Nonlinear Evolution of the r-Modes in Neutron Stars

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(Received 31 October 2000)

The evolution of a neutron-star r-mode driven unstable by gravitational radiation is studied here using numerical solutions of the full nonlinear fluid equations. The dimensionless amplitude of the mode grows to order unity before strong shocks develop which quickly damp the mode. In this simulation the star loses about 40% of its initial angular momentum and 50% of its rotational kinetic energy before the mode is damped. The nonlinear evolution causes the fluid to develop strong differential rotation which is concentrated near the surface and poles of the star.

DOI: 10.1103/PhysRevLett.86.1152 PACS numbers: 04.40.Dg, 04.30.Db, 97.60.Jd

The r-modes of all rotating stars are driven towards instability by gravitational radiation (GR) reaction [1,2], and the strength of this destabilizing force is sufficient to dominate over internal dissipation in hot, rapidly rotating neutron stars [3]. The growth time scale of the instability is about 40 s for neutron stars with millisecond rotation periods. Thus it is generally expected that GR will cause the dimensionless amplitude of the most unstable ($m = 2$) r-mode to grow to order unity within about ten minutes of the birth of such a star. The emission of GR by this process removes angular momentum and rotational kinetic energy from the star. The strength of the GR emitted and the time scale for spinning down the young neutron star depend critically on the amplitude to which the r-mode grows. Initial estimates assumed that the amplitude would grow to order unity before some unknown process would saturate the mode. With this saturation amplitude, a neutron star spins down to about one-tenth of its maximum angular velocity in about one year, and the GR from this event might be detectable by LIGO II [4]. Stergioulas and Font [5] find no saturation of the r-modes even at large amplitudes in their nonlinear numerical study. Thus at present no one really knows what process will saturate the r-modes or how large their amplitudes will grow. Here we investigate the growth of the r-modes by solving numerically the nonlinear hydrodynamic equations driven by GR reaction.

Neutron stars are compact objects with reasonably strong gravitational fields, $GM/R \approx 0.2c^2$. While these objects may contain fluid moving at fairly large velocities, $v^2 \lesssim GM/R$, Newtonian theory is expected to describe them up to errors of order GM/Rc^2. For simplicity then we study here the nonlinear evolution of the r-modes using the Newtonian equations:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{1}$$

$$\rho(\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v}) = -\vec{\nabla} p - \rho \vec{\nabla} \Phi + \rho \vec{F}_{\mathrm{GR}}, \tag{2}$$

where $\vec{v}$ is the fluid velocity, ρ and p are, respectively, the density and pressure, Φ is the Newtonian gravitational potential, and $\vec{F}_{\mathrm{GR}}$ is the GR reaction force. The gravitational potential is determined by Poisson's equation,

$$\nabla^2 \Phi = 4\pi G \rho. \tag{3}$$

The GR reaction force $\vec{F}_{\mathrm{GR}}$ due to a time-varying current quadrupole (the dominant multipole for the r-modes) can be written (see Blanchet [6] and Rezzolla *et al.* [7])

$$F^x_{\mathrm{GR}} - iF^y_{\mathrm{GR}} = -\kappa i(x + iy)[3v^z J^{(5)}_{22} + zJ^{(6)}_{22}], \tag{4}$$

$$F^z_{\mathrm{GR}} = -\kappa\, \mathrm{Im}\left\{(x + iy)^2\left[3\, \frac{v^x + iv^y}{x + iy} J^{(5)}_{22} + J^{(6)}_{22}\right]\right\}, \tag{5}$$

where $J^{(n)}_{22}$ represents the nth time derivative of J_{22},

$$J_{22} = \int \rho r^2 \vec{v} \cdot \vec{Y}^{B*}_{22} d^3x, \tag{6}$$

and $\vec{Y}^B_{22} = \hat{r} \times r\vec{\nabla} Y_{22}/\sqrt{6}$ is the magnetic-type vector spherical harmonic. In slowly rotating stars the $m = 2$ r-mode projects onto J_{22} and no other J_{lm}. The parameter κ that appears in Eqs. (4) and (5) sets the strength of the GR reaction force and has the value $\kappa = 32\sqrt{\pi}\, G/(45\sqrt{5}\, c^7)$ in general relativity theory. For practical reasons (see further discussion below), we take κ to be about 4500 times this value.

We solve Eqs. (1)–(3) numerically in a rotating reference frame using the computational algorithm developed at Louisiana State University (LSU) to study a variety of astrophysical hydrodynamic problems [8]. Briefly, the code performs an explicit time integration of the equations using a finite-difference technique that is accurate to second order both in space and time and uses techniques very similar to those of the familiar ZEUS code [9].

We find that as written, Eqs. (4) and (5) for $\vec{F}_{\mathrm{GR}}$ are nearly useless in a numerical evolution. The problem is the large number of time derivatives of J_{22} that appear there. In the case of a pure mode with frequency ω this problem is easily solved: $J^{(n)}_{22} = (i\omega)^n J_{22}$. Even when the amplitude of the r-mode becomes large, we expect the fluid motion to be dominated by periodic motions at the fundamental frequency of the r-mode. Thus, we expect the normal-mode expressions for the time derivatives of the

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multipole moments to be reasonably accurate even in the nonlinear regime. It is easy to evaluate J_{22} from Eq. (6), and $J_{22}^{(1)}$ can also be expressed as an integral over the fluid variables using Eqs. (1) and (2):

$$J_{22}^{(1)} = \int \rho[\vec{v} \cdot \vec{\nabla}(r^2 \vec{Y}_{22}^{B*}) \cdot \vec{v} - r^2 \vec{\nabla}\Phi \cdot \vec{Y}_{22}^{B*}]\, d^3x \,. \tag{7}$$

Thus we evaluate the time derivatives needed in Eqs. (4) and (5) using $J_{22}^{(5)} = \omega^4 J_{22}^{(1)}$ and $J_{22}^{(6)} = -\omega^6 J_{22}$. We have verified that using these approximations our numerical code accurately reproduces the analytical description of the r-mode evolution and growth due to GR reaction in slowly rotating models.

In order to monitor the nonlinear evolution of an r-mode, it will be helpful to introduce nonlinear generalizations of the amplitude and frequency of the mode. Since the r-mode projects primarily onto the multipole moment J_{22}, we define the nonlinear amplitude to be

$$\alpha = \frac{8\pi R_0 |J_{22}|}{\Omega_0 \int \rho r^4 d^3x} \,, \tag{8}$$

where Ω_0 is the initial angular velocity and R_0 is the radius of the corresponding nonrotating stellar model. This α is normalized to reduce to the standard definition used in perturbations of slowly rotating stars [3]. We must also define a generalization of the frequency of the r-mode. For a small-amplitude mode the time derivative of J_{22} is proportional to the frequency: $J_{22}^{(1)} = i\omega J_{22}$. Thus we are led to define the following nonlinear generalization of the r-mode frequency:

$$\omega = -\frac{|J_{22}^{(1)}|}{|J_{22}|} \,. \tag{9}$$

These expressions for α and ω are very stable numerically since they are expressed as integrals [10].

We have studied the growth of an r-mode as described above by solving numerically the nonlinear hydrodynamic equations for a rotating stellar model represented on a $64 \times 128 \times 128$ cylindrical grid. We prepare initial data for this evolution by constructing first a rigidly rotating equilibrium stellar model. For simplicity we use the polytropic equation of state $p = K\rho^2$, which approximates the features of more realistic neutron-star models reasonably well. The model most extensively studied here is very rapidly rotating, with angular velocity $\Omega_0 = 0.635\sqrt{\pi G \bar{\rho}_0}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. The ratio of the initial rotational kinetic energy to gravitational potential energy for this model is $T/|W| = 0.101$. At the initial time we take the fluid density to be that of this equilibrium stellar model, while the fluid velocity is taken to include a small amplitude r-mode perturbation:

$$\vec{v} = \Omega_0 \vec{\varphi} + \alpha_0 R_0 \Omega_0 \left(\frac{r}{R_0}\right)^2 \mathrm{Re}(\vec{Y}_{22}^{B}) \,. \tag{10}$$

In our simulation we take the initial r-mode amplitude to be $\alpha_0 = 0.1$.

According to perturbation theory [3], the ratio of the GR growth time to the r-mode pulsation period is expected to be $\tau_{\mathrm{GR}}/P_r = 5.64 \times 10^4$ for the stellar model studied here. At this rate, growth of the r-mode is far too slow to be studied effectively via an explicit hydrodynamic simulation. To overcome this, we artificially increase κ in Eqs. (4) and (5) to a value such that $\tau_{\mathrm{GR}}/P_r \approx 12.6$; i.e., the strength of our GR reaction is about 4500 times stronger than it should be. This made the growth rate of the r-mode large enough so that it could be followed up to nonlinear amplitude with a reasonable amount of computing time, yet kept it slow relative to the dynamical and sound-crossing times.

Figure 1 shows the $\mathrm{Re}(J_{22})$ that results from the evolution of this system. Time in these figures is given in units of the initial rotation period of the star: $P_0 = 2\pi/\Omega_0$. Figure 1 illustrates that the evolution of J_{22} is dominated by the sinusoidal r-mode oscillations. Figure 2 illustrates the evolution of the r-mode amplitude in this simulation. We see that the growth is exponential (as predicted by perturbation theory) until $\alpha \approx 2$. Then some nonlinear process limits the growth; α peaks at $\alpha = 3.35$ and then decreases rapidly. The evolution of the r-mode frequency ω defined by Eq. (9) is illustrated in Fig. 3. The evolution of ω is quite smooth when the amplitude of the

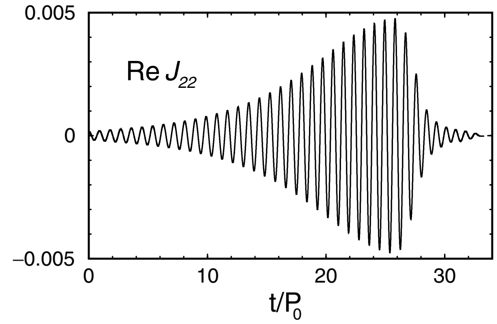

FIG. 1. Evolution of the current quadrupole moment J_{22}.

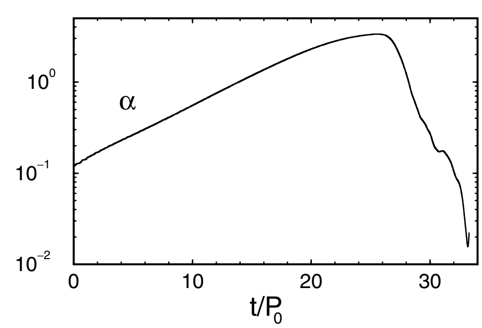

FIG. 2. Nonlinear evolution of the r-mode amplitude α.

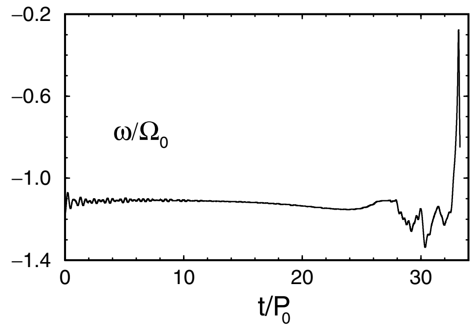

FIG. 3. Evolution of the r-mode frequency.

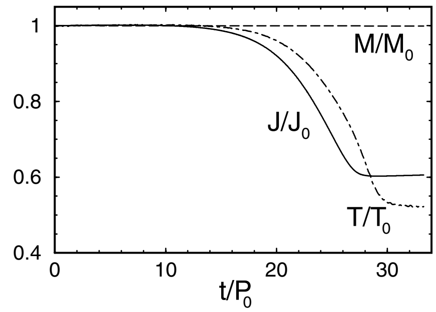

FIG. 5. Evolution of the total angular momentum J/J_0 (solid curve), mass M/M_0 (dashed line), and kinetic energy T/T_0 (dot-dashed curve) of the star.

r-mode is large: $\alpha \gtrsim 0.5$. At early ($t \lesssim 10P_0$) and at late ($t \gtrsim 28P_0$) times when the r-mode amplitude is small, we see that other modes also make noticeable contributions to J_{22}, and hence to ω.

Several authors [11] have suggested that the nonlinear evolution of an unstable r-mode could generate differential rotation. This differential rotation would amplify existing magnetic fields in the star, and these in turn might significantly affect the evolution of the r-mode. We have explored this possibility by monitoring the average differential rotation $\Delta\Omega$. We find that $\Delta\Omega$ grows to $\Delta\Omega \approx 0.41\bar{\Omega}$ at time $t \approx 28P_0$ and then decreases. Here $\bar{\Omega}$ is the ratio of the angular momentum to the moment of inertia of the star. But the average value of $\Delta\Omega/\bar{\Omega}$ may be misleading. Figure 4 illustrates the spatial dependence of the azimuthally averaged angular velocity $\Omega(\varpi, z) = \int \Omega \, d\varphi/2\pi$ at the time $t = 25.6P_0$. We see that the differential rotation is confined mostly to a thin shell of material near the surface of the star and is particularly concentrated near each polar cap. The bulk of the material in the star remains fairly rigidly rotating. Thus it appears that the magnetic fields generated by this differential rotation may have strong effects locally (e.g., leading to the ejection of material at the surface or along the rotation axis) but may not affect the global behavior of the r-mode as much as if the differential rotation were distributed more uniformly.

Energy and angular momentum are removed from the star by GR emission from the r-mode according to the expression [7,12]

$$\frac{dE}{dt} = \frac{|\omega|}{2}\frac{dJ}{dt} = -\frac{128\pi}{225}\frac{G}{c^7}\kappa\omega^6|J_{22}|^2. \qquad (11)$$

Figure 5 illustrates the evolution of the total angular momentum J, the total mass M, and the kinetic energy of the fluid T. We see that M is essentially unchanged, while J decreases by about 40%. The numerical evolution of J agrees with Eq. (11) to within a few percent [13]. Figure 5 reveals that the kinetic energy of the star, T, continues to decrease for about two rotation periods after the rate of emission of J into GR falls to zero.

What nonlinear process is responsible for limiting the growth of the r-mode? Figure 5 reveals that the kinetic energy of the star, T, continues to decrease even after the rate of emission of J into GR falls to zero. This implies that the energy stored in the r-mode is *not* being radiated away as GR or being transferred to other macroscopic modes. For if it were, T would be more or less conserved once the GR losses become small. Rather we see that at $t \approx 26P_0$, as the amplitude of the r-mode peaks, the predicted evolution of the total energy E due to GR loss (the dashed curve in Fig. 6) significantly diverges from the

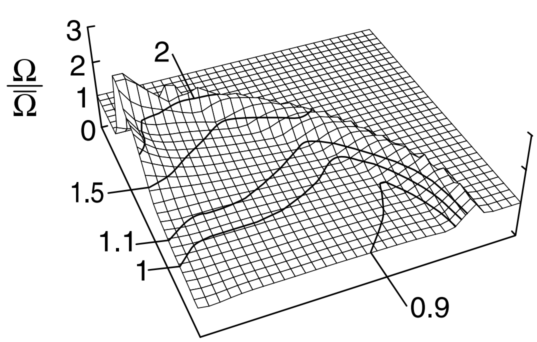

FIG. 4. Spatial dependence of the azimuthally averaged angular velocity $\Omega(\varpi, z)/\bar{\Omega}$. Rotation axis is the left edge of the figure; equatorial plane is the bottom edge.

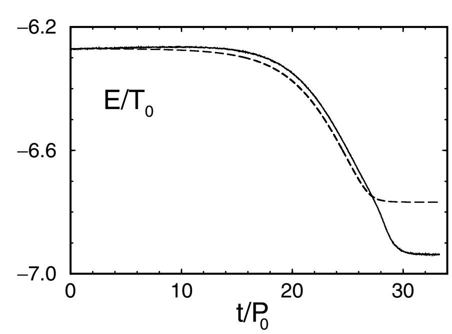

FIG. 6. Evolution of the total energy E divided by the initial rotational kinetic energy T_0. The solid curve is the actual numerical evolution; the dashed curve is the predicted evolution due to GR losses alone using Eq. (11).

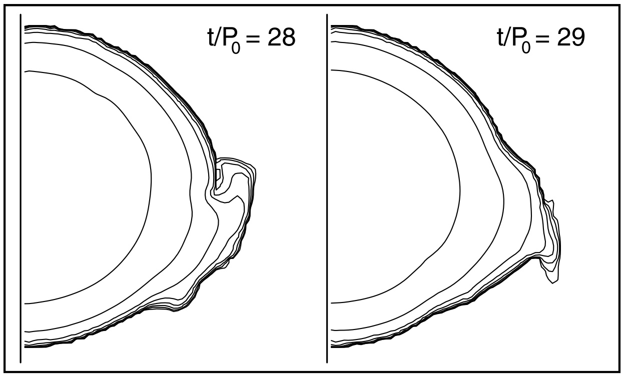

FIG. 7. Density contours (at $10^{-n/2}\rho_{\max}$ with $n = 1, 2, \ldots$) in selected meridional planes at times $t = 28P_0$ and $29P_0$ illustrate the breaking of surface waves. Shocks at the leading edges of these waves appear to be the primary mechanism that suppresses the r-mode.

numerical evolution of E. This nonconservation of E is due to the formation of shocks associated with the breaking of surface waves as illustrated in Fig. 7 [14].

In summary then, we find that an r-mode can grow to a relatively large amplitude under the influence of the GR reaction force. The hydrodynamic mechanism that acts to suppress the r-mode in our simulation is the formation of shocks near the surface of the star. Since the GR reaction force in our simulation is too strong by a factor of 4500, it is still possible that slower hydrodynamic processes (like the transfer of energy to other modes) could limit the r-mode at smaller values of α. It is also possible that the coupling of the r-mode to g-modes in real neutron-star matter (but absent from our barotropic simulation) could also limit the growth at smaller α. However, if shock formation turns out to be the dominant suppression mechanism, then we expect the peak amplitude to be relatively insensitive to the strength of the GR coupling [15]. In this case the dimensionless amplitude of the mode will peak at about $\alpha = 3.4$. This implies that the spindown of the star by GR will occur in about one-tenth ($\propto 1/\alpha^2_{\max}$) the time previously estimated [3].

During our simulation about 40% of the initial angular momentum and 50% of the initial rotational kinetic energy is radiated away as GR. Most of this energy is radiated in a much narrower frequency band $\Delta f \approx 0.05f$ than had been expected [4]. Further, the frequency of the r-mode is about 20% smaller than that predicted by simple perturbation theory. Both of these effects tend to make the GR emitted by this process more easily detectable by LIGO. Even at the end of our simulation the star is still rather rapidly rotating, and we presume that GR will again drive the r-mode unstable and a second episode of spindown will occur. During our simulation about 16% of the initial rotational kinetic energy of the star is dissipated by the shocks. This energy will be converted to heat in a real neutron star and this would delay cooling and the formation of a crust. A detailed thermal analysis will have to be carried out to determine exactly what other effects this thermal energy may have.

We thank J. Friedman, B. Owen, K. Thorne, and R. Wagoner for helpful discussions. We also thank H. Cohl, J. Cazes, and especially P. Motl for contributions to the LSU hydrodynamic code. This research was supported by NSF Grants No. PHY-9796079, No. AST-9987344, No. AST-9731698, No. PHY-9900776, and No. PHY-9907949 and NASA Grants No. NAG5-4093 and No. NAG5-8497. We thank NRAC for computing time on NPACI facilities at SDSC where tests were conducted, and we thank CACR for access to the HP V2500 computers at Caltech, where the primary simulations were performed.

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[13] The difference between the actual and the predicted evolution of E (at early times) and J was due primarily to numerical error introduced by the Coriolis term in the rotating-frame evolution equations. A higher-order expression for this term has now been implemented, and even better agreement is expected in future simulations.
[14] In the barotropic hydrodynamic code used in this simulation, the structure of a shock is correctly evolved, but the energy dissipated in the shock is simply ignored.
[15] We have verified that $\alpha_{\max}$ is insensitive to the value of κ by running an additional simulation. We find that $\alpha_{\max}$ decreases by 0.36% when κ is increased by the factor 4/3.