Stability of the *r*-modes in white dwarf stars

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The stability of the *r*-modes in rapidly rotating white dwarf stars is investigated. Improved estimates of the growth times of the gravitational-radiation driven instability in the *r*-modes of the observed DQ Her objects are found to be longer (probably considerably longer) than 6×10^9 yr. This rules out the possibility that the *r*-modes in these objects are emitting gravitational radiation at levels that could be detectable by LISA. More generally it is shown that the *r*-mode instability can only be excited in a very small subset of very hot ($T\gtrsim10^6$ K), rather massive ($M\gtrsim0.9M_\odot$) and very rapidly rotating ($P_{\rm min}\leqslant P\lesssim1.2P_{\rm min}$) white dwarf stars. Further, the growth times of this instability are so long that these conditions must persist for a very long time ($t\gtrsim10^9$ yr) to allow the amplitude to grow to a dynamically significant level. This makes it extremely unlikely that the *r*-mode instability plays a significant role in any real white dwarf stars. [S0556-2821(99)01818-4]

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I. INTRODUCTION

Recently it was discovered by Andersson [1] and Friedman and Morsink [2] that gravitational radiation tends to drive the *r*-modes of all rotating stars unstable. Lindblom, Owen, and Morsink [3] subsequently calculated the strength of the gravitational radiation coupling to these modes and found it to be strong enough to overcome viscous dissipation in hot rapidly rotating neutron stars. This calculation was confirmed using better estimates of the dissipative coupling by Andersson, Kokkotas, and Schutz [4] and Lindblom, Mendell, and Owen [5]. Thus it is now generally expected that this *r*-mode instability will limit the rotation rates of hot young rapidly rotating neutron stars. The excess angular momentum in these stars will be carried away by gravitational radiation, and these stars will be spun down to more slowly rotating stable configurations within about one year of their births. The gravitational radiation emitted during this process may eventually be detectable by the Laser Interferometric Gravitational Wave Observatory (LIGO) (see Owen *et al.* [6] and Brady and Creighton [7]).

The possibility that this *r*-mode instability plays a role in other astrophysical systems has also been proposed. For example, the possibility that this instability might play a role in limiting the angular velocities of old and relatively cold neutron stars spun up by accretion was considered by Bildsten [8], Andersson, Kokkotas, and Stergioulas [9], and Levin [10]. It has also been proposed that this instability might play a role in rapidly rotating white dwarf stars by Andersson, Kokkotas, and Stergioulas [9] and Hiscock [11]. This last possibility will be investigated more thoroughly in this paper. A set of rapidly rotating white dwarf models and the time scales for the gravitational-radiation driven *r*-mode instability in these stars are computed in Sec. II. The possibility that this instability plays a significant role in the observed DQ Her objects is examined in Sec. III. It is shown that the minimum growth time for this instability in these objects is 6×10^9 yr, and that the actual growth time is almost certainly much longer than this. Thus, the *r*-mode instability does not play any significant role in these objects, and gravitational radiation from the *r*-modes in these objects will not be observable by the Laser Interferometer Space Antenna (LISA). Section IV evaluates the effects of viscosity on the *r*-modes in white dwarf stars. It is shown that all white dwarfs with core temperatures cooler than about 10^6 K are stable, that all white dwarfs with masses less than $0.9M_\odot$ are stable, and that all white dwarfs with rotation periods longer than $1.2P_{\rm min}$ (where $P_{\rm min}$ is the minimum rotation period of that star) are stable. It is shown that white dwarf stars cool too quickly (in the absence of accretion) for the *r*-mode instability to grow to significant levels in any star. And finally, it is shown that this instability can only play a significant role in white dwarf stars that are maintained at very high core temperatures by accretion and that remain at nearly maximal rotation rates for a period of time that exceeds about 10^9 yr. It seems very unlikely that these conditions will ever be met in any real white dwarf stars.

II. RAPIDLY ROTATING WHITE DWARFS

In order to study to properties of the *r*-modes in white dwarf stars, simple models have been constructed using the equation of state of a zero-temperature degenerate Fermi gas with $Z/A=\frac{1}{2}$ that is appropriate for carbon-oxygen white dwarfs [12]. Families of rigidly rotating stars were constructed using the numerical techniques described by Ipser and Lindblom [13] for finding very rapidly rotating and highly non-spherical models. A constant mass family of rotating models was constructed by successively spinning up a non-rotating model until its angular velocity was equal (or very nearly equal) to the frequency of the equatorial orbit located just at the surface of the star. In Fig. 1 are presented the rotation periods $P_{\rm min}$ of these maximally rotating white dwarf models as a function of the mass of the star. Also shown in Fig. 1 is a set of points that represent the minimum rotation periods for white dwarf stars as determined by Hachisu [14]. The numerical models constructed here used a spherical grid that allowed a much finer determination of the location of the surface of the star than the models constructed by Hachisu. Nevertheless, there is good agreement between

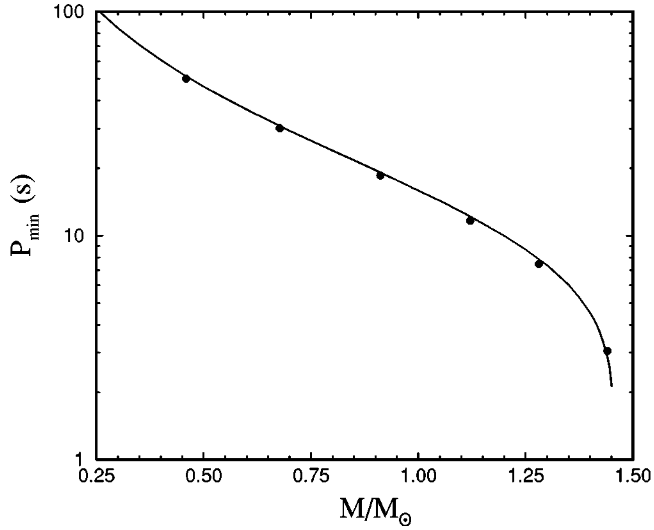

FIG. 1. Minimum rotation periods (in seconds) for rigidly rotating white dwarf stars based on the equation of state of a zero-temperature degenerate Fermi gas with $Z/A=\frac{1}{2}$.

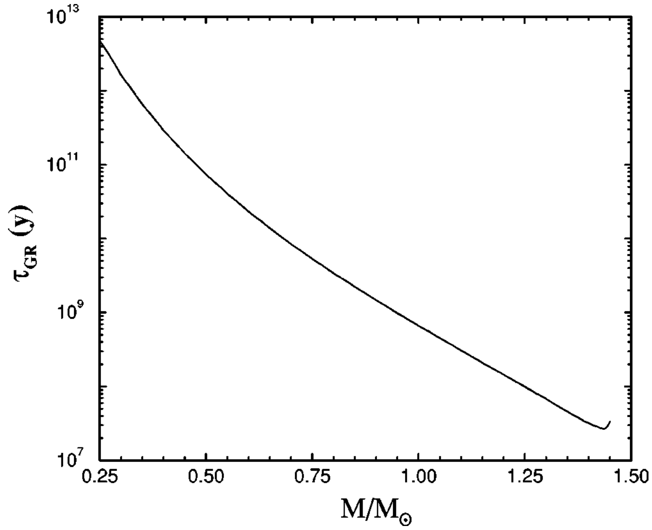

FIG. 2. Gravitational radiation growth times (in years) of the r-modes for white dwarf stars rotating at their maximum possible rotation rates.

our calculations of the limiting rotation periods of these stars.

The growth time of the gravitational-radiation driven r-mode instability is estimated using the expressions derived by Lindblom, Owen, and Morsink [3]. In particular the growth time τ_{GR} for the dominant $m=2$ mode is

$$\frac{1}{\tau_{GR}}=\frac{2\pi}{25}\left(\frac{4}{3}\right)^8\frac{G}{c}\int_0^R\rho\left(\frac{r\Omega}{c}\right)^6 dr, \tag{2.1}$$

where ρ and Ω are the density and angular velocity of the equilibrium stellar model, and G and c are Newton's constant and the speed of light respectively. This is only the lowest order term in the expansion for $1/\tau_{GR}$ in powers of the angular velocity. To lowest order therefore it is sufficient to evaluate this integral using the density function ρ for the non-rotating model of a given mass. The gravitational radiation growth time τ_{GR} defined in Eq. (2.1) is proportional to Ω^{-6}. It is convenient to define a characteristic growth time $\tilde{\tau}_{GR}$ therefore for the star rotating at the maximum possible angular velocity. In particular then the gravitational radiation growth time is related to this characteristic growth time by

$$\tau_{GR}=\tilde{\tau}_{GR}\left(\frac{P}{P_{\rm min}}\right)^6. \tag{2.2}$$

The values of the characteristic growth time $\tilde{\tau}_{GR}$ for a range of white dwarf masses are presented in Fig. 2.

III. DQ HER OBJECTS

Recently Andersson, Kokkotas, and Stergioulas [9] and Hiscock [11] have suggested that the instability of the r-modes driven by gravitational radiation may play an important astrophysical role in the DQ Her type cataclysmic variable systems. These objects emit radiation with periodic luminosity variations that have been interpreted as the rotation periods of rapidly rotating white dwarf stars. The systems having the shortest periods are WZ Sge with period 27.9 s, AE Aqr with 33.1 s, V533 Her with 63.6 s, and DQ Her with 71.1 s [15]. If these represent the rotation periods of the white dwarfs (an interpretation that is not universally accepted [16,17]), then Fig. 1 shows that these systems could (depending on their masses) contain very rapidly rotating white dwarf stars. The gravitational radiation growth times of the r-modes in these systems are smallest—for a star of given mass—for the most rapidly rotating stars. It can also be shown [by brute force numerical examination of the data in Fig. 2 together with Eq. (2.2)] that the gravitational radiation growth times of the r-modes in these systems are also smallest—for stars of given rotation period—for that star which has the smallest mass. Hence it is straightforward to show from the data in Figs. 1 and 2 that the minimum possible gravitational radiation growth times for the r-modes in the DQ Her objects are 6.7×10^9 yr for WZ Sge, 1.5×10^{10} yr for AE Aqr, 3.7×10^{11} yr for V533 Her, and 6.6×10^{11} yr for DQ Her.

The estimates of the gravitational radiation growth times given above are almost certainly significant underestimates. First, internal fluid dissipation (i.e. viscosity) has been completely neglected. This will always act to increase these time scales. Second, the assumption that these are maximally rotating white dwarf stars is quite questionable and consequently the gravitational radiation growth times are expected to be much longer than the values given above. For example, the most recent mass determination of the white dwarf in WZ Sge is only $0.3M_\odot$ [17].[1] The minimum gravitational

[1]A white dwarf with rotation period of 27.9 s must have at minimum a mass of $0.725M_\odot$. Therefore, if the rotation period of WZ Sge is 27.9 s, then its mass must be greater than $0.725M_\odot$. Conversely, if its mass is less than $0.725M_\odot$ as suggested by recent mass determinations, then the observed 27.9 s period must be a harmonic of its fundamental rotation period or be caused by a pulsation or other phenomenon unrelated to its rotation.

radiation growth time for a star of mass $0.3M_\odot$ is 1.6×10^{12} yr. The most recently measured mass for AE Aqr is $0.8M_\odot$ [18] and for DQ Her is $0.6M_\odot$ [19]. These values imply that these objects are rotating significantly below their maximum rotation rates: $P/P_{\rm min}=1.4$ for AE Aqr and $P/P_{\rm min}=2.0$ for DQ Her. The gravitational radiation growth times for these systems would then be 2.4×10^{10} yr for AE Aqr and 1.3×10^{12} yr for DQ Her. And further, recent observations indicate that the rotation period of the white dwarf in DQ Her is 142.2 s rather than 71.1 s [16]. This increases the gravitational growth time for this star to 8.1×10^{13} yr.

In order for the instability in the *r*-modes to play a significant dynamical role in the evolution of a system, the dimensionless amplitude of the mode must grow to a value of order unity. For example, the gravitational radiation amplitudes computed by Hiscock [11] are based on a presumed balance between accretion and gravitational-radiation torques on the star. Using the equations in Owen *et al.* [6], it is easy to show that (in a rapidly rotating star) this balance requires the dimensionless amplitude of the *r*-mode to be a constant of order unity multiplied by $(\tau_{GR}\dot{M}/M)^{1/2}$. Given the observed accretion rates, the amplitudes of the *r*-modes would have to be of order unity in these systems to maintain this balance. The extreme lengths of the gravitational-radiation time scales in these systems means that there is not enough time for the amplitude of the *r*-mode to grow to such a large value. So even if viscosity were unimportant and all white dwarfs were unstable to the gravitational-radiation driven *r*-mode instability, this instability cannot be playing any significant role in the observed DQ Her objects. Gravitational radiation from the *r*-mode instability in these objects will not be detectable by LISA.

IV. VISCOUS DISSIPATION

Shear viscosity tends to suppress the gravitational radiation driven instability in the *r*-modes. In particular the growth time τ of the mode becomes

$$\frac{1}{\tau}=\frac{1}{\tau_{GR}}-\frac{1}{\tau_V}, \tag{4.1}$$

where τ_V is the viscous time scale. For the $m=2$ *r*-mode of primary interest to us here, τ_V is given by the expression [3]

$$\tau_V=\tfrac{1}{5}\int_0^R \rho r^6 dr\left(\int_0^R \eta r^4 dr\right)^{-1}, \tag{4.2}$$

where η is the shear viscosity. The dominant form of shear viscosity in hot white dwarf stars is expected to be electron scattering with the ion liquid. Nandkumar and Pethick [20] provide the following analytical fit for the shear viscosity in a ^{12}C ion liquid:

$$\eta=\frac{10^6\rho_6^{2/3}}{(1+1.62\rho_6^{2/3})I_2} \tag{4.3}$$

where I_2 is

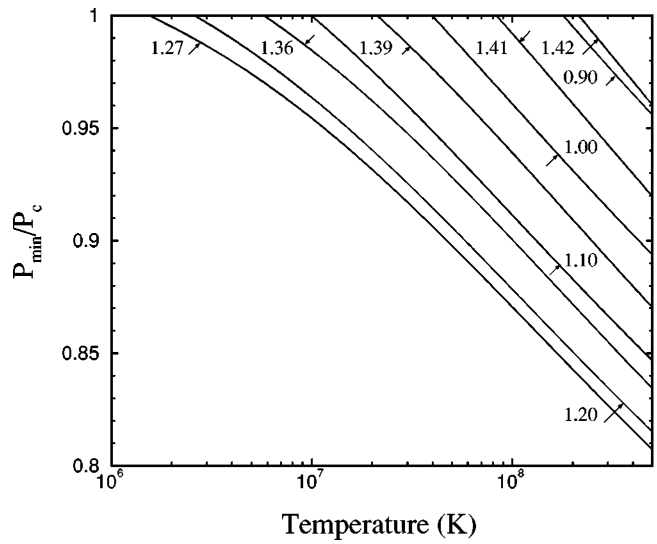

FIG. 3. Critical rotation periods P_c where the gravitational radiation instability in the *r*-modes first sets in: stars with $P<P_c$ are unstable. The mass of the white dwarf (in units of $M_\odot$) labels each curve.

$$I_2=0.667\log(1.32+0.103T_6^{1/2}\rho_6^{-1/6})+0.611-\frac{0.475+1.12\rho_6^{2/3}}{1+1.62\rho_6^{2/3}}. \tag{4.4}$$

The density ρ_6 and temperature T_6 are to be given here in units of 10^6 g/cm^3 and 10^6 K respectively. This formula for η is sufficiently accurate (i.e. to within about 10%) for the densities above 10^6 g/cm^3 which dominate the integral in the denominator of Eq. (4.2).

The growth time τ defined in Eq. (4.1) is negative for slowly rotating stars: $P\gg P_{\rm min}$. In this case viscosity dominates and the *r*-mode instability is suppressed. For more rapidly rotating stars, however, this expression may become positive and hence the *r*-modes unstable. The critical rotation period P_c where the instability first sets in is determined by setting $1/\tau=0$. It follows then that

$$\frac{P_{\rm min}}{P_c}=\left(\frac{\tilde{\tau}_{GR}}{\tau_V}\right)^{1/6}. \tag{4.5}$$

This critical rotation period P_c has been evaluated for the white dwarf models discussed in Sec. II and a range of core temperatures appropriate for hot white-dwarf stars. These results are depicted graphically in Fig. 3. These results show that no white dwarf is unstable to the *r*-mode instability when its core temperature falls below 10^6 K. Further, no white dwarf star with mass smaller than $0.9M_\odot$ is subject to the *r*-mode instability if its core temperature is below 2×10^8 K, the maximum core temperature expected to occur in accreting white dwarf systems [21,22]. No white dwarf with rotation period greater than $1.2P_{\rm min}$ is unstable to the *r*-mode instability if its core temperature is below 2×10^8 K.

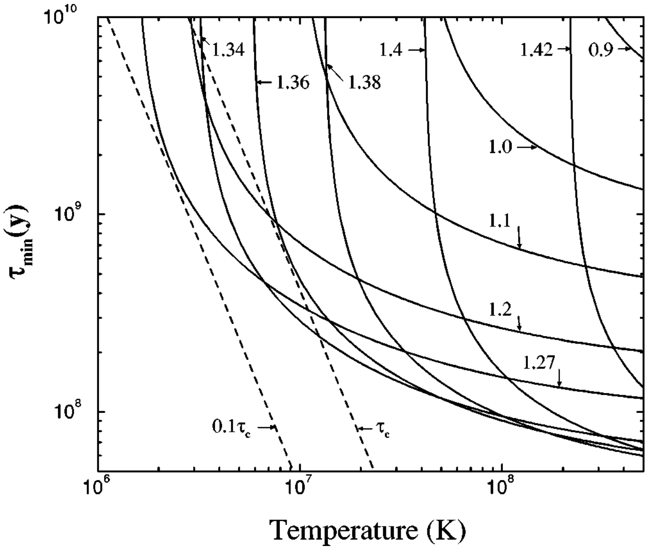

FIG. 4. Minimum growth times τ_{min} for the gravitational radiation driven instability of the r-modes for white dwarf stars. The mass of the star (in units of $M_\odot$) labels each curve. The dashed curves depict the time τ_c needed for a white dwarf to cool to a given core temperature.

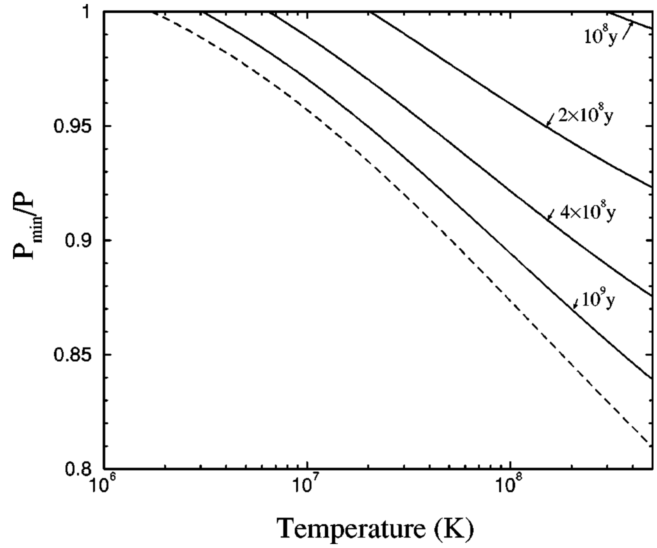

FIG. 5. Rotation periods P (as a function of temperature) for 1.3 $M_\odot$ white dwarf stars where the growth time of the $m=2$ r-mode has the prescribed value. The dashed curve is the critical rotation period P_c where the growth time is infinite.

To investigate further whether the r-mode instability might be playing a role in some white dwarf stars, the growth times τ for these modes are examined. The expression for this time scale, Eq. (4.1), can be re-written in terms of the quantities graphed in Figs. 1, 2 and 3:

$$\tau = \tilde{\tau}_{GR}\left[\left(\frac{P_{min}}{P}\right)^6 - \left(\frac{P_{min}}{P_c}\right)^6\right]^{-1}. \tag{4.6}$$

The time scale τ becomes infinite as $P \to P_c$ and is minimum for $P = P_{min}$. Thus it is useful to define the smallest possible growth time τ_{min} for a star of given mass and temperature:

$$\tau_{min} = \tilde{\tau}_{GR}\left[1 - \left(\frac{P_{min}}{P_c}\right)^6\right]^{-1}. \tag{4.7}$$

The values of these minimum growth times are depicted in Fig. 4. It follows that no white dwarf has an r-mode instability with growth time shorter than 7×10^7 yr if its core temperature is below 2×10^8 K.

In the absence of accretion a white dwarf star will quickly cool. Its core temperature drops to the value T in approximately the time τ_c given by the expression [23]

$$\tau_c \approx 1.3\times10^{11} T_6^{-5/2} \text{ yr}. \tag{4.8}$$

The dashed curves in Fig. 4 represent this cooling time τ_c and also $0.1\tau_c$. It follows that most of the minimum r-mode instability growth times are longer than τ_c. In these stars, the amplitude of the r-mode will grow by less than a factor of e before the star cools and the instability is suppressed. No star has a minimum instability growth time that is shorter than $0.1\tau_c$. Thus, the r-mode instability will never have time to grow to a dynamically important level in any white dwarf except during the time it is heated by accretion.

If a white dwarf is heated by accretion, this heating can last for only a finite time, given approximately by $\tau_A = \Delta M/\dot{M}$ where ΔM is the total amount of accreted material and $\dot{M}$ is the accretion rate. Since the r-mode instability is never effective for stars with $M < 0.9M_\odot$, we see that the r-mode instability can only act for a time when $\Delta M < 0.6M_\odot$. Further, we see from Fig. 4 that $\tau_{min} \gtrsim 7\times10^7$ yr for stars with $T \lesssim 2\times10^8$ K, the maximum core temperature reached in models of accretion onto white dwarfs [21,22]. In order for the r-mode instability to have time to grow to a dynamically significant level then, it would be necessary to have $\tau_A \gtrsim 10\tau_{min}$. This implies that the accretion rate must satisfy $\dot{M} \lesssim 9\times10^{-10} M_\odot$ yr^{-1}. Given this low accretion rate, this argument can be strengthened slightly. For white dwarfs with $\dot{M} \lesssim 10^{-9} M_\odot$ yr^{-1} the core temperature is only expected to reach about 8×10^7 K [21,24] during accretion and so, from Fig. 4, $\tau_{min} \gtrsim 10^8$ yr. For stars with core temperatures in this range, Fig. 4 implies that instability can only occur on a time scale fast enough to act effectively within the age of the universe for stars in the mass range $1.05 \lesssim M \lesssim 1.4M_\odot$ and so $\Delta M \lesssim 0.35M_\odot$. Thus, the upper limit on the accretion rate can be improved slightly to $\dot{M} \lesssim 3.5\times10^{-10} M_\odot$ yr^{-1}.

One further difficulty to achieving a dynamically significant r-mode instability in white dwarf stars is illustrated in Fig. 5. The minimum growth times τ_{min} shown in Fig. 4 only pertain to stars that are maximally rotating $P = P_{min}$. The actual growth times of these modes increase rapidly for more slowly rotating stars, as illustrated in Fig. 5 for the $1.3M_\odot$ stellar model. For example, the growth time increases from about 10^8 yr for the maximally rotating model with $T = 2\times10^8$ K to about 10^9 yr for the still very rapidly rotating model with $P = 1.15P_{min}$. Thus the very hot ($T \gtrsim 3\times10^6$ K) and very rapid-rotation conditions ($P_{min} \lesssim P \lesssim 1.15P_{min}$) needed to allow the growth of the r-mode must be maintained for a very long time. This can only be achieved by a very low and steady accretion rate that adds (or removes) angular momentum from the star at just the rate needed to maintain maximal rotation as the star's mass increases.[2] In the most favorable case (i.e. by keeping temperatures above 10^8 K, rotation periods less than $1.05P_{\rm min}$ and the mass of the star near $1.3M_\odot$) these special conditions need last only about 10^9 yr in order to allow the amplitude of the mode to grow by a factor of $e^{10}\approx 2\times 10^4$. In the more typical and less favorable cases this type of accretion would be needed for 10^{10} yr or longer to achieve this amplification of the mode. It seems rather unlikely that these special conditions have ever been met in any real white dwarf stars.

[2]The angular momentum of a maximally rotating white dwarf star decreases as its mass increases for stars with $M\gtrsim 1.1M_\odot$ [14].

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