

NONLINEAR DEVELOPMENT OF THE SECULAR BAR-MODE INSTABILITY IN ROTATING NEUTRON STARS

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ABSTRACT

We have modeled the nonlinear development of the secular bar-mode instability that is driven by gravitational radiation reaction (GRR) forces in rotating neutron stars. In the absence of any competing viscous effects, an initially uniformly rotating axisymmetric $n = 1/2$ polytropic star with a ratio of rotational to gravitational potential energy $T/|W| = 0.181$ is driven by GRR forces to a barlike structure, as predicted by linear theory. The pattern frequency of the bar slows to nearly zero, that is, the bar becomes almost stationary as viewed from an inertial frame of reference as GRR removes energy and angular momentum from the star. In this “Dedekind-like” state, rotational energy is stored as motion of the fluid in highly noncircular orbits inside the bar. However, in less than 10 dynamical times after its formation the bar loses its initially coherent structure as the ordered flow inside the bar is disrupted by what appears to be a purely hydrodynamic short-wavelength “shearing”-type instability. The gravitational waveforms generated by such an event are determined, and an estimate of the detectability of these waves is presented.

Subject headings: gravitational waves — hydrodynamics — instabilities — stars: neutron

1. INTRODUCTION

As Chandrasekhar (1969) and Tassoul (1978) have discussed in depth, if a star rotates sufficiently fast, to a point where the ratio of rotational to gravitational potential energy in the star $T/|W| \gtrsim 0.27$, it will encounter a dynamical instability that will result in the deformation of the star into a rapidly spinning, barlike structure. Although it was originally identified in configurations that are uniformly rotating and uniform in density, more generalized analyses made it clear that the dynamical bar-mode instability should arise at approximately the same critical value of $T/|W|$ in centrally condensed and differentially rotating stars (e.g., Ostriker & Bodenheimer 1973). A number of groups have used numerical hydrodynamic techniques to follow the nonlinear development of this barlike structure in the context of the evolution of protostellar gas clouds (Tohline et al. 1985; Durisen et al. 1986; Williams & Tohline 1988; Pickett et al. 1998; Cazes & Tohline 2000) and in the context of rapidly rotating neutron stars (New et al. 2000; Brown 2000). Very recently, numerical simulations by Centrella et al. (2001) and Shibata et al. (2002, 2003) have shown that low-order non-axisymmetric instabilities can become dynamically unstable at much lower values of $T/|W|$ in stars that have rather extreme distributions of angular momentum. Through linear stability analyses, Karino & Eriguchi (2003) and Watts et al. (2004) have attempted to show the connection between these instabilities and the classical bar-mode instability discovered in stars with less severe distributions of angular momentum. In our present analysis we will not be directly investigating the onset or development of these dynamical instabilities.

Classical stability studies have also indicated that a uniformly rotating (or moderately differentially rotating) star with $T/|W| \gtrsim 0.14$ should encounter a secular instability that will tend to deform its structure into a barlike shape if the star is

subjected to a dissipative process capable of redistributing angular momentum within its structure. The nonlinear development of this secular instability has not previously been modeled in a fully self-consistent manner, so it is not yet clear whether stars that encounter this type of instability will evolve to a structure that has a significant barlike distortion. In this paper, we present results from a numerical simulation that has been designed to follow the nonlinear development of the secular bar-mode instability in a rapidly rotating neutron star. A force due to gravitational radiation reaction (GRR) serves as the dissipative mechanism that drives the secular development of the bar mode. By following the development of the bar to a nonlinear amplitude and calculating the rate at which angular momentum and energy are lost from the system through gravitational radiation, we are able to provide a quantitative estimate of the distance to which such a gravitational wave source could be detected by existing and planned experiments, such as the Laser Interferometer Gravitational-Wave Observatory (LIGO).

Chandrasekhar (1970) was the first to discover that gravitational radiation reaction forces can excite the secular bar-mode instability in uniformly rotating, uniform-density stars with incompressible equations of state (i.e., the Maclaurin spheroids). This work was generalized by Friedman & Schutz (1978) and Comins (1979a, 1979b) to show that the GRR instability extends to stars with any equation of state and to other nonaxisymmetric modes with higher azimuthal mode numbers ($m > 2$). Managan (1985), Imamura et al. (1985), Ipser & Lindblom (1990), and Lai & Shapiro (1995) have all shown that the critical value of $T/|W|$ at which the GRR secular instability in the ($m = 2$) bar-mode sets in does not depend sensitively on the polytropic index of the equation of state or the differential rotation law of the star. These stability analyses have also been generalized to systems in which the star is governed by relativistic, rather than purely Newtonian, hydrodynamics and gravitational fields (Friedman

1978; Lindblom & Hiscock 1983; Cutler 1991; Cutler & Lindblom 1992; Stergioulas & Friedman 1998; Shapiro & Zane 1998; Di Girolamo & Vietri 2002).

Lindblom & Detweiler (1977) first showed that viscous processes within compact stars can act to suppress the GRR-driven secular bar-mode instability. Various physical viscosities have been considered: for example, shear viscosity due to electron and neutron scattering (Flowers & Itoh 1976; Cutler & Lindblom 1987), shear viscosity due to neutrino scattering (Kazanas & Schramm 1977; Lindblom & Detweiler 1979; Thompson & Duncan 1993), bulk viscosity due to weak nuclear interactions (Jones 1971; Sawyer 1989; Ipser & Lindblom 1991; Yoshida & Eriguchi 1995), or “mutual friction” effects in a superfluid (Lindblom & Mendell 1995). In the present work, we will not be investigating the influence of viscous processes on the GRR-driven bar-mode instability, focusing instead on following the nonlinear development of the bar mode in purely inviscid systems.

Detweiler & Lindblom (1977) and Lai & Shapiro (1995) have made efforts to follow the nonlinear evolution of rotating neutron stars that are susceptible to the secular (or the dynamical) bar-mode instability by using energy and angular momentum conservation to construct a sequence of quasi-equilibrium, ellipsoidal configurations. Here we follow the GRR-driven evolution of the bar mode in an even more realistic way by integrating forward in time the coupled set of nonlinear partial differential equations that govern dynamical motions in nonrelativistic fluids and by including a post-Newtonian radiation reaction force term in the equation of motion. Reviews of this and other instabilities that are expected to arise in young neutron stars have been written by Lindblom (1997, 2001), Stergioulas (2003), and Andersson (2003).

2. METHODS

Using the Hachisu (1986) self-consistent field technique, we constructed two initial equilibrium stellar models governed by Newtonian gravity and an $n = 1/2$ polytropic equation of state; that is, the gas pressure p and density ρ were related through the expression $p = K\rho^{1+1/n}$, where K is a constant. Model “SPH” was initially nonrotating and, hence, spherically symmetric; model “ROT181” was initially axisymmetric and uniformly rotating with a ratio of rotational to gravitational potential energy $T/|W| = 0.181$. Other properties of these two initial models are detailed in Table 1; M is the mass of the star, R_{eq} and R_{pole} are the star’s equatorial and polar radii, respectively, $\bar{\rho}$ is the star’s mean density, Ω_{rot} is the angular velocity of rotation, and J is the star’s total angular momentum. Columns (2) and (4)

TABLE 1
INITIAL MODEL PARAMETERS

PARAMETER (1)	MODEL SPH		MODEL ROT181	
	Code (2)	cgs (3)	Code (4)	cgs (5)
c	3.122	3.00×10^{10}	1.634	3.00×10^{10}
M	1.359	2.80×10^{33}	0.1716	2.80×10^{33}
K	0.7825	1.83×10^{-10}	0.1281	1.83×10^{-10}
R_{eq}	0.8413	1.25×10^6	0.6102	1.97×10^6
R_{pole}	0.8413	1.25×10^6	0.2746	8.86×10^5
$\bar{\rho}$	0.5460	3.42×10^{14}	0.4922	2.39×10^{14}
Ω_{rot}	0.0	0.0	0.9705	5.52×10^3
J	0.0	0.0	0.01632	1.58×10^{49}

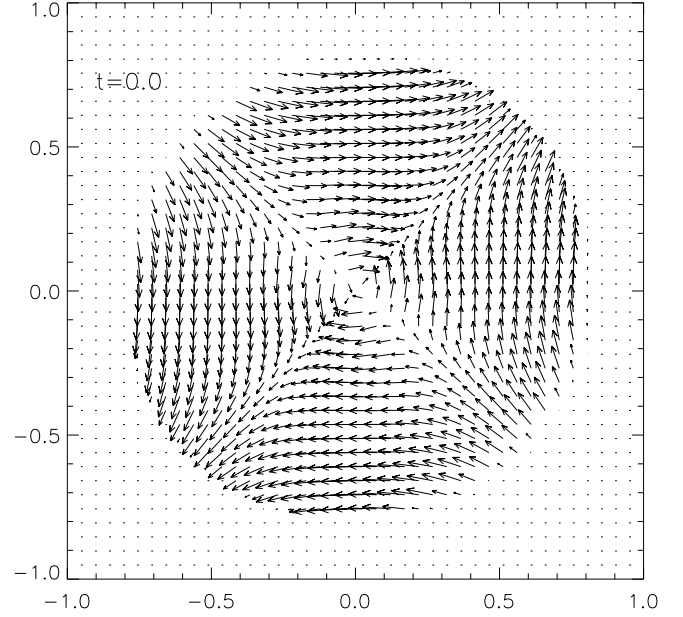


FIG. 1.—Velocity vectors in the equatorial plane of model SPH at time $t = 0$, but after the nonrotating model was perturbed by the $\ell = m = 2$ bar-mode eigenfunction drawn from linear theory.

of Table 1 give the values of these various quantities in dimensionless code units, where we have assumed the gravitational constant, the star’s central density, and the radial extent of the computational grid are all equal to unity (i.e., $G = \rho_c = \varpi_{\text{grid}} = 1$). Columns (3) and (5) of Table 1 give the value of each quantity in cgs units, assuming both stars have $M = 1.4 M_{\odot}$ and $K = 1.83 \times 10^{-10} \text{ cm}^8 \text{ g}^{-2} \text{ s}^{-2}$. (This value of K produces a spherical $1.4 M_{\odot}$ star with a radius of 12.5 km, which is characteristic of a neutron star.)

Each model was introduced into our hydrodynamic code along with a low-amplitude, nonaxisymmetric perturbation that was designed to closely approximate the eigenfunction of the $\ell = m = 2$ “bar mode” in a spherical $n = 1/2$ polytrope (Ipser & Lindblom 1990). As an illustration, Figure 1 shows the perturbed velocity field that was introduced in the equatorial plane of model SPH along with a low-amplitude, barlike distortion in the density that oriented the bar along the vertical axis. Then, the nonlinear hydrodynamic evolution of each model was followed using the numerical simulation techniques described in detail by Motl et al. (2002). More specifically, we integrated forward in time a finite-difference approximation of the following coupled set of partial differential equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \rho \nabla (\Phi + \kappa \Phi_{\text{GR}}), \quad (2)$$

$$\frac{\partial \tau}{\partial t} + \nabla \cdot (\tau \mathbf{v}) = 0, \quad (3)$$

$$\nabla^2 \Phi = 4\pi G \rho, \quad (4)$$

where \mathbf{v} is the fluid velocity, Φ is the Newtonian gravitational potential, $\tau \equiv (\epsilon \rho)^{1/\Gamma}$ is the entropy tracer, ϵ is the specific internal energy, $p = (\Gamma - 1)\epsilon \rho$, and $\Gamma = 1 + 1/n = 3$. Because the models were initially constructed using a polytropic

index $n = 1/2$ and they were evolved using an adiabatic form of the first law of thermodynamics (eq. [3]) with an adiabatic exponent $\Gamma = 3$, the models effectively maintained uniform specific entropy at a value specified by the initial model's polytropic constant, K .

In the equation of motion, equation (2), we included the post-Newtonian approximation to the gravitational radiation reaction potential produced by a time-varying, $\ell = m = 2$ mass quadrupole moment (Ipser & Lindblom 1991),

$$\Phi_{\text{GR}} \equiv -\sqrt{\frac{2\pi}{375}} \left(\frac{G}{c^5}\right) \varpi^2 e^{2i\phi} D_{22}^{(5)}, \quad (5)$$

where $D_{22}^{(5)}$ is the fifth time derivative of the quadrupole moment and c is the speed of light. For modeling purposes, a dimensionless coefficient κ was affixed to the radiation reaction potential term in the equation of motion. By adjusting the value of κ , we could readily remove or artificially enhance the effect of this non-Newtonian GRR force.

As implemented on our cylindrical computational mesh (ϖ, ϕ, z), D_{22} and its first time derivative were evaluated using the expressions

$$D_{22} = \sqrt{\frac{15}{32\pi}} \int \rho \varpi^2 e^{-2i\phi} d^3x, \quad (6)$$

$$D_{22}^{(1)} = \sqrt{\frac{15}{8\pi}} \int \rho \varpi (v_\varpi - iv_\phi) e^{-2i\phi} d^3x. \quad (7)$$

Following Lindblom et al. (2001, 2002) we have assumed that the quadrupole moment has a time dependence of the form $D_{22} \propto e^{-i\omega_{22}t}$; hence

$$D_{22}^{(n)} = (-i\omega_{22})^n D_{22}, \quad (8)$$

where at any instant in time the complex eigenmode frequency $\omega_{22} = \omega_r + i\omega_i$ can be determined by taking the ratio $D_{22}^{(1)}/D_{22}$. Thus, in equation (5) we set

$$D_{22}^{(5)} = |\omega_{22}|^4 D_{22}^{(1)}. \quad (9)$$

3. PREDICTIONS OF LINEAR THEORY

Using the linear perturbation techniques described by Ipser & Lindblom (1990, 1991), we determined that in model SPH $\omega_r = \text{Re}(\omega_{22}) = \pm(1.21 \pm 0.01)\Omega_0$ and, if the model is scaled to a mass of $1.4 M_\odot$ and a radius of 12.5 km, $\omega_i = \text{Im}(\omega_{22}) = -(1.00 \pm 0.01) \times 10^{-3} \kappa \Omega_0$, where $\Omega_0 \equiv (\pi G \bar{\rho})^{1/2} = 1.308$ in dimensionless code units. (The uncertainty is estimated from the values that are determined numerically by the linear perturbation method for different radial resolutions.) In model SPH, therefore, we should expect the amplitude of the bar mode to damp exponentially on a timescale $\tau_{\text{GR}} \equiv 1/|\omega_i| = 193\kappa^{-1}\tau_{\text{patt}}$, where $\tau_{\text{patt}} \equiv 2\pi/|\omega_r| = 3.97$ is the pattern period in dimensionless code units.

Linear perturbation analyses have not yet provided quantitative values of the bar-mode eigenfrequency in rapidly rotating $n = 1/2$ polytropes. From the information given in Ipser & Lindblom (1990, 1991), however, we expect that in model ROT181, (1) ω_r/Ω_0 should be positive but close to zero, as viewed from an inertial reference frame; and (2) ω_i/Ω_0 should be slightly positive, that is, the mass quadrupole moment should grow exponentially, but on a timescale that is very long

compared to the *damping* time predicted for the nonrotating model, SPH. More specifically, if $|\omega_r|/\Omega_0$ proves to be an order of magnitude smaller in model ROT181 than it is in model SPH, then we should expect τ_{GR} to be $\sim 10^5$ times larger, because the amplitude of the GRR driving term in equation (2) is proportional to ω_{22}^5 .

4. BAR-MODE EVOLUTIONS

4.1. Model SPH

Initially, the perturbation applied to model SPH had an amplitude $D_0 \equiv |D_{22}(t=0)| \approx 10^{-3}$ and a velocity field (see Fig. 1) designed to excite the “backward moving” $\ell = m = 2$ bar mode, that is, the mode for which $\omega_r < 0$. We followed the evolution of the model on a cylindrical grid with a resolution of $66 \times 128 \times 130$ zones in ϖ, ϕ , and z , respectively, and with the coefficient of the radiation reaction force term in equation (2) set to the value $\kappa = 20$. The effect of this was to shorten the timescale for the exponential decay by a factor of 20, to a predicted value of $\tau_{\text{GR}} = 9.63\tau_{\text{patt}}$. By shortening the decay timescale in this manner, we were able to significantly reduce the amount of computational resources that were required to follow the decay of the bar mode while maintaining a decay rate that was slow compared to the characteristic dynamical time of the system, $\Omega_0^{-1} \approx 0.2\tau_{\text{patt}}$. This is the same technique that was successfully employed by Lindblom et al. (2001, 2002) in an earlier investigation of the r -mode instability in young neutron stars.

Figure 2 displays the key results from our SPH model evolution. The solid curves in the top two panels display ω_r and ω_i as a function of time through just over nine pattern periods. Each of these frequencies oscillate about a fairly well-defined mean value: $\langle \omega_r \rangle \approx -1.56 = -1.19\Omega_0$ and $\langle \omega_i \rangle \approx -0.03 = -0.023\Omega_0$. Oscillations about these mean values initially had an amplitude $\sim \pm 0.05 = 0.038\Omega_0$, indicating that our initial nonaxisymmetric perturbation did not excite a pure eigenmode, but these oscillations decreased in amplitude somewhat as the evolution proceeded. Our measured values of ω_r and ω_i are within 3% and 15%, respectively, of the values predicted from linear theory (see § 3). The solid curve in the bottom panel of Figure 2 shows in a semilog plot the behavior of $|D_{22}|$ with time.

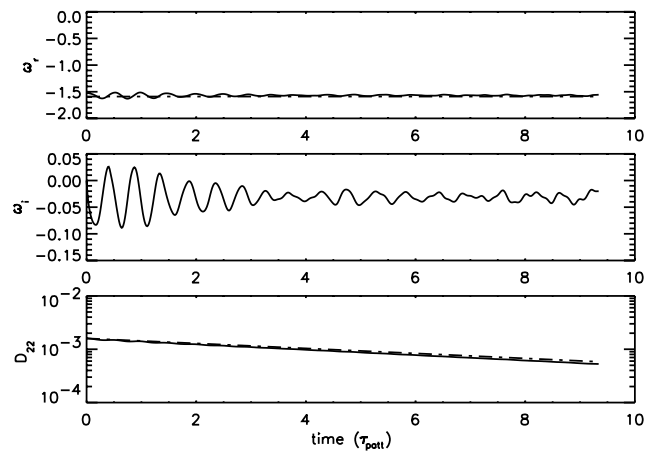


FIG. 2.—Time evolution of the amplitude $|D_{22}|$ (bottom) and the real (top) and imaginary (middle) components of the $\ell = m = 2$ bar-mode frequency from model SPH (solid curves). Time is shown in units of the predicted pattern period, $|D_{22}|$ has been normalized to MR_{eq}^2 , and frequencies are shown in dimensionless code units. Dash-dotted lines show predictions of linear theory.

TABLE 2
SIMULATION RESULTS

Model (1)	Ω_0 (2)	κ (3)	ω_r (4)	ω_i (5)	τ_{GR} (6)	ω_r/Ω_0 (7)	$\omega_i/(\Omega_0\kappa)$ (8)
SPH	1.308	2.00×10^4	-1.56	-0.03	34	-1.19	-1.1×10^{-3}
ROT181.....	1.488	1.75×10^5	0.27	0.08	12	0.18	3.1×10^{-7}

(Note that in this figure $|D_{22}|$ has been normalized to MR_{eq}^2 .) There is a clear exponential decay with a measured damping time (given by the slope of the solid curve) of $\tau_{\text{GR}} \approx 8.45\tau_{\text{patt}}$. This decay time is completely consistent with the measured value of $\langle\omega_i\rangle$ that we have obtained from the middle panel of Figure 2 and, again, within 15% of the predicted value (illustrated by the solid dash-dotted line in the top panel of the figure). The somewhat larger discrepancy in the measured value of ω_i is most probably due to the fact that the GRR formalism used here was derived under the assumption that $|\omega_i| \ll |\omega_r|$. Since ω_i is caused by the Φ_{GR} potential, which is proportional to the fifth power of the frequency, fractional discrepancies that are of the order of $5|\omega_i/\omega_r| \approx 0.1$ are not unexpected.

The first row of numbers in Table 2 summarizes these simulation results. Specifically, columns (4), (5), and (6) list the values of ω_r , ω_i , and τ_{GR} that have been drawn directly from Figure 2; all three numbers are given in dimensionless code units. In the last two columns of this table, the real and imaginary frequencies have been reexpressed in units of the dynamical frequency, Ω_0 . In the last column, we also have adjusted ω_i by the factor of κ in order to show the frequency (and associated growth rate) as it would appear in a real neutron star, where the GRR force would not be artificially exaggerated.

4.2. Model ROT181

4.2.1. Radiation Reaction with $\kappa = 1.75 \times 10^5$

Model ROT181 was introduced into our hydrodynamic code with a nonaxisymmetric perturbation in the density that had the

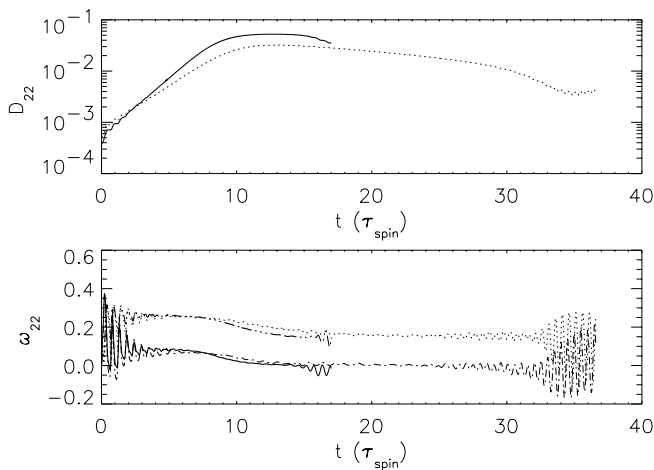


FIG. 3.—Time evolution of the amplitude $|D_{22}|$ (top) and the $\ell = m = 2$ bar-mode frequency ω_{22} (bottom) from two rapidly rotating models. Time is shown in units of the initial rotation period $\tau_{\text{spin}} = 2\pi/\Omega_{\text{rot}}$ of the model, $|D_{22}|$ has been normalized to MR_{eq}^2 , and frequencies are shown in dimensionless code units. Curves that terminate at approximately $17\tau_{\text{spin}}$ display data from model ROT181, and curves that extend past $35\tau_{\text{spin}}$ show data from the lower resolution model ROT179. In the bottom panel, both the real (e.g., dash-dotted curve for model ROT181) and imaginary (e.g., solid curve for model ROT181) components of ω_{22} are displayed.

same structure as the perturbation that was introduced into model SPH. Because we expected the natural oscillation frequency of the bar mode to be close to zero (as viewed from an inertial reference frame), however, we did not perturb the velocity field of the model. We followed the evolution of model ROT181 on a cylindrical grid with a resolution of $130 \times 128 \times 98$ zones in ϖ , ϕ , and z , respectively, and with the coefficient of the radiation reaction force term set to $\kappa = 1.75 \times 10^5$. Note that fewer vertical grid zones were required than in model SPH because model ROT181 was significantly rotationally flattened, but more radial zones were used than in model SPH in order to allow room for model ROT181 to expand radially during the nonlinear-amplitude phase of its evolution. A much larger value of κ was selected because, as explained earlier, the natural growth rate of the bar mode in model ROT181 was expected to be orders of magnitude smaller than the decay rate measured in model SPH.

Figures 3 and 4 display some of the key results from this ROT181 model evolution. The bottom panel of Figure 3 shows the time-dependent behavior of the real (dash-dotted curve) and imaginary (solid curve) components of ω_{22} in our code's dimensionless frequency units; the solid curve in the top panel displays the time-dependent behavior of $|D_{22}|$ normalized to MR_{eq}^2 . Figure 4 shows how the global parameters $T/|W|$ (solid curve) and J (dashed curve) evolve with time. The behavior of the model can be best described in the context of three different evolutionary phases: “early,” $0 \leq t/\tau_{\text{spin}} \lesssim 7$; “intermediate,” $7 \lesssim t/\tau_{\text{spin}} \lesssim 12$; and “late,” $t/\tau_{\text{spin}} \gtrsim 12$, where $\tau_{\text{spin}} \equiv 2\pi/\Omega_{\text{rot}} = 6.47$ in dimensionless code units.

During the model's early evolution, both components of the frequency ω_{22} oscillate about well-defined mean values: $\langle\omega_r\rangle \approx 0.27 = 0.181\Omega_0$ and $\langle\omega_i\rangle \approx 0.08 = 0.054\Omega_0$. [Following Ipser & Lindblom (1991), we define Ω_0 in terms of the mean density $\bar{\rho}_0$ of a spherical star that has the same M and K as model ROT181, that is, $\Omega_0 \equiv (\pi G \bar{\rho}_0)^{1/2} = 1.488$ in dimensionless code units; see

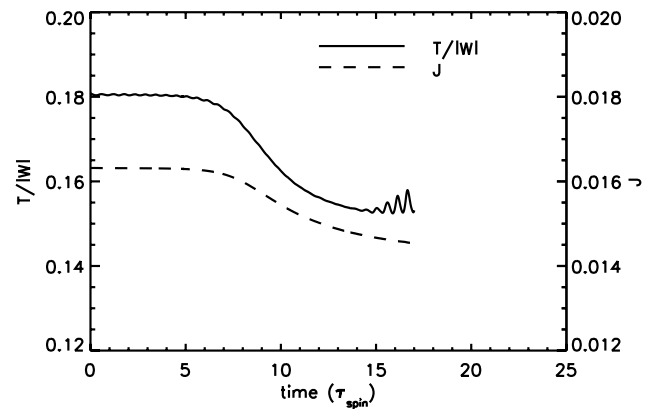


FIG. 4.—Time evolution of the angular momentum J and the energy ratio $T/|W|$ from model ROT181. Here J is in dimensionless code units, and time is shown in units of the initial rotation period of the model. During the intermediate phase of the evolution, both quantities noticeably drop as angular momentum is lost via the GRR force term in the equation of motion.

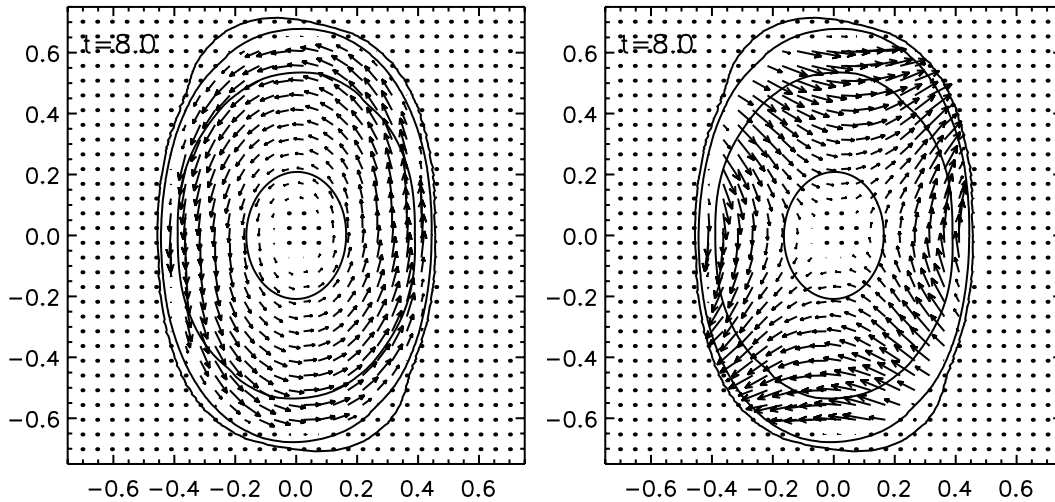


FIG. 5.—Structure of model ROT181 at time $t = 8\tau_{\text{spin}}$, during the intermediate phase of its evolution. In both panels, solid curves are isodensity contours in the equatorial plane, while vectors illustrate the equatorial-plane velocity flow field as viewed from a frame rotating with a specific frequency $\Omega_{\text{frame}} = \langle\omega_r\rangle/2$ (left) or $\Omega_{\text{frame}} = \Omega_{\text{rot}}$ (right).

Table 2.] During this same phase of the evolution, both J and $T/|W|$ remain fairly constant, but $|D_{22}|$ increases exponentially with a growth time (obtained from the slope of the displayed curve) $\tau_{\text{GR}} \approx 1.85\tau_{\text{spin}}$. This growth time is completely consistent with the measured value of $\langle\omega_i\rangle$, from which we would expect $\tau_{\text{GR}}/\tau_{\text{spin}} = \langle\omega_i\rangle^{-1}(\Omega_{\text{rot}}/2\pi) = 1.93$. The second row of numbers in Table 2 summarizes these simulation results.

After approximately seven rotation periods, the amplitude of $|D_{22}|$ begins to saturate, and the model deforms into a clearly visible barlike configuration with an axis ratio measured in the equatorial plane of approximately 2:1 (see Fig. 5). The barlike structure is initially spinning with a frequency given by $\langle\omega_r\rangle/2$, as measured during the early phase of the ROT181 evolution. This pattern frequency of the bar is a factor of 7.2 smaller than the rotation frequency Ω_{rot} of the model in its initial, axisymmetric state, so it is not surprising that the bar also exhibits sizable internal motions—it has a “Dedekind-like” structure. Figure 5 illustrates the structure of the model at this time. Both panels contain the same set of equatorial-plane isodensity contours delineating the bar, along with a set of velocity vectors depicting the fluid flow inside the bar. On the left-hand side, the velocity vectors are drawn in a frame corotating with the bar (i.e., rotating at the frequency $\langle\omega_r\rangle/2$) to illustrate the elliptical streamlines of fluid flow within the Dedekind-like bar; on the right-hand side, the velocity vectors are drawn in a frame rotating at the frequency Ω_{rot} . When viewed in this latter frame, one sees a global velocity structure that is very similar to the flow field depicted in Figure 1, that is, it resembles the natural eigenfunction of the $\ell = m = 2$ bar mode that was derived by perturbation analysis for nonrotating spherical stars, such as our model SPH. We note that this velocity structure developed spontaneously in model ROT181, as the initial model contained no velocity perturbation.

During this intermediate phase of the model’s evolution the bar remains a robust configuration, but its pattern frequency slows as the system loses approximately 10% of its angular momentum (through gravitational radiation) and $T/|W|$ drops to a value of ~ 0.156 . It is particularly interesting to note that during this phase of the evolution the GRR driving term in the equation of motion reaches a maximum and then drops as rapidly as it initially rose; this is illustrated in Figure 6, where we have plotted the time-dependent behavior of the product

$|\omega_{22}|^5 |D_{22}|$. Although the bar maintains a nonlinear structure, i.e., $|D_{22}|$ remains large, during this intermediate phase of the model’s evolution Φ_{GR} drops quickly in concert with a decrease in the frequency $|\omega_{22}|$.

During the late phase of the model ROT181 evolution, the Dedekind-like bar began to lose its coherent structure. Small-scale fluctuations in the density and velocity fields developed throughout the volume of the bar, and these fluctuations grew in amplitude on a dynamical timescale. Even vertical oscillations developed throughout the model, disrupting both the vertically stratified planar flow and reflection symmetry through the equatorial plane that persisted throughout the early and intermediate phases of the model’s evolution. After approximately $15\tau_{\text{spin}}$, the model was no longer a recognizable bar, although it remained decidedly nonaxisymmetric, showing density and velocity structure on a wide range of scales in all three dimensions. Figure 7 provides a snapshot of model ROT181’s structure at $t = 19.9\tau_{\text{spin}}$ during the late phase of its evolution. (Actually, Fig. 7 is drawn from the late phase of a revised evolution of model ROT181, which was evolved further in time; see § 4.2.3 for details.) Isodensity contours reveal a nonaxisymmetric

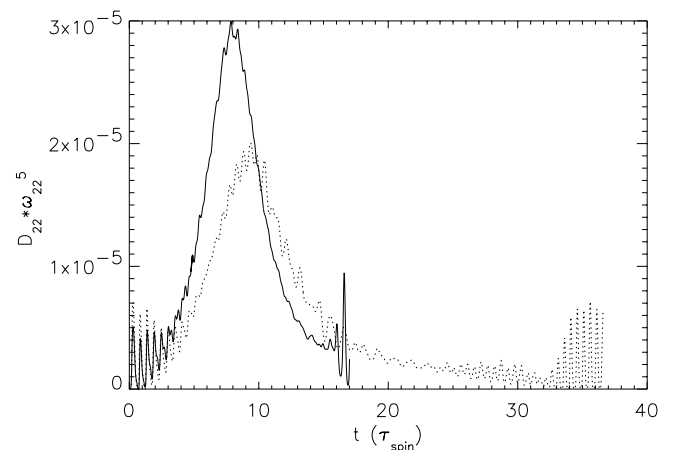


FIG. 6.—From model ROT181 (ROT179), the solid (dotted) curve depicts the time evolution of the product $\omega_{22}^5 |D_{22}|$, which indicates the strength of Φ_{GR} in the equation of motion. Time is shown in units of the initial rotation period.

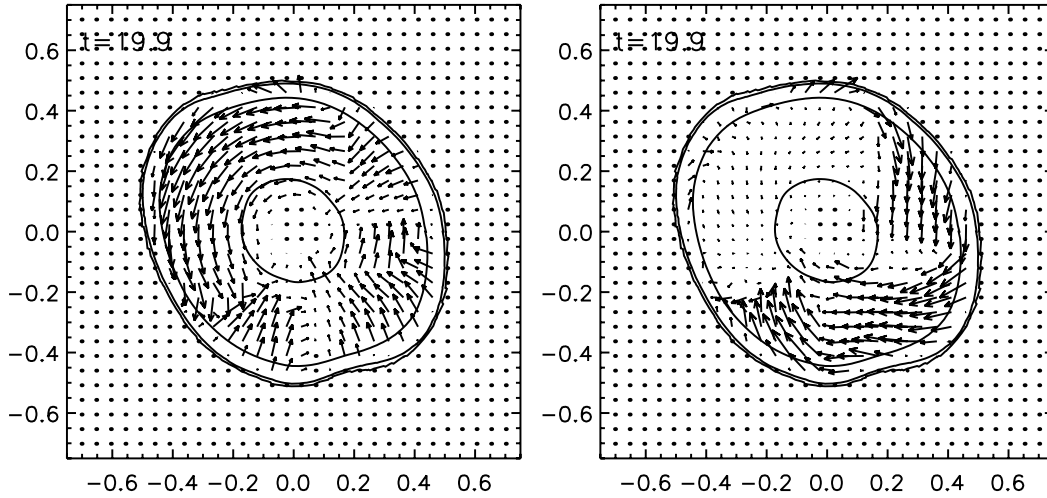


FIG. 7.—Neutron star’s structure at time $t = 19.9\tau_{\text{spin}}$ during the late phase of the revised model ROT181 evolution. In both panels, solid curves are isodensity contours in the equatorial plane, while vectors illustrate the equatorial-plane velocity flow field as viewed from a frame rotating with a specific frequency $\Omega_{\text{frame}} = 0$ (left) or $\Omega_{\text{frame}} = \Omega_{\text{rot}}$ (right).

structure that no longer can be described simply as a bar, and, when viewed from a frame rotating at a frequency Ω_{rot} (right panel), the flow field is seen to be more complex than in the bar.

4.2.2. Detectability of Gravitational Wave Radiation

A rapidly spinning neutron star located in our Galaxy (and perhaps anywhere in the Local Group of galaxies) that acquires the type of nonlinear-amplitude barlike structure that developed in model ROT181 will produce gravitational radiation at a frequency and amplitude that should soon be detectable by gravitational wave detectors such as LIGO (Abramovici et al. 1992; Abbott et al. 2004), VIRGO (Acernese et al. 2002), GEO600 (Willke et al. 2002; Gossler et al. 2002), or TAMA300 (Tagoshi et al. 2001). As our simulation shows, however, both the amplitude and pattern frequency of the bar, and hence the strength and observed frequency of the gravitational radiation, will vary with time. To illustrate this, Figure 8 depicts the evolution of model ROT181 across a “strain-frequency” diagram, which is often referenced by the experimental relativity community when discussing detectable sources of gravitational radiation. Specifically, the dimensionless strain $h_{\text{norm}} \equiv (h_+^2 + h_\times^2)^{1/2}$, where h_+ and h_\times are the two polarization states of gravitational waves. For an observer located a distance r along the axis ($\theta = 0, \phi = 0$) of a spherical coordinate system with the origin located at the center of mass of the system, we have $h_+ = (G/c^4)(1/r)(\dot{\mathcal{I}}_{xx} - \dot{\mathcal{I}}_{yy})$ and $h_\times = (G/c^4)(2/r)\dot{\mathcal{I}}_{xy}$, where the reduced moment of inertia $\mathcal{I}_{lm} \equiv \int \rho(x_l x_m - \frac{1}{3}\delta_{lm} x_k x_k) dx^3$. To obtain the strain values h_{norm} shown in Figure 8, we have assumed $r = 10$ kpc, and the time derivative of each reduced moment of inertia was evaluated numerically using the method recommended by Finn & Evans (1990). Model ROT181’s evolutionary trajectory in this diagram is strikingly similar to the trajectory that was predicted by Lai & Shapiro (1995; see their Fig. 4) using a much simpler, approximate model for the development of the secular bar-mode instability in young neutron stars.

In order to estimate the distance to which a gravitational wave source of this type would be detectable by a gravitational wave interferometer, such as LIGO, we could integrate under the curve in Figure 8, taking into account the amount of time that the source spends in each frequency band. Because we have artificially amplified the strength of the GRR force, however, our model evolves through frequency space along the curve

shown in Figure 8 much more rapidly than would be expected for a real neutron star that experiences this type of instability; hence our model cannot be used directly to estimate the length of time that such a source would spend near each frequency. However, Owen & Lindblom (2002) have outlined a method by which the detectability of a source can be estimated from a knowledge of ΔJ , the total angular momentum that is radiated away from the source via gravitational radiation. Specifically, the signal-to-noise ratio (S/N) that could be achieved by optimal filtering can be estimated from the expression,

$$(S/N)^2 \approx \frac{4G}{5\pi c^3 r^2} \frac{|\Delta J|}{f S_h(f)}, \quad (10)$$

where m is the azimuthal quantum number ($m = 2$ for the bar mode), r is the distance to the source, and $S_h(f)$ is the power spectral density of the detector noise at frequency f . From our model ROT181 evolution, we find $\Delta J = 1.67 \times 10^{48}$ g cm² s⁻¹. When it reaches its design sensitivity, LIGO’s

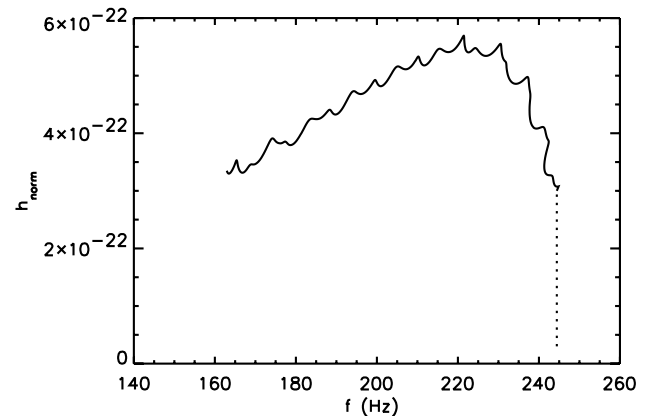


FIG. 8.—Evolution of model ROT181 in a strain-frequency diagram from $6\tau_{\text{spin}}$ to $11\tau_{\text{spin}}$ (solid curve). As is schematically illustrated by the vertical dotted line, the amplitude h_{norm} of the gravitational wave signal initially grows at a constant frequency, $f = \omega_r/(2\pi) \approx 240$ Hz. As energy and angular momentum are radiated from the system, the frequency drops monotonically, and the strain reaches a maximum amplitude then steadily declines.

4 km interferometer noise curve¹ should exhibit $(S_h)^{1/2} \approx 3 \times 10^{-23} \text{ Hz}^{-1/2}$ at $f = 220 \text{ Hz}$, which is the characteristic frequency of the spinning bar in model ROT181. From expression (10), we therefore estimate that a source of the type we are modeling will be detectable by LIGO with a $S/N \gtrsim 8$ out to a distance of 2 Mpc. (During LIGO's S3 science run in late 2003, the 4 km LHO interferometer had already come within a factor of 2 of this design sensitivity.) With Advanced LIGO [using sapphire test masses, the projected noise curve² gives $(S_h)^{1/2} \approx 2.0 \times 10^{-24} \text{ Hz}^{-1/2}$ at $f = 220 \text{ Hz}$] we estimate that this type of source will be detectable with $S/N \gtrsim 8$ out to 32 Mpc.

Of course, the detectability of gravitational waves generated by the secular bar-mode instability will also depend on the frequency with which such events occur nearby. To estimate an event rate we can draw on the discussion of Kokkotas (2004), where an estimate was made of the event rate of the dynamical bar-mode instability in young neutron stars. Since the conditions required for the onset of the secular bar-mode instability ($T/|W| \gtrsim 0.14$) are almost as extreme as the conditions required for the onset of the dynamical bar-mode instability ($T/|W| \gtrsim 0.27$), it would be very surprising if the two event rates were not similar. If we assume that only young neutron stars can be rotating rapidly enough to be susceptible to either bar-mode instability, and if we assume that a neutron star can form only from the collapse of the core of a massive star, then a reasonable upper limit on the rate of these events will be given by the event rate of Type II supernovae, that is, 1–2 per century per gas-rich galaxy (Cappellaro et al. 1999). (Another scenario is that rapidly rotating neutron stars form from the accretion-induced collapse of white dwarfs. However, according to Liu [2002] the frequency of such events is orders of magnitude lower than the event rate of Type II supernovae.) Adopting a local galaxy density of $n_g \approx 0.01 \text{ Mpc}^{-3}$ (Kalogera et al. 2001), we should expect $\lesssim 30$ Type II supernovae each year out to 32 Mpc. Not all Type II supernovae will produce neutron stars (Kokkotas [2004] estimates, for example, that 5%–40% of supernova events produce black holes instead), and only a fraction f_{rot} of neutron stars will be formed with sufficient rotational energy to be susceptible to a bar-mode instability, so the predicted event rate should be reduced accordingly. A naive estimation based on angular momentum conservation during core collapse suggests that virtually all newly born neutron stars will be formed rapidly rotating, and therefore, $f_{\text{rot}} \sim 1$; this is the direction Kokkotas (2004) leans. However, models of axisymmetric core collapse (Tohline 1984; Dimmelmeier et al. 2002a, 2002b; Ott et al. 2004) indicate that the ratio of energies $T/|W|$ in a newly formed neutron star is quite sensitive to the equation of state of the core during its collapse, and it is easy to imagine physical scenarios in which appropriately rapidly rotating neutron stars will rarely be formed; therefore, $f_{\text{rot}} \ll 1$. At the present time it is not clear which picture is more correct, but adopting the more optimistic view, it should be possible for LIGO to detect on the order of 10 such events each year.

4.2.3. Model Convergence

In an effort to determine whether the Dedekind-like bar structure was destroyed during the late phase of the ROT181 model

¹ The projected noise curve for LIGO's 4 km interferometers (published as part of the LIGO Science Requirements Document [SRD]) and the actual noise curve achieved by the 4 km interferometer at the LIGO Hanford Observatory (LHO) during the S3 science run can be obtained from http://www.ligo.caltech.edu/~lazz/distribution/LSC_Data/.

² Projected noise curves for the Advanced LIGO design using either sapphire or silica test masses can be obtained from <http://www.ligo.caltech.edu/advLIGO>.

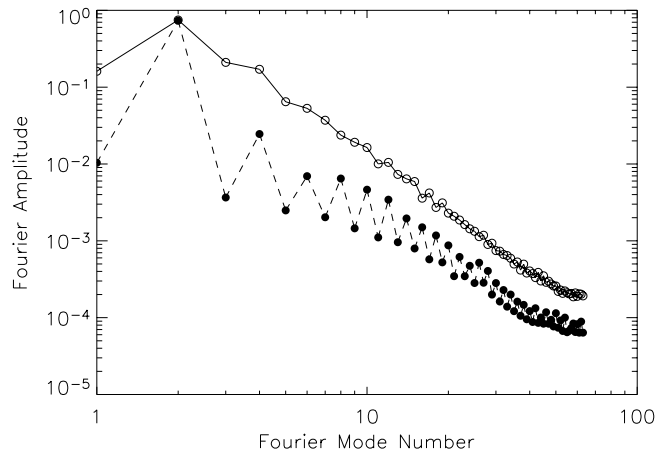


FIG. 9.—Spectrum of the Fourier-mode amplitude of the azimuthal density distribution at time $t = 10\tau_{\text{spin}}$ (filled circles), when the bar was well developed, and at time $t = 20\tau_{\text{spin}}$ (open circles), after the higher order modes destroyed the coherent bar in the revised evolution of model ROT181. To guide the eye, amplitudes determined for various modes at the same time are connected by straight line segments.

evolution as a result of physically realistic hydrodynamic processes or by a radiation reaction force that was artificially too large, we set $\kappa = 0$ and then reran the last segment of the simulation, starting from $t = 11\tau_{\text{spin}}$. This “revised” evolution produced results that were qualitatively identical to the late phase of the GRR-driven evolution. That is, the bar was destroyed by the dynamical development of velocity and density structure on a wide range of scales in all three dimensions. In an effort to quantitatively describe this relatively complex structure, Figure 9 shows a representation of the azimuthal Fourier-mode amplitudes of the model’s density distribution at two points in time: $t = 10\tau_{\text{spin}}$, when the bar was well developed; and $t = 20\tau_{\text{spin}}$, after the higher order nonaxisymmetric structure was well developed. (Note that the late phase of this revised evolution was followed somewhat farther in time than the original model ROT181 evolution described in § 4.2.1.) At the earlier time, only the $m = 2$ amplitude contained a significant amount of power, and all odd amplitudes were smaller than their even neighbors. At the later time the Fourier-mode amplitudes appear to be related to one another by a simple power law, indicating that power has been spread smoothly over all resolvable length scales.

As an additional test of the reliability of our results, we repeated our rotating model evolution on a computational grid that had a factor of 2 poorer resolution in each of the three spatial dimensions; specifically, the new simulation was performed on a grid with $66 \times 64 \times 66$ zones in ϖ , ϕ , and z , respectively. On this lower resolution grid, it was not possible to begin the evolution from precisely the same initial state as model ROT181. However, we were able to construct a uniformly rotating $n = 1/2$ polytrope with $T/|W| = 0.179$ (only 1% less than the corresponding initial energy ratio of model ROT181). We introduced a nonaxisymmetric density perturbation that produced approximately the same initial mass quadrupole moment amplitude $|D_{22}|$ as in model ROT181, and throughout the evolution the coefficient of the radiation reaction force term in equation (2) was set to $\kappa = 1.75 \times 10^5$, as in model ROT181. Hereafter, we refer to this lower resolution evolution as model ROT179.

The key results of this lower resolution evolution are illustrated by the curves in Figures 3 and 6 that extend beyond $\tau_{\text{spin}} = 35$. As shown in the bottom panel of Figure 3, the real

and imaginary components of the eigenfrequency ω_{22} identified during model ROT179's early evolution were nearly identical to the values measured in model ROT181. As shown in the top of Figure 3, $|D_{22}|$ grew exponentially up to a nonlinear value, leveled off, then slowly decayed as the bar mode's pattern frequency slowed. As shown in Figure 6, the strength of the GRR force grew steadily up to a maximum value that was somewhat lower in amplitude and somewhat delayed in time compared to model ROT181, and then almost as rapidly it dropped by an order of magnitude, as in model ROT181. Finally, during the late phase of model ROT179's evolution, the bar's coherent structure was destroyed by the development of dynamical structure on much smaller scales, just as was observed during the late phase of model ROT181's evolution. (This phenomenon is evidenced in Fig. 3 by the rapid oscillations in both components of ω_{22} at times $t \gtrsim 32\tau_{\text{spin}}$.) In this lower resolution evolution, however, the small-scale dynamical structure took roughly twice as long to develop as in model ROT181. The somewhat slower initial growth of the bar mode and the bar's lower peak nonlinear amplitude can both be attributed directly to this simulation's coarser spatial resolution. The delay in development of the smaller scale structure was almost certainly due, in part, to our inability to resolve structure on the smallest scales in model ROT179, but the delay may also have occurred, in part, because the bar itself was never as pronounced as in model ROT181. Similar behavior has been observed in simulations that have analyzed the long-term stability of r -mode oscillations in young neutron stars (Gressman et al. 2002).

5. SUMMARY AND CONCLUSIONS

Using nonrelativistic numerical hydrodynamic techniques coupled with a post-Newtonian treatment of GRR forces, we have simulated the nonlinear development of the secular bar-mode instability in a rapidly rotating neutron star. In each simulation we have artificially enhanced the strength of the GRR force term in the equation of motion (by selecting values of the parameter $\kappa > 1$) in order to be able to follow the secular development of the bar with a reasonable amount of computing resources. In each case, however, κ was set to a small enough value that the amplitude of the mass quadrupole moment changed slowly compared to the dynamical timescale of the system, thus ensuring that the system as a whole remained in dynamical equilibrium. We first tested our simulation technique by studying the evolution of the $\ell = m = 2$ bar mode in a nonrotating neutron star model (model SPH). The developing bar mode exhibited an azimuthal oscillation frequency within 3% of the frequency predicted by linear theory, and the amplitude of the bar mode damped (as predicted) at a rate that was within 15% of the rate predicted by linear theory.

Next, we evolved a rapidly rotating model (model ROT181), which was predicted by linear theory to be unstable toward the growth of the bar mode. From the early "linear-amplitude" phase of this model's evolution, we measured the bar mode's azimuthal oscillation frequency and its exponential growth rate; the values are summarized in Table 2. The oscillation frequency $\langle \omega_r \rangle / \Omega_0$ was almost an order of magnitude smaller than in model SPH, and $\langle \omega_i \rangle / (\Omega_0 \kappa)$ was 4 orders of magnitude smaller than (and had the opposite sign of) the value measured in model SPH. Both of these frequency values reflect the fact that model ROT181 was rotating only slightly faster than the marginally unstable model (predicted to have $T/|W| \approx 0.14$), in which both components of ω_{22} should be precisely zero. We watched the unstable bar mode grow up to and saturate at a sufficiently large

nonlinear amplitude that the barlike distortion was clearly visible in two- and three-dimensional plots of isodensity surfaces. This nonlinear barlike structure persisted for several rotation periods, and during this intermediate phase of the ROT181 model evolution we tracked the frequency and amplitude of the gravitational radiation that should be emitted from the configuration because of its time-varying mass quadrupole moment. Our model's evolution in a strain-frequency diagram closely matches the evolutionary trajectory predicted by Lai & Shapiro (1995), lending additional credibility to their relatively simple (and inexpensive) way of predicting the evolution of such systems as well as to our first attempt to model such an evolution using nonlinear hydrodynamic techniques. During the late phase of our model ROT181 evolution, the bar lost its coherent structure and the system evolved to a much more complex nonaxisymmetric configuration. The general features of this late phase of the evolution were reproduced when the simulation was rerun on a coarser computational grid and even when the GRR forces were turned off. Therefore, while the size and shape of the intermediate phase Dedekind-like structure of our model may well have been influenced strongly by the excessive strength of the GRR force used in our simulation, it appears as though the final complex "turbulent" phase of the evolution was governed by purely hydrodynamic phenomena.

It is not clear what physical mechanism was responsible for the development of the small-scale structure and subsequent destruction of the bar during the late phase of the evolution of model ROT181. Because the bar's structure was Dedekind-like, that is, fluid inside the bar was moving along elliptical streamlines with a mean frequency that was significantly higher than the bar pattern frequency, it is tempting to suggest that the small-scale structure arose because of differential shear. However, according to Hawley et al. (1999) Coriolis forces are able to stabilize differentially rotating astrophysical flows against shearing instabilities even in accretion disks in which the shear is much stronger than in our Dedekind-like bar (see, however, Longaretti 2002 for an opposing argument). Furthermore, other models of differentially rotating astrophysical bars (Cazes & Tohline 2000; New et al. 2000) do not appear to be susceptible to the dynamical instability that destroyed the bar in our ROT181 model evolution. We suspect, instead, that the late-time behavior of model ROT181 results either from nonlinear coupling of various oscillatory modes within the star or from an "elliptic flow" instability similar to the one identified in laboratory fluids that are forced to flow along elliptical streamlines. The dissipative effect of mode-mode (actually, three-mode) coupling has been examined in depth by Schenk et al. (2002) and Arras et al. (2003) in the context of the r -mode instability in young neutron stars, and Brink et al. (2004) have shown the connection between this process and the rapid decay of the r -mode in extended numerical evolutions, such as the ones performed by Gressman et al. (2002). However, this phenomenon has not yet been studied to the same degree in relation to the $\ell = m = 2$ f -mode. Lifschitz & Lebovitz (1993), Lebovitz & Lifschitz (1996), and Lebovitz & Saldanha (1999) have demonstrated that the elliptic flow instability seen in laboratory fluids is likely to arise in self-gravitating ellipsoidal figures of equilibrium, especially if they have Dedekind-like internal flows. Additional analysis and, very likely, additional nonlinear simulations will be required before we are able to determine which (if either) of these mechanisms was responsible for the destruction of the bar in our ROT181 model evolution.

Our nonlinear simulation of model ROT181 demonstrates that when a rapidly rotating neutron star becomes unstable to

the secular bar-mode instability, the barlike distortion may grow to nonlinear amplitude and thereby become a strong source of gravitational radiation. However, it will not be a long-lived continuous wave source, as one might optimistically have expected; in our simulation, the nonlinear-amplitude bar survived fewer than ten rotation periods. In a real neutron star the GRR forces will be much weaker than those of our simulation, so we expect the bar mode to grow and persist for many more rotation periods. However, we also expect the amplitude of the bar mode to saturate at a much lower amplitude in a real neutron star. Nevertheless, we expect the bar mode to persist in rapidly rotating neutron stars long enough to allow gravitational radiation to remove sufficient angular momentum for them to relax into a secularly stable equilibrium state. Thus, the amount of angular momentum radiated away in real neutron stars should be comparable to that in our simulation. While such astrophysical systems may not be the easiest sources to detect with broadband gravitational wave detectors such as LIGO because the frequency of the emitted radiation will change steadily with time, our estimates suggest that gravitational waves arising from the excited secular bar-mode instability in rapidly rotating neutron stars could well be detectable in the not too distant future from neutron stars as far away as 32 Mpc.

After submitting this paper for publication, we became aware that Shibata & Karino (2004) have just completed an investigation similar to the one presented here in which they have used post-Newtonian simulations to study the nonlinear development of the secular bar-mode instability in rapidly rotating neutron stars. Their initial models were differentially rotating $n = 1$ ($\Gamma = 2$) polytropes with $0.2 \lesssim T/|W| \lesssim 0.26$. The early and

intermediate phases of their model evolutions agree well with the results of our model ROT181 evolution, that is, the bar mode grew exponentially at rates consistent with the predictions of linear theory and reached a nonlinear amplitude, producing an ellipsoidal star of moderately large ellipticity. The strength of the GRR force used in our simulations was considerably larger than theirs. This may explain why the bar mode grows to a larger amplitude and why, in turn, there is a more significant decrease in the pattern frequency of the bar as it evolves toward a Dedekind-like configuration in our simulation. This may also explain why the bar-mode structure was ultimately destroyed by short-wavelength disturbances in our evolutions, while such turbulence had not yet developed in theirs.

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