

Model waveform accuracy requirements for the Allen χ^2 discriminatorLee Lindblom¹ and Curt Cutler^{2,3}¹*Center for Astrophysics and Space Sciences, University of California at San Diego,
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This paper derives accuracy standards for model gravitational waveforms required to ensure proper use of the Allen χ^2 discriminator in gravitational wave data analysis. These standards are different from previously established requirements for detection and waveform parameter measurement based on signal-to-noise optimization. We present convenient formulas for evaluating and interpreting the contribution of model errors to measured values of this χ^2 statistic. The new accuracy standards derived here are needed to ensure the reliability of measured values of the Allen χ^2 statistic, both in their traditional role as vetoes and in their current role as elements in evaluating the significance of candidate detections.

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I. INTRODUCTION

For most potential astrophysical sources of gravitational waves (GWs), including the orbital inspiral and merger of two black holes, the exact solutions of Einstein's equations that describe them are not known. Therefore matched-filter searches for the GWs emitted by these systems must rely on approximate model waveforms. Standard approximation methods include: the post-Newtonian approximation, the effective one-body approximation, the large-mass-ratio approximation, and numerical relativity. What are the requirements on the accuracy of these approximate gravitational waveforms set by the practical needs of GW data analysts? In previous work, one of us (L.L.) examined in detail how waveform inaccuracy impacts signal-to-noise ratios (SNRs) [1] and derived sufficient conditions on waveform accuracy to ensure that detection rates and waveform parameter measurements are not significantly affected by waveform errors. In addition one of us (C.C.) developed formulas relating the systematic errors in the inferred physical parameters of a binary inspiral waveform (e.g., the masses of the two bodies) to the model errors in the waveform [2]. For both detection and parameter-estimation purposes, a reasonable goal for theoretical waveform modelers is to insure that errors (e.g., false dismissals or parameter-estimation errors) due to the intrinsic detector noise dominate over errors due to inaccurate waveform models.

In this paper we consider the requirements on waveform accuracy needed for the use of the Allen χ^2 discriminator in GW data analysis. This χ^2 discriminator was introduced by Allen [3] to provide a veto against instrumental glitches in GW detectors that, because of their large amplitude, give a high matched-filter SNR value but which do not actually resemble the waveforms used as search templates. This χ^2 discriminator measures how well the frequency-domain

structure of a putative GW signal agrees with the frequency-domain structure of the model waveform used to detect it. In current LIGO data analysis this Allen χ^2 discriminator is *not* used by itself as a veto on candidate GW signals; i.e., there is no threshold value of this χ^2 such that candidates with higher values are simply discarded. Instead, the Allen χ^2 statistic is now used along with the standard matched-filter SNR to produce a *reweighted* SNR that is used to assess the statistical significance of candidate detections [4]. This reweighted SNR is more effective for estimating this significance in the presence of realistic non-Gaussian noise in the data than the standard SNR [3,5].

The accuracy standards derived here are quite general, requiring only that errors in the model waveform have no more effect on the Allen χ^2 discriminator than statistical noise in the detector. These standards are important to insure that the values of this χ^2 statistic currently used in the assessment of the significance of candidate signals are reliable. And we feel that these waveform accuracy requirements will continue to be relevant for future uses of the Allen χ^2 discriminator in GW searches for the following reasons. (i) In the future other uses of this χ^2 statistic in GW data analysis may be developed in which the influence of model errors could be even more important. (ii) We have derived a number of simple, convenient formulas describing how model waveform error affects the Allen χ^2 statistic, and these could be useful in developing future data analysis applications. (iii) It has often been suggested that (some version of) the Allen χ^2 statistic could be used for model verification (e.g., is general relativity the correct theory of gravitation, and is the observed inspiral waveform actually produced by black holes as opposed to some more exotic type of compact object like a boson star) [6]. (iv) The accuracy standards associated with the Allen χ^2 statistic should be useful to waveform modelers right now by

providing a new, simple figure of merit for assessing model waveform accuracy.

The remainder of this paper is organized as follows. In Sec. II we briefly review relevant basic background material on GW data analysis. In Sec. III we derive expressions describing how model gravitational waveform errors affect measured values of the Allen χ^2 statistic and then derive our Allen χ^2 discriminator-based requirement on waveform accuracy.

II. GW DATA ANALYSIS: BACKGROUND

This section contains short summaries of some relevant background material on GW data analysis: matched-filter methods for GW searches, the Allen χ^2 discriminator, and previous work on how inaccuracies in model gravitational waveforms impact GW data analysis. A more comprehensive discussion of gravitational wave data analysis can be found in many references, including for example Creighton and Anderson [7] and references cited therein. An up-to-date summary of how data are being analyzed in the first advanced LIGO observing runs is given in Abbott *et al.* [8].

A. Matched-filter searches

Let $h_e(t, \lambda_e)$ denote the exact gravitational waveform from a particular astrophysical source with physical parameters λ_e . It is most convenient to describe the matched-filter approach to GW data analysis in terms of the Fourier transforms of the waveforms. Let $h_e(f, \lambda_e)$ denote the Fourier transform of the exact waveform:

$$h_e(f, \lambda_e) = \int_{-\infty}^{\infty} h_e(t, \lambda_e) e^{-2\pi i f t} dt. \quad (1)$$

Signals are detected in the noisy output data stream from a GW detector by searching for model waveforms $h_m(f, \lambda_m)$ that provide a sufficiently good match to the exact waveform of the signal embedded in that data. This matching is done by projecting the Fourier transforms of model waveforms onto the GW signal using the noise-weighted (complex) inner product $\langle h_e | h_m \rangle$, defined by

$$\langle h_e | h_m \rangle = 4 \int_0^{\infty} \frac{h_e(f) h_m^*(f)}{S_n(f)} df, \quad (2)$$

where $S_n(f)$ is the one-sided power spectral density of the detector strain noise.

Matched-filter searches for GW signals begin by looking for a model waveform $h_m(\lambda_m)$ that agrees with the signal $h_e(\lambda_e)$ to some level of accuracy. One measure of this agreement is the SNR $\rho_m(\lambda_m)$, defined by

$$\rho_m^2(\lambda_m) = \frac{|\langle h_e | h_m(\lambda_m) \rangle|^2}{\langle h_m(\lambda_m) | h_m(\lambda_m) \rangle}. \quad (3)$$

The quantity ρ_m measures the projection of the signal h_e onto the model waveform h_m , using the noise-weighted inner product given in Eq. (2). Thus ρ_m measures the component of h_e described by the model waveform h_m in units of the noise level of the detector. The best-fit waveform model for a particular signal h_e is obtained by adjusting the model parameters λ_m to maximize ρ_m . In LIGO GW searches using matched-filter methods, candidate signals are required to meet some minimal threshold for ρ_m (in each of at least two detectors). This minimal detection threshold has been set at $\rho_m \gtrsim 5.5$, for example, in recent initial LIGO searches for compact binary signals [9] as well as the current advanced LIGO GW searches using these methods, i.e., the PyCBC analysis [4,8].

The parameters λ_m include some that represent the intrinsic physical characteristics of the gravitational wave source (e.g., the masses and spins of the black holes in a compact binary system), plus extrinsic parameters, such as the relative orientations of the source and the detector. The model waveform h_m can also be multiplied by an arbitrary complex scale factor without changing the measured SNR defined in Eq. (3). This complex scale can be written as a real amplitude A_0 and phase ϕ_0 : $A_0 e^{i\phi_0}$. We are free to choose these scale parameters in any way we wish. Here it is convenient to fix the amplitude A_0 so that the model waveform has the same overall scale as the observed signal h_e by requiring

$$\rho_m^2 = \langle h_m | h_m \rangle. \quad (4)$$

Similarly, it is convenient to fix the phase parameter ϕ_0 by requiring that it match the complex phase of the observed signal by requiring

$$\langle h_e | h_m \rangle = \langle h_m | h_e \rangle. \quad (5)$$

We will assume in the analysis that follows that these model waveform scale parameters have been chosen in this way according to Eqs. (4) and (5).

B. The Allen χ^2 discriminator

Allen [3] was the first to propose using the χ^2 discriminator in GW data analysis. Allen's χ^2 statistic measures how well the frequency dependence of a detected signal agrees with that of the model waveform used to detect it. Once a candidate signal is identified whose measured SNR ρ_m exceeds some minimal detection threshold, the optimal model waveform h_m , normalized using Eqs. (4) and (5), is written as a sum of p mutually orthogonal components, $h_m = \sum_{k=1}^p h_m^k$. Each component waveform has support only in the frequency range $f_{i-1} \leq f \leq f_i$, chosen so that $\langle h_m^k | h_m^k \rangle = \langle h_m | h_m \rangle / p$. The (renormalized) root-mean-square deviation χ_r^2 of these component signal-to-noise quantities from their expected values is given by

$$\chi_r^2 = \frac{p}{2p-2} \frac{1}{\rho_m^2} \sum_{k=1}^p \left| \langle h_e | h_m^k \rangle - \frac{\langle h_e | h_m \rangle}{p} \right|^2. \quad (6)$$

The expectation value and standard variation of the quantity χ_r^2 (assuming stationary Gaussian detector noise) are given by the standard expressions for a system having $2p-2$ degrees of freedom (cf. Allen [3]):

$$\langle \chi_r^2 \rangle = 1 \pm \frac{1}{\sqrt{p-1}}. \quad (7)$$

The expressions given here are written for an arbitrary number of frequency bins p . The choice $p=16$ was typical in initial LIGO searches (e.g., see Ref. [10]), while choosing p in a way that depends on the properties of the waveform model, like $p = [0.4(f_{\text{peak}}/\text{Hz})^{2.3}]$, is also being used in advanced LIGO searches [8].

Allen's original idea was to use the χ^2 discriminator to veto candidate signals having $\chi_r^2 > \chi_{\text{th}}^2$, for some appropriately chosen threshold χ_{th}^2 . It was used effectively in this way, for example, to reject large non-Gaussian noise glitches in the analysis of the initial LIGO S5 data [5]. The Allen χ^2 discriminator continues to play a role in GW data analysis, but its use now is less direct. Candidate signals having sufficiently large SNR ρ_m must now satisfy several criteria before they are considered true gravitational wave events. One of these criteria is a significance test that estimates the probability the optimal model waveform h_m also matches detector noise alone. The significance of a candidate event is determined by comparing its measured reweighted SNR $\hat{\rho}_m$ (defined below) to those obtained from a very large number of detector noise samples. (For the purpose of this test, the detector noise is simulated using time-shifted data from the detector.) This reweighted SNR $\hat{\rho}_m$ reduces the standard SNR ρ_m for events having larger than expected values of χ_r^2 :

$$\hat{\rho}_m = \begin{cases} \rho_m & \text{if } \chi_r^2 \leq 1, \\ \rho_m / [\frac{1}{2}(1 + (\chi_r^2)^3)]^{1/6} & \text{if } \chi_r^2 > 1. \end{cases} \quad (8)$$

It therefore serves as a filter that can effectively remove large non-Gaussian noise glitches by substantially reducing their effective SNR, but it does this in a softer way than using χ_r^2 as a strict veto.

C. Waveform accuracy and false dismissal rates

In this section we review the impact of model waveform errors on false dismissal rates. The best-fit model waveform h_m will differ from the exact h_e by an amount $\delta h = h_m - h_e$ that represents an error in the model waveform. These errors may arise either from errors in the model waveform parameters λ_m or from intrinsic errors in the model waveform itself (e.g., errors from the numerical relativity code used to produce it). The largest SNR that could be achieved

in the absence of any model waveform error ($\delta h = 0$) is the optimal SNR $\rho_o = \langle h_e | h_e \rangle^{1/2}$. Gravitational wave searches using matched-filter methods will miss some fraction of the real signals unless the measured SNR ρ_m is close to the optimal ρ_o . It is straightforward to determine how ρ_m depends on the waveform error δh :

$$\rho_m^2 = \rho_o^2 \left[1 - \frac{\langle \delta h | \delta h \rangle}{\langle h_m | h_m \rangle} + \mathcal{O}(\delta h^4) \right], \quad (9)$$

where we have assumed the waveform error δh is small in the sense that $|\delta h| \ll |h_m|$. This expression uses the fact that

$$0 = \langle \delta h | h_m \rangle, \quad (10)$$

which follows as a consequence of the model waveform scale-factor normalization conditions in Eqs. (4) and (5). It follows from Eq. (9) that model waveform errors must be limited by

$$\frac{\langle \delta h | \delta h \rangle}{\langle h_m | h_m \rangle} < 2\epsilon_{\text{max}}, \quad (11)$$

for some ϵ_{max} to ensure that the measured SNR ρ_m does not differ significantly from the optimal ρ_o . This result was derived in Ref. [1]. We note that while the complex inner product used in this paper is different from the real inner product Eq. (3) of Ref. [1], the criterion above is actually the same, since both the numerator and denominator in Eq. (11) are real.

Previous studies [1] have shown that the parameter ϵ_{max} determines the fraction of real signals that would be missed in GW searches. The exact requirement on the value of the parameter ϵ_{max} that appears in Eq. (11) is determined by the false dismissal rate that will be tolerated in a particular search and the details of the data analysis procedure being used. If we assume the model waveform parameters λ_m have been adjusted to give the optimal fit to the observed signal, then the errors in the model waveform, δh , must be limited using Eq. (11) with $\epsilon_{\text{max}} = 0.035$ to ensure that no more than about 10% of real signals are missed [1].

In actual matched-filter searches for GWs from compact binary systems, the model waveform parameters λ_m are usually limited to a discrete grid of points. It is the combination of grid-spacing errors and intrinsic model waveform errors that determine the false dismissal rate. The intrinsic waveform accuracy requirement for searches that use discrete grids and a 10% false dismissal probability must therefore be even more stringent than $\epsilon_{\text{max}} = 0.035$. For typical LIGO template bank searches where the maximum mismatch between waveforms in the template bank is $\epsilon_{\text{MM}} = 0.03$, the appropriate value for ϵ_{max} is $\epsilon_{\text{max}} = 0.005$; see Ref. [1] for more details.

III. WAVEFORM ACCURACY FOR THE ALLEN χ^2 DISCRIMINATOR

How do inaccuracies in approximate model waveforms affect the value of χ_r^2 defined in Eq. (6)? Let $\delta h = \sum_k \delta h^k$, where δh^k denotes the component of the model waveform error in the k th frequency bin. We find it helpful to reexpress the quantity $\langle h_e | h_m^k - h_m / p \rangle$ that appears in the definition of χ_r^2 in Eq. (6) in the following way,

$$\begin{aligned} \langle h_e | h_m^k - h_m / p \rangle &= \langle h_m - \delta h | h_m^k - h_m / p \rangle \\ &= -\langle \delta h^k | h_m \rangle. \end{aligned} \quad (12)$$

The derivation of this expression depends on using the model waveform normalization conditions given in Eqs. (4) and (5). Using this expression, it is straightforward to determine how waveform errors δh affect the value of χ_r^2 :

$$\delta \chi_r^2 = \frac{p}{2p-2} \frac{1}{\rho_m^2} \sum_{k=1}^p |\langle \delta h^k | h_m \rangle|^2. \quad (13)$$

If we define

$$\delta h_{\parallel}^k = h_m^k \sqrt{\frac{p^2}{2p-2} \frac{\langle \delta h^k | h_m \rangle}{\langle h_m | h_m \rangle}}, \quad (14)$$

then our expression for $\delta \chi_r^2$ can be written even more simply:

$$\delta \chi_r^2 = \sum_{k=1}^p \langle \delta h_{\parallel}^k | \delta h_{\parallel}^k \rangle. \quad (15)$$

We point out that Allen [3] derived an analogous expression [i.e., his Eq. (6.18)] for the variation of his original χ^2 due to errors in the model waveform parameters λ_m . The current definition of χ_r^2 in Eq. (6) differs from Allen's original in significant ways: Allen's original χ^2 only measured the frequency dependence of differences in the amplitudes, but not differences in phase, between the observed and model waveforms. And Allen did not consider the possibility of intrinsic waveform errors in his analysis. Equation (15) is significantly more general than Allen's expression and is therefore essentially new.

A reasonable requirement on the accuracy of model waveforms used to evaluate χ_r^2 is that $\delta \chi_r^2$ be smaller than typical random variations in χ_r^2 due to Gaussian noise in the detector, i.e., $\delta \chi_r^2 \leq 1/\sqrt{p-1}$ from Eq. (7). This requirement on the intrinsic model waveform error is given by

$$\sum_{k=1}^p \langle \delta h_{\parallel}^k | \delta h_{\parallel}^k \rangle < \frac{1}{\sqrt{p-1}}. \quad (16)$$

This expression makes it clear that the Allen χ^2 discriminator imposes different accuracy requirements than those

needed for detection: Equation (16) places restrictions on δh_{\parallel}^k instead of δh itself. Also, since the right side of Eq. (16) is independent of the signal's SNR, the *relative* waveform accuracy $\delta h/h$ required by Eq. (16) is more stringent for higher-SNR signals. This suggests that the model waveforms intended for general use in gravitational wave data analysis should be tested with respect to both of these requirements.

These waveform accuracy requirements, Eqs. (11) and (16), can also be expressed in a more intuitive way. We define real quantities ψ and φ that represent the (log of the) amplitude and the phase of the frequency-domain waveforms, respectively:

$$h_e = e^{\psi_e + i\varphi_e}, \quad (17)$$

$$h_m = e^{\psi_e + \delta\psi + i\varphi_e + i\delta\varphi}. \quad (18)$$

The waveform modeling error $\delta h = h_m - h_e$ can therefore be written in the form

$$\delta h = h_e (e^{\delta\psi + i\delta\varphi} - 1) \quad (19)$$

$$\approx h_m [\delta\psi + i\delta\varphi + \mathcal{O}(\delta h^2)]. \quad (20)$$

(We assume that $|\delta h| \ll |h_m|$ and keep only the lowest order terms in δh in the following analysis.) Using these expressions, the left side of Eq. (11) (the detection waveform accuracy requirement) becomes

$$\frac{\langle \delta h | \delta h \rangle}{\langle h_m | h_m \rangle} = \int_0^\infty \frac{4\delta h^*(f)\delta h(f)}{S_n(f)\langle h_m | h_m \rangle} df \quad (21)$$

$$= \int_0^\infty (\delta\psi^2 + \delta\varphi^2) w(f) df, \quad (22)$$

where the weight function $w(f)$ is defined by

$$w(f) = \frac{4h_m^*(f)h_m(f)}{S_n(f)\langle h_m | h_m \rangle}. \quad (23)$$

This signal-to-noise weighting function satisfies the usual normalization condition $1 = \int_0^\infty w(f) df$. It will be useful to denote $w(f)$ -weighted averages of quantities in the following way,

$$\bar{Q} \equiv \int_0^\infty Q(f) w(f) df. \quad (24)$$

Then Eq. (11) can be rewritten in terms of the amplitude and phase errors:

$$\frac{\langle \delta h | \delta h \rangle}{\langle h_m | h_m \rangle} = \overline{\delta \psi^2} + \overline{\delta \varphi^2} \leq 2\epsilon_{\max}. \quad (25)$$

Next we want to express our Allen χ^2 -based model waveform accuracy requirement, Eq. (16), in terms of the waveform amplitude and phase errors. Therefore we decompose those waveform errors into amplitude and phase errors, $\delta\psi_k$ and $\delta\varphi_k$, having support in each frequency bin labeled by the index k :

$$\delta h^k = h_m(\delta\psi_k + i\delta\varphi_k). \quad (26)$$

The projection $\langle \delta h^k | h_m \rangle$ that appears in the definition of δh_{\parallel}^k , Eq. (14), can be written in terms of the amplitude and phase errors as

$$\frac{\langle \delta h^k | h_m \rangle}{\langle h_m | h_m \rangle} = 4 \int_0^\infty \frac{\delta h^k h_m^*}{S_n \langle h_m | h_m \rangle} df \quad (27)$$

$$= 4 \int_0^\infty \frac{(\delta\psi_k + i\delta\varphi_k) h_m h_m^*}{S_n \langle h_m | h_m \rangle} df \quad (28)$$

$$= \overline{\delta\psi_k} + i\overline{\delta\varphi_k}. \quad (29)$$

Using Eqs. (14) and (15), the effects of waveform error on $\delta\chi_r^2$ can therefore be written as

$$\delta\chi_r^2 = \frac{p\langle h_m | h_m \rangle}{2p-2} \sum_{k=1}^p [(\overline{\delta\psi_k})^2 + (\overline{\delta\varphi_k})^2]. \quad (30)$$

Equation (16) can therefore be rewritten as a requirement on the signal- and detector-noise-weighted averages of the waveform amplitude and phase errors:

$$\sum_{k=1}^p [(\overline{\delta\psi_k})^2 + (\overline{\delta\varphi_k})^2] \leq \frac{2\sqrt{p-1}}{p\langle h_m | h_m \rangle}. \quad (31)$$

We note that the sums $\overline{\delta\psi} = \sum_k \overline{\delta\psi_k}$ and $\overline{\delta\varphi} = \sum_k \overline{\delta\varphi_k}$ vanish, $\overline{\delta\psi} = \overline{\delta\varphi} = 0$, as a consequence of the waveform normalization conditions in Eqs. (4) and (5).

As an example, we examine these waveform accuracy requirements for typical values of p and ϵ_{\max} used in LIGO data analysis: $p = 16$ and $\epsilon_{\max} = 0.005$. In this case Eqs. (25) and (31) reduce to

$$\sqrt{\overline{\delta\psi^2} + \overline{\delta\varphi^2}} \lesssim 0.1, \quad (32)$$

$$\sqrt{\sum_{k=1}^p [(\overline{\delta\psi_k})^2 + (\overline{\delta\varphi_k})^2]} \lesssim 0.1 \frac{7}{\sqrt{\langle h_m | h_m \rangle}}. \quad (33)$$

The right sides of these two inequalities are comparable for $\rho_m = \sqrt{\langle h_m | h_m \rangle} = 7$, but, as noted above, the Allen χ^2 -based requirement becomes more restrictive for stronger sources.

We recall that our Allen χ^2 -based waveform accuracy requirement was derived assuming that the model waveform parameters have been optimized to maximize ρ_m^2 . If the Allen χ^2 discriminator were to be used as a strict veto of candidate signals identified in a template bank search with discretely spaced model waveform parameters, then the effects of model waveform parameter mismatch would also have to be taken into account. This would likely decrease the model waveform accuracy error tolerance for the Allen χ^2 discriminator, as it does with the detection accuracy requirements (cf. Ref. [1]). However those new requirements would depend critically on how this χ^2 discriminator is used, e.g., how veto thresholds are set. Since the Allen χ^2 discriminator is not presently being used in this way, however, we have forgone this analysis here.

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