissipative to the dynamical time scales. Studies have shown that this ratio is extremely small in neutron stars.[50]

The imaginary part of the frequency can be evaluated numerically using eq. (25). All that is needed is the dissipation-free eigenfunction of the mode, and the thermodynamic functions η and ζ that describe the viscous forces in the stellar fluid. The viscosity coefficients have been evaluated for neutron star matter,[51,52] and these quantities are given approximately by

$$\zeta = 6.0 \times 10^{-59} \left(\frac{\rho}{\omega}\right)^2 T^6, \tag{26}$$

$$\eta = 6.0 \times 10^{6} \left(\frac{\rho}{T}\right)^2 . \tag{27}$$

Note that these viscosities depend on the thermodynamic temperature T of the star. The bulk viscosity ζ is proportional to T^6 and becomes very large when the temperature of the star is high. The shear viscosity η is proportional to T^{-2} so it becomes large when the temperature is low. These two types of viscosity are comparable in neutron stars when $T \approx 10^9$K. Viscosity tends to suppress the gravitational radiation instability in rotating stars. Hence it is clear that these viscous forces will be very effective in suppressing this instability in very hot and very cool neutron stars.

To determine which rotating stars are unstable, the imaginary parts of the frequencies of their modes must be evaluated using eq. (25). The modes with the lowest values of l and m couple most strongly to gravitational radiation, while the viscous coupling increases as l and m increase. The viscous forces tend to suppress the gravitational radiation driven secular instability. Thus, the only modes that are likely to be unstable in these stars are those with relatively small values of l and m. In practice the viscous forces are always found to suppress the gravitational instability in modes with $m \geq 6$. In sufficiently slowly rotating stars all of the modes (that have been examined) are stable. It is useful therefore to define the critical angular velocity $\Omega_{\rm crit}$ where some mode first becomes unstable, that is where

$$0 = \frac{1}{\tau(\Omega_{\rm crit})}. \tag{28}$$

Figure 6 illustrates the critical angular velocities for a range of neutron star temperatures.[53] The critical angular velocity is displayed in units of $\Omega_{\rm max}$ the maximum angular velocity for which there exists an equilibrium stellar model. Figure 6 reveals that in very cool neutron stars, $T < 10^7$K, the critical angular velocity is identical to $\Omega_{\rm max}$. Thus, the viscous forces completely suppress the gravitational radiation instability in these stars. Similarly in hot neutron stars, $T > 10^{10}$K, the bulk viscosity suppresses the instability. Only neutron stars with temperatures in the range $10^7 < T < 10^{10}$K are subject to the gravitational radiation driven secular instability. Further, this instability only occurs in the most rapidly rotating stars. Even for the most extreme case, $T \approx 2 \times 10^9$K, only those stars with angular velocities greater than about $0.96\Omega_{\rm max}$ may be subject to the gravitational radiation driven secular instability. Figure 6 illustrates that there is only a moderate dependence of $\Omega_{\rm crit}$ on the mass of the star. (More massive stars couple more strongly to gravitational radiation and hence have somewhat lower $\Omega_{\rm crit}$.)

The discussion of the effects of dissipation up to this point has been based on Newtonian hydrodynamics, with the effects of gravitational radiation added as a small correction. Some work has been done, however, to estimate the effects of general relativistic dynamics on these results. Figure 7 illustrates the critical angular velocities based on a calculation that uses the post-Newtonian frequencies for the modes as described in Section 4.[53] This calculation shows that post-Newtonian effects tend to enhance the gravitational radiation instability in these stars. This increases the range of temperatures where this instability may set in, and lowers the critical angular velocities to about $0.91\Omega_{\rm max}$ in the most extreme case for $1.4M_\odot$ stars. The effects of post-Newtonian hydrodynamics on these stability results are quite striking. It illustrates the need for us to press on to a more accurate fully relativistic analysis of this problem.

The earliest studies of the secular instabilities of rotating stars were concerned with the viscosity driven instability,[6] rather than the gravitational ra-

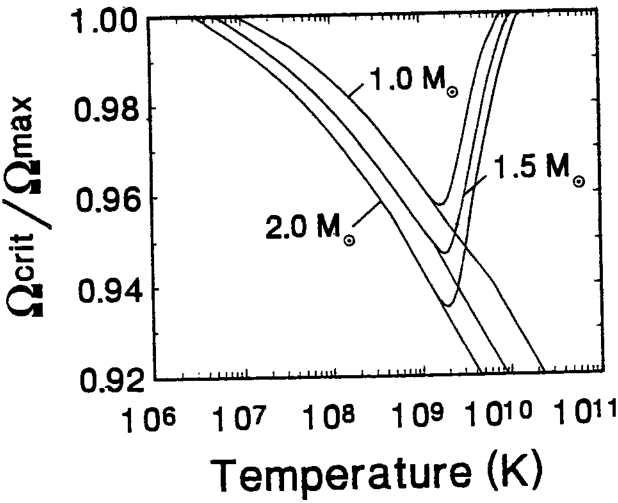

Figure 6: The temperature dependence of the critical angular velocities of neutron stars. The critical angular velocities $\Omega_{\rm crit}$ are expressed in terms of $\Omega_{\rm max}$ the maximum angular velocity for which there exists an equilibrium neutron star model.

diation driven instability discussed extensively here. The viscosity driven instability occurs in a different set of modes, but the formalism described here can easily be turned to study it. Such studies reveal that the viscosity driven secular instability probably does not play any role in neutron stars at all. The principle reason is that the viscosity driven secular instability only occurs in stars with very stiff equations of state. In stars with polytropic equations of state, $p = \kappa\rho^\gamma$, the adiabatic index γ must exceed 2.237 for a viscosity driven secular instability to exist at all.[54] The equation of state of real neutron star matter appears to be not quite stiff enough. Analysis has shown that the viscosity driven instability does not occur in any of thirteen realistic equations of state for $1.4M_\odot$ neutron star models.[36,55] These realistic equations of state become stiffer at higher densities, however. In a few of the stiffest equations of state, it has been found that the most massive neutron star models are subject to this instability in the most rapidly rotating models. It remains to be seen whether the actual equation of state in neutron stars is stiff enough to allow this viscosity driven instability, and whether this instability plays any role in

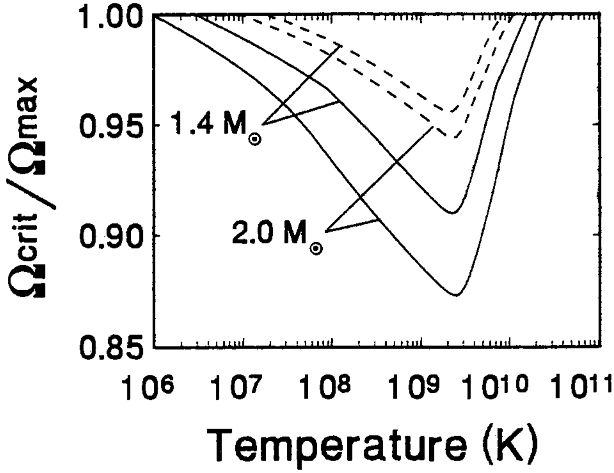

Figure 7: The temperature dependence of the critical angular velocities of neutron stars using Newtonian (dashed curves) and post-Newtonian (solid curves) gravitation and hydrodynamics.

the astrophysics of real neutron stars.

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