Phase transitions and the mass-radius curves of relativistic stars

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(Received 2 March 1998; published 22 June 1998)

The properties of the mass-radius curves of relativistic stellar models constructed from an equation of state with a first-order phase transition are examined. It is shown that the slope of the mass-radius curve is continuous unless the discontinuity in the density at the phase transition point has a certain special value. The curve has a cusp if the discontinuity is larger than this value. The curvature of the mass-radius curve becomes singular at the point where the high density phase material first appears. This singularity makes the mass-radius curve appear on large scales to have a discontinuity in its slope at this point, even though the slope is in fact continuous on microscopic scales. Analytical formulas describing the behavior of these curves are found for the simple case of models with two-zone uniform-density equations of state. [S0556-2821(98)02114-6]

PACS number(s): 04.40.Dg, 26.60.+c, 95.30.Sf, 97.60.Jd

I. INTRODUCTION

The structures of spherically symmetric stellar models are usually described in general relativity theory in terms of the functions $\rho(r)$, $p(r)$, and $m(r)$: the total energy density, the pressure, and the "mass" contained within a sphere of radius r. These functions satisfy Einstein's equation for a static spherically symmetric spacetime with fluid source, which may be reduced to the following pair of ordinary differential equations [1]:

$$\frac{dp}{dr} = -(\rho+p)\frac{m+4\pi r^3 p}{r(r-2m)}, \tag{1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{2}$$

Consider the families of solutions to these equations, each of which is determined by a different equation of state $\rho = \rho(p)$. For each equation of state and for each value of the central pressure, there exists a unique solution of Eqs. (1),(2) that is non-singular at the center of the star [2]. Thus for each equation of state there exists a one-parameter family of stellar models parametrized by p_c the central pressure of the star. A large class of equations of state have stellar models with finite total radii, $p(R)=0$, and finite total masses $M = m(R)$. The discussion here is limited to these equations of state [3]. The collection of total masses $M(p_c)$ and radii $R(p_c)$ for a given equation of state is called the mass-radius curve: $[M(p_c), R(p_c)]$. Each equation of state determines a unique mass-radius curve, and conversely it appears (although the argument [4] falls short of being a rigorous proof) that each mass-radius curve determines a unique equation of state. Thus there is hope that the high density equation of state of neutron star matter may one day be determined by measurements of the macroscopic mass-radius curve of these stars.

This paper is concerned with analyzing the features of the mass-radius curve for the case of an equation of state with a first-order phase transition. Such a phase transition may well be a feature of the equation of state of real neutron star matter. Pion condensation [5] and/or quark deconfinement [6] might well provide the mechanism that drives such a phase transition. This paper does not focus on the microphysics of the mechanism that may trigger such a transition, but rather the consequences that such a transition might have on the observable macroscopic equilibrium structures of neutron stars. In particular this paper investigates how the properties of such a phase transition might be read from the structure of the mass-radius curve.

Consider equations of state that are smooth except at one value of the pressure p_t where the energy density undergoes a simple discontinuity

$$\rho_- \equiv \lim_{p\uparrow p_t} \rho(p) < \lim_{p\downarrow p_t} \rho(p) \equiv \rho_+ \tag{3}$$

as illustrated in Fig. 1. It is convenient to parameterize the magnitude of the discontinuity in the equation of state by the dimensionless quantity Δ:

$$\Delta = \frac{\rho_+ - \rho_-}{\rho_- + p_t}. \tag{4}$$

Figure 2 illustrates the mass-radius curves for the equations of state shown in Fig. 1 (simple polytropes $p \propto \rho^2$ with a density discontinuity inserted at the pressure p_t). The mass scale, M_t, and radius scale, R_t, used here are the total mass

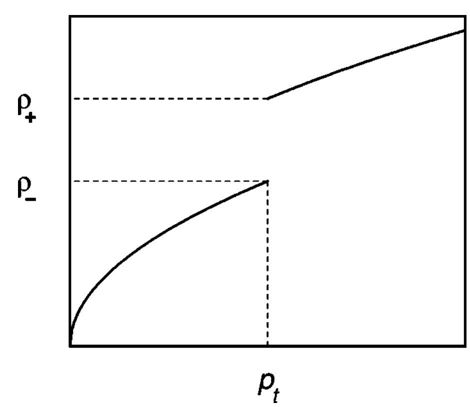

FIG. 1. Equation of state with a first-order phase transition at the pressure $p = p_t$.

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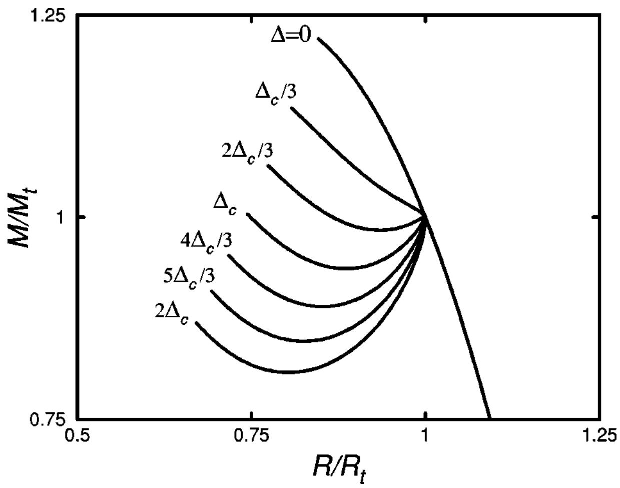

FIG. 2. Mass-radius curves for equations of state with first order phase transitions. The different curves correspond to different values of the parameter $\Delta=(\rho_+-\rho_-)/(\rho_-+p_t)$.

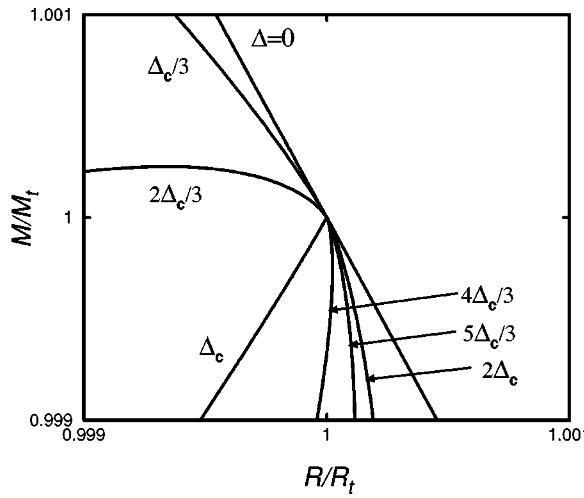

FIG. 3. Mass-radius curves for equations of state with first order phase transitions. This figure represents the same stellar models depicted in Fig. 2 but on a much finer scale.

and radius of the stellar model with central pressure p_t. The quantity Δ_c that is used to scale the discontinuity in the equation of state is defined by

$$\Delta_c=\frac{\rho_-+3p_t}{2(\rho_-+p_t)}. \tag{5}$$

(Note that $1/2\leq\Delta_c<3/2$.) Figure 2 illustrates that a first-order phase transition makes the mass-radius curve bend sharply at the critical point (M_t,R_t) where the high density phase material first appears in the core of the star.

Figure 2 makes it appear that mass-radius curves have finite discontinuities in their slopes at the point where the higher density phase material first enters the star. Further, it appears that the magnitude of the discontinuity in the slope is determined by the parameter Δ that measures the magnitude of the phase transition. Thus one might hope that an expression can be derived which determines the properties of the phase transition (e.g. the value of Δ) in terms of some features (e.g. the change in slope) of the mass-radius curve in a neighborhood of the point (M_t,R_t). This hope is diminished, however, on closer examination of these curves. Figure 3 illustrates the same set of mass-radius curves as shown in Fig. 2, however, on a much finer scale. Figure 3 shows that the slopes of all of the curves are in fact continuous at the point (M_t,R_t), except for the special case with $\Delta=\Delta_c$. The mass-radius curves for equations of state with $\Delta>\Delta_c$ reverse direction at the point (M_t,R_t), however their slopes are continuous there. The curves of models with strong first-order phase transitions have cusps at the critical point. This microscopic continuity of the slope makes it impossible to find a purely local relationship between the properties of the phase transition and the magnitude of the macroscopic bend that occurs in the mass-radius curves, as illustrated in Fig. 2.

The continuity of the slope of the mass-radius curve even in the presence of first-order phase transitions was first discovered in the context of Newtonian stellar models by Ramsey [7] and Lighthill [8]. The corresponding result for relativistic models was demonstrated in analogy with the Newtonian analysis by Seidov [9]. A more complete and somewhat more rigorous derivation of this fact is presented in the Appendix here for arbitrary relativistic stellar models. These analyses demonstrate that a special value of the magnitude of the phase transition is $\Delta=\Delta_c$, with Δ_c defined in Eq. (5). For stronger phase transitions, $\Delta>\Delta_c$, the mass-radius curve reverses direction at the critical point, and probably triggers the onset of instability in the stellar models immediately above this point.

The general analysis of Lighthill and Seidov shows that the slope of the mass-radius curve is continuous (in almost all cases) even at the critical stellar model where the influence of a phase transition is first felt. This result, however, raises more questions than it answers. The ''typical'' mass-radius curves displayed in Figs. 2, 3 show that the phase transition does have a very profound effect on the slope of these curves in a very small neighborhood of the critical stellar model. How does the phase transition change the curvature of these curves on very small scales in a neighborhood of the critical model, while leaving it relatively unaffected on larger scales? In Sec. II a more detailed analysis of the structure of the mass-radius curve in the neighborhood of a critical point is undertaken in an attempt to understand this behavior. An analysis is given there of the simple case of equations of state having two uniform-density zones: $\rho=\rho_-$ for $p<p_t$ and $\rho=\rho_+$ for $p>p_t$. Analytical expressions are derived for the mass-radius curve for these models in a small neighborhood of the critical point (M_t,R_t). These expressions show that the phase transition causes the curvature of the mass-radius curve to diverge at this point, even though its slope is well defined and continuous there. This singular part of the curvature causes the mass-radius curves in these simple models to bend on relatively small scales, much like the more realistic ones depicted in Figs. 2, 3. Unfortunately, the two-zone models are too simple to model accurately the behaviors of the mass-radius curves of more realistic equations of state. A more complicated analysis is needed, but that analysis is deferred to a future investigation.

II. TWO-ZONE UNIFORM-DENSITY MODELS

The general solution of Einstein's equation representing a static spherical uniform-density star was first found by Schwarzschild [10]. Let ρ_i denote the constant density of the star. Then the general solution to Eqs. (1),(2) can be written

$$m(r)=\frac{4\pi}{3}\rho_i r^3+a_i, \tag{6}$$

$$p(r)+\rho_i=b_i f_i(r)\left[1+4\pi b_i\int_{c_i}^{r} r' f_i^3(r')dr'\right]^{-1}, \tag{7}$$

where a_i, b_i and c_i are arbitrary constants, and $f_i(r)$ is defined as

$$f_i(r)=\left(1-\frac{8\pi}{3}\rho_i r^2-\frac{2a_i}{r}\right)^{-1/2}. \tag{8}$$

More complicated stellar models composed of concentric uniform-density layers may also be constructed by combining together the basic solutions given in Eqs. (6),(7). These laminated models satisfy Eqs. (6),(7) with ρ_i the fluid density within a particular layer. The regularity of the global solution is assured by choosing c_i to be the inner radius of the i^{th} layer, and the constants a_i and b_i to make $p(r)$ and $m(r)$ continuous at $r=c_i$.

Now consider the stellar models composed of material having a simple two-zone uniform-density equation of state:

$$\rho=\begin{cases}\rho_- & \text{for } p<p_t,\\ \rho_+ & \text{for } p>p_t.\end{cases} \tag{9}$$

This is the simplest equation of state having a first order phase transition. The family of stellar models associated with this equation of state is easily obtained from Eqs. (6),(7).

For stars with small central pressures $p_c<p_t$, the solutions are the standard interior Schwarzschild models [10]. These may be obtained from the general expressions above by setting $a_i=c_i=0$ and $b_i=p_c+\rho_-$. The total masses and radii of these models are determined by solving Eqs. (6),(7) for the points where $p(R)=0$ and $M=m(R)$. As functions of the central pressure these solutions are

$$R^2(p_c)=\frac{3}{8\pi\rho_-}\left[1-\left(\frac{\rho_-+p_c}{\rho_-+3p_c}\right)^2\right], \tag{10}$$

$$M(p_c)=\frac{4\pi}{3}\rho_- R^3(p_c). \tag{11}$$

Thus the models with small central pressures have the well-known cubic mass-radius curve. The critical model in this family is the one having $p_c=p_t$ with total mass $M_t=M(p_t)$ and total radius $R_t=R(p_t)$. Expressions for these critical values are given by Eqs. (10),(11) with $p_c=p_t$.

The stars with large central pressures, $p_c>p_t$, have two concentric layers. The inner layer is composed of high density material with $\rho=\rho_+$, and the outer layer of lower density material with $\rho=\rho_-$. The structure of the inner core is determined from Eqs. (6),(7) with $\rho_i=\rho_+$, $a_i=c_i=0$, and $b_i=p_c+\rho_+$. The radius, r_t, of this inner core is determined by solving Eq. (7) for the point where $p_t=p(r_t)$. As a function of the central pressure, p_c, this core radius is

$$r_t^2=\frac{3}{8\pi\rho_+}\left\{1-\left[\frac{(\rho_++p_c)(\rho_++3p_t)}{(\rho_++p_t)(\rho_++3p_c)}\right]^2\right\}. \tag{12}$$

The outer envelopes of these models are determined again by Eqs. (6),(7) with $\rho_i=\rho_-$, $a_i=4\pi(\rho_+-\rho_-)r_t^3/3$, $b_i=(\rho_-+p_t)\sqrt{1-8\pi\rho_+ r_t^2/3}$, and $c_i=r_t$. The quadrature indicated in Eq. (7) can be expressed in terms of standard elliptic integral functions for this case, but that representation does not offer any particular insight for our purposes here.

The total masses and total radii of the two-zone uniform-density models are found by solving Eqs. (6),(7) for the points where $p(R)=0$ and $M=m(R)$. These equations can not be solved analytically even for these simple two-zone models. However, the nature of the solutions near the critical model can be studied by means of power series expansions. The small parameter $s=(r_t/R_t)^2$ (which vanishes as $p_c\downarrow p_t$) can be used to expand the various quantities (i.e., b_i and f_i) that appear in Eq. (7). The integration that appears in Eq. (7) is performed term by term and the resulting equation is solved for the total radius of the star $p(R)=0$. The resulting series expansion for R is used to evaluate the total mass $M=m(R)$ using Eq. (6) to the same order of approximation:

$$\frac{R(s)}{R_t}=1+r_1 s+r_{3/2}s^{3/2}+r_2 s^2+\mathcal{O}(s^{5/2}), \tag{13}$$

$$\frac{M(s)}{M_t}=1+m_1 s+m_{3/2}s^{3/2}+m_2 s^2+\mathcal{O}(s^{5/2}), \tag{14}$$

where the expansion coefficients r_1, m_1, etc. are given by

$$r_1=\frac{\Delta_c-\Delta}{8\Delta_c^3}, \tag{15}$$

$$r_{3/2}=\frac{\Delta(3-6\Delta_c^2-8\Delta_c^4)}{8\Delta_c^4(3-2\Delta_c)}, \tag{16}$$

$$r_2=\frac{9\Delta(4\Delta_c-\Delta)(4\Delta_c^2-1)}{128\Delta_c^6(3-2\Delta_c)}-\frac{r_1^2}{2}, \tag{17}$$

$$m_1=3r_1, \tag{18}$$

$$m_{3/2}=\frac{\Delta(9-18\Delta_c^2-8\Delta_c^4)}{8\Delta_c^4(3-2\Delta_c)}, \tag{19}$$

$$m_2=3(r_2+r_1^2), \tag{20}$$

and where Δ and Δ_c are defined in Eqs. (4) and (5) respectively. These series expansions give a reasonably good approximation of the mass-radius curve near the critical model as illustrate in Fig. 4. The series agree with the exact (numerically determined) mass-radius curves to within 1% for models whose masses and radii differ from the critical values

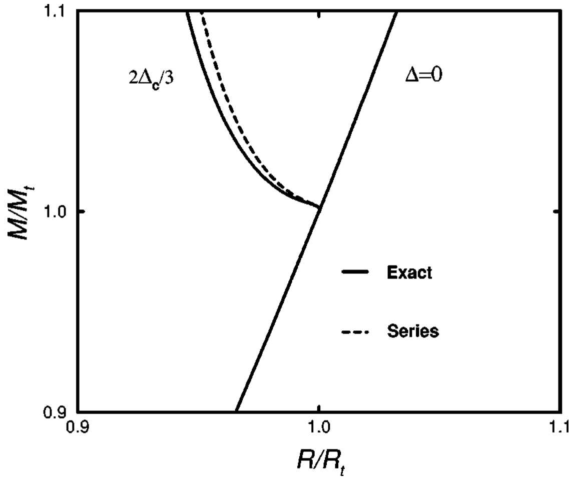

FIG. 4. Mass-radius curve for the two-zone uniform-density equation of state. The series expansion for the curve (dashed line) is compared to the exact (solid line) for a model with $\Delta=2\Delta_c/3$ and $p_t/\rho_-=0.5$.

by up to about 10% in the worst case examined. (The series converge most poorly for $\Delta=\Delta_c$ among those cases examined.) This level of accuracy is good enough to account for the interesting features of the mass-radius curve near the critical point. Figure 4 illustrates that the series exhibit the same apparent discontinuity in slope on large scales as the exact curves.

The tangent vector to the mass-radius curves is determined by differentiating the expressions for $x(s)\equiv R(s)/R_t$ and $y(s)\equiv M(s)/M_t$ in Eqs. (13),(14):

$$\begin{pmatrix} dx/ds \\ dy/ds \end{pmatrix} = \begin{pmatrix} r_1+\frac{3}{2}r_{3/2}s^{1/2}+2r_2 s \\ m_1+\frac{3}{2}m_{3/2}s^{1/2}+2m_2 s \end{pmatrix}. \tag{21}$$

This expression illustrates that the slope of the mass radius curve is continuous even at the critical model. This follows from the fact that the tangent vector computed from just above the critical model using Eq. (21) is $r_1(dx/dp_c)^{-1}$ times that computed from just below using Eq. (11). Just above the critical point the tangent vector is proportional to the quantity r_1 defined in Eq. (15). r_1 is positive for weak phase transitions $\Delta<\Delta_c$ but negative for strong transitions $\Delta>\Delta_c$. Thus the tangent vector and hence the mass-radius curve itself reverses direction when a strong phase transition occurs. This confirms the general continuity analysis of Ramsey, Lighthill, and Seidov for the simple case of the two-zone uniform-density models.

While the general analysis of the structure of the mass-radius curve fails for the case of a phase transition with $\Delta=\Delta_c$, the analysis here for the simple two-zone uniform-density models succeeds even in this case. The terms proportional to s in Eqs. (13),(14) vanish when $\Delta=\Delta_c$, and therefore s is not a good affine parameter for the mass-radius curve at $s=0$ in this case. Instead, the appropriate parameter is $\lambda=s^{3/2}$. In this case the tangent vector evaluated at the critical model is $(dx/d\lambda,dy/d\lambda)=(r_{3/2},m_{3/2})$. This vector is *not* proportional to the tangent vector just below the critical model, and so the slope of the mass-radius curve is *not* continuous in this case.

In order to discuss the magnitude of the change in slope that occurs at the critical model it is necessary to adopt a metric structure for the space of masses and radii. This makes it possible to define the inner products of tangent vectors (and so define angles) and also more generally to discuss the curvatures of these curves. There is no canonical choice for this metric, and therefore no absolute intrinsic meaning can be given to angles or curvatures that are computed. Nevertheless these quantities are useful tools for understanding the features of the mass-radius curves seen in Figs. 2,4. Thus the metric used to display those figures is adopted: the flat metric with Cartesian coordinates $x\equiv R/R_t$ and $y\equiv M/M_t$.

Return now to the kink that occurs in the mass-radius curve for the case of a phase transition with $\Delta=\Delta_c$. The angle between the slopes above and below the transition point can be determined (using the metric defined above) from the inner product between the tangent vectors. The resulting angle θ depends solely and monotonically on the ratio p_t/ρ_-:

$$\cos\theta=\frac{r_{3/2}+3m_{3/2}}{\sqrt{10(r_{3/2}^2+m_{3/2}^2)}}. \tag{22}$$

This θ varies from about 4.4° for $p_t/\rho_-=0$ to about 170.4° for $p_t/\rho_-=\infty$. This formula is a simple example of the kind of relationship that one had hoped to find relating the parameters of the phase transition and the macroscopic structure of the mass-radius curve. In this special case (phase transitions with $\Delta=\Delta_c$) the magnitude of the kink in the mass-radius curve determines the ratio p_t/ρ_-. Unfortunately, this formula is not universal even for phase transitions with $\Delta=\Delta_c$. The magnitude of the kink displayed in Fig. 3 does not satisfy this equation for example. The general form of this relationship must depend on other features of the equation of state (e.g. $d\rho/dp$ at the transition point) that are not present in the simple two-zone uniform-density models.

The curvature of any of the mass-radius curves can be evaluated by differentiating the unit tangent vector along the trajectory of the curve. The resulting acceleration is equal to the inverse of the radius of curvature of the curve. For a general curve in a flat two-dimensional space, this acceleration is given by

$$a=\left(\frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{d^2y}{ds^2}\frac{dx}{ds}\right)\left[\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right]^{-3/2}. \tag{23}$$

This expression is invariant under changes in the parametrization of the curve, but not on the assumed metric of the mass-radius space. It is straightforward to evaluate this expression using the series expansions, Eqs. (13),(14), for the curve:

$$a=\frac{3(3r_{3/2}-m_{3/2})}{40\sqrt{10}r_1^2}\times\left[\frac{1}{s^{1/2}}-\frac{9(r_{3/2}+3m_{3/2})}{20r_1}\right]-\frac{3}{5\sqrt{10}}+\mathcal{O}(s^{1/2}). \quad (24)$$

The first term in Eq. (24) is proportional to Δ and therefore vanishes when there is no phase transition. The second term is a pure number that is independent of the parameters of the phase transition. The second term is therefore the curvature of the mass-radius curve just below the phase transition point. The first term includes a part proportional to $1/s^{1/2}$ which diverges at the critical model. This infinite curvature causes the mass-radius curve to bend sharply in the neighborhood of the critical point.

The analysis here shows that the mass-radius curves of stellar models with first-order phase transitions have infinite accelerations at the critical model where the high density phase first appears. This acceleration causes these curves to bend sharply, appearing on large scales to have a discontinuous slope at this point. Analytical formulas, Eqs. (13),(14), have been derived that describe quantitatively the structures of these curves for models with simple two-zone uniform-density equations of state as illustrated in Fig. 4. These formulas also account in a qualitative way for the behavior of the mass-radius curves of more realistic equations of state, as illustrated in Figs. 2, 3. The quantitative description of the more realistic mass-radius curves (as would be needed to analyze the measured masses and radii of real neutron stars) requires the derivation of the analogs of Eqs. (13),(14) for a general equation of state. This generalization is not a simple extension of the analysis presented here, and is deferred to a future investigation.

ACKNOWLEDGMENTS

I thank B. Schmidt, J. Friedman, J. Ipser, J. Isenberg and A. K. M. Masood-ul-Alam for helpful conversations concerning this work. I also thank B. Schutz and the Max Plank Institut für Gravitationsphysik (Albert Einstein Institut), Potsdam for their hospitality during a visit in which a portion of this work was completed. This research was supported by grants PHY-9796079 from the National Science Foundation, and NAG5-4093 from the National Aeronautics and Space Administration.

APPENDIX: CONTINUITY OF THE SLOPE OF THE MASS-RADIUS CURVE

This appendix presents the argument that the slope of the mass-radius curve is continuous even at the point where the phase transition first sets in (except for the case $\Delta=\Delta_c$). The discussion here is more complete and somewhat more rigorous than Seidov's original [9].

This argument can be made a little simpler by introducing a somewhat unusual representation of the equations of stellar structure. The structure of a spherical star in general relativity is usually expressed in terms of the functions $m(r,p_c)$, $p(r,p_c)$ and $\rho(r,p_c)$ satisfying Eqs. (1),(2). The analysis here is concerned with how these functions behave as p_c varies near the point p_t where a phase transition occurs. Unfortunately the functions $m(r,p_c)$, $p(r,p_c)$, and especially $\rho(r,p_c)$ are not smooth when a phase transition is present. The density function $\rho(r,p_c)$ in particular is discontinuous at the boundary of the inner core of high density phase material, and the position of this discontinuity changes as p_c is varied. Thus, approximate expressions for these functions in terms of power series expansions near the critical point (as needed in Seidov's analysis) are somewhat awkward. This difficulty is simplified by considering the structure of the star in terms of the ''inverses'' of these functions [4]: e.g. $m(p,p_c)$ and $r(p,p_c)$. Since the pressure is a monotonic function of the radius in these models, this inversion is always possible. These functions are smooth in their dependence on p for fixed p_c, and so it is more straightforward to approximate them with power series expansions.

It is also useful to introduce a slightly different set of basic variables to describe the structures of stars instead of the usual p, m, and r. It is preferable to use the thermodynamic enthalpy function

$$h(p)=\int_0^p\frac{dp'}{\rho+p'}, \quad (A1)$$

in place of the pressure as the independent variable in this representation of the problem, because it makes the differential equations non-singular at the surface of the star. Similarly, it is somewhat preferable to use the functions $u=r^2$ and $v=m/r$ as dependent variables because they are smoother functions of h near the centers of the stars. The straightforward translations of the standard structure equations (1), (2) into this new set of variables gives

$$\frac{du}{dh}=-\frac{2u(1-2v)}{4\pi up(h)+v}\equiv U(u,v,h), \quad (A2)$$

$$\frac{dv}{dh}=-(1-2v)\frac{4\pi u\rho(h)-v}{4\pi up(h)+v}\equiv V(u,v,h). \quad (A3)$$

In these Eqs. (A2),(A3) the functions $p(h)$ and $\rho(h)$ are determined from the chosen equation of state $\rho=\rho(p)$ and Eq. (A1). They are therefore explicitly known functions once a particular equation of state has been selected. This version of the equations has several nice features. First, the use of h as the independent variable makes the domain where the solution is defined, $[0,h_c]$ where h_c is the value of h at the center of the star, known before the solution is found rather than after. Second, the total radius of the star is determined simply by evaluating the function u at the surface of the star $h=0$, instead of solving the usual surface equation $p(R)=0$. Third, the use of h as independent variable makes the equations non-singular at the surface of the star. And fourth, the use of u and v as dependent variables make the solutions near $h=h_c$ smoother than the usual functions m and r.

Consider the one-parameter family of solutions to these equations constructed from a single equation of state: $u(h,\lambda)$ and $v(h,\lambda)$, where λ is the parameter that distinguishes the individual members of the family. Each member of this family satisfies the usual boundary conditions, both at the center of the star $h=h_c$,

$$u[h_c(\lambda),\lambda]=v[h_c(\lambda),\lambda]=0, \tag{A4}$$

and at the surface of the star $h=0$,

$$u(0,\lambda)=R^2(\lambda), \tag{A5}$$

$$v(0,\lambda)=\frac{M(\lambda)}{R(\lambda)}, \tag{A6}$$

where $M(\lambda)$ is the total mass and $R(\lambda)$ is the total radius of the model with parameter λ. The choice of the particular parametrization is arbitrary; however, it is convenient to insist that each member of the family have a unique central pressure p_c and a unique central value of the enthalpy h_c. Thus, either of these quantities could be used as the parameter λ.

Near the centers of these stars, the solutions to the structure equations can be given analytically as power series expansions. When the equation of state is smooth [i.e., when $\rho(h)$ and $p(h)$ are smooth functions] then $u(h,\lambda)$ and $v(h,\lambda)$ have the expansions:

$$u(h,\lambda)=\frac{3(h_c-h)}{2\pi(\rho_c+3p_c)}+\mathcal{O}((h_c-h)^2), \tag{A7}$$

$$v(h,\lambda)=\frac{2\rho_c(h_c-h)}{\rho_c+3p_c}+\mathcal{O}((h_c-h)^2). \tag{A8}$$

The right sides of Eqs. (A7),(A8) depend on λ implicitly. The central value of h depends on which member of the one-parameter family is being considered, thus $h_c=h_c(\lambda)$. The choice of parametrization is arbitrary however. Thus $h_c(\lambda)$ is an arbitrarily monotonic function. The quantities ρ_c and p_c also depend on λ in the obvious ways: $\rho_c=\rho[h_c(\lambda)]$, etc.

Next consider the situation where the equation of state is smooth, except at a certain phase transition point $h=h_t$. Assume that the density has a finite discontinuity at this point:

$$\rho_-\equiv\lim_{h\uparrow h_t}\rho(h)<\lim_{h\downarrow h_t}\rho(h)\equiv\rho_+ . \tag{A9}$$

This is simply the restatement of Eq. (3) in terms of h instead of p. The pressure function $p(h)$ is C^0 at this point as a consequence of Eq. (A1), but it has a finite discontinuity in its first derivative there. One of the important facts that is needed in this analysis is the continuity of the functions u and v. The functions $u(h,\lambda)$ and $v(h,\lambda)$ are C^0 functions of h for fixed λ and C^0 functions of λ for fixed values of h. These continuity conditions are reasonably easy to establish. First consider the continuity of $u(h,\lambda)$ and $v(h,\lambda)$ as functions of h for fixed λ. The right sides of Eqs. (A2),(A3) are smooth functions of u and v (for $u>0$ and $v>0$) and are continuous functions of h except when $h=h_t$. When $h=h_t$ the right side of Eq. (A3) has a finite discontinuity as described in Eq. (A9). If u and v were discontinuous for some value of h, then the left sides of Eqs. (A2),(A3) would be singular there. But the right sides are finite for $h<h_c(\lambda)$, so $u(h,\lambda)$ and $v(h,\lambda)$ must be continuous for all $h<h_c(\lambda)$ for fixed λ.

Next consider the continuity of $u(h,\lambda)$ and $v(h,\lambda)$ as functions of λ for fixed h. Assume that the parameter λ is chosen so that $h_c(\lambda)$ is smooth and monotonically increasing. Let λ_t denote the critical value of the parameter for which $h_c(\lambda_t)=h_t$. For $\lambda<\lambda_t$ the expansions in Eqs. (A7),(A8) show that $u(h,\lambda)$ and $v(h,\lambda)$ are continuous in λ at least in a small neighborhood of the center of the star where $h=h_c$. The differential Eqs. (A2),(A3) are non-singular outside of this neighborhood. The standard theorems [11] insure that the solutions to such non-singular equations depend continuously on their boundary values. These boundary values as determined by Eqs. (A7),(A8) can be applied a small distance away from the singular point $h=h_c$. Thus, $u(h,\lambda)$ and $v(h,\lambda)$ are continuous functions of λ for fixed h, at least for $\lambda<\lambda_t$. When $\lambda>\lambda_t$ a similar argument insures the continuity of $u(h,\lambda)$ and $v(h,\lambda)$ in the cores of the stars where $h\geqslant h_t$. Thus $u(h_t,\lambda)$ and $v(h_t,\lambda)$ are continuous functions of λ for $\lambda>\lambda_t$. These functions can now be considered as the boundary values for $u(h,\lambda)$ and $v(h,\lambda)$ in the domain $h<h_t$. In this domain the standard theorems again apply, so the continuity of the boundary values [i.e., $u(h_t,\lambda)$ and $v(h_t,\lambda)$] guarantee the continuity of $u(h,\lambda)$ and $v(h,\lambda)$ as functions of λ for fixed h. The only troublesome point is at $\lambda=\lambda_t$.

Consider the stellar models with λ just above the critical point. These models consist of a very small central core of material of the higher-density phase, $\rho\geqslant\rho_+$, and the vast majority of the material in the lower-density phase. In the limit $\lambda\downarrow\lambda_t$ the size and mass of this central core of material goes to zero. This limit can be seen analytically in the expansions given in Eqs. (A7),(A8). In this limit what remains is a star composed entirely of matter in the lower-density phase, except for the single point at the center of the star. At this single central point the material remains in the higher-density phase. But, the matter at this single point does not effect the structure of the star at all. The solutions to the structure Eqs. (A2),(A3) are not changed if the equation of state is changed only at a single value of h. Thus the function $u_\downarrow(h)=\lim_{\lambda\downarrow\lambda_t}u(h,\lambda)$ is identical to the function that describes a stellar model consisting entirely of lower density material with $h_c=h_t$: $u_\uparrow=\lim_{\lambda\uparrow\lambda_t}u(h,\lambda)$. A similar argument applies to $v(h,\lambda)$. Thus the functions $u(h,\lambda)$ and $v(h,\lambda)$ are continuous functions of λ even for $\lambda=\lambda_t$.

In order to understand the structure of the mass-radius curve in stars having a first-order phase transition, the structure of stars having a very small central core of the high density phase material must be analyzed in some detail. The central cores of such models are described by the series solutions given in Eqs. (A7),(A8):

$$u_+(h,\lambda)=\frac{3(h_c-h)}{2\pi(\rho_c+3p_c)}+\mathcal{O}((h_c-h)^2), \tag{A10}$$

$$v_+(h,\lambda)=\frac{2\rho_c(h_c-h)}{\rho_c+3p_c}+\mathcal{O}((h_c-h)^2), \qquad (A11)$$

for $h_t<h<h_c(\lambda)$. The outer envelopes of these stars are composed of material from the lower density phase. In the stars of interest here—those with only very small cores of high density material—the structure of the outer envelope is nearly identical to the structure of a star composed entirely of low density phase material. Thus the inner region of this outer envelope may be approximated as

$$u_-(h,\lambda)=\frac{3(h_c-h)}{2\pi(\rho_-+3p_c)}+\delta u(h)(h_c-h_t)+\mathcal{O}((h_c-h)^2), \qquad (A12)$$

$$v_-(h,\lambda)=\frac{2\rho_-(h_c-h)}{\rho_-+3p_c}+\delta v(h)(h_c-h_t)+\mathcal{O}((h_c-h)^2), \qquad (A13)$$

where δu and δv are solutions to the linearized structure equations. Quite generally, these linearized structure equations have the form

$$\frac{d\,\delta u}{dh}=\frac{\partial U}{\partial u}\delta u+\frac{\partial U}{\partial v}\delta v, \qquad (A14)$$

$$\frac{d\,\delta v}{dh}=\frac{\partial V}{\partial u}\delta u+\frac{\partial V}{\partial v}\delta v, \qquad (A15)$$

where $U(u,v,h)$ and $V(u,v,h)$ are the functions defined in Eqs. (A2),(A3). For our purposes here it is sufficient to evaluate these functions using the first order terms in the expansions for u and v given in Eqs. (A7),(A8). In this case the functions of interest to us have the forms:

$$\frac{\partial V}{\partial u}=2\pi(\rho_-+p_c)\frac{\partial U}{\partial u}=-\frac{2\pi(\rho_-+p_c)\rho_-}{\rho_-+3p_c}\frac{1}{h_c-h}+\mathcal{O}((h_c-h)^0), \qquad (A16)$$

$$\frac{\partial V}{\partial v}=2\pi(\rho_-+p_c)\frac{\partial U}{\partial v}=\frac{3(\rho_-+p_c)}{2(\rho_-+3p_c)}\frac{1}{h_c-h}+\mathcal{O}((h_c-h)^0). \qquad (A17)$$

The resulting form of Eqs. (A14),(A15) can be integrated analytically. The general solution is

$$\delta u(h)=A+\frac{B}{(h_c-h)^{1/2}}, \qquad (A18)$$

$$\delta v(h)=\frac{4\pi\rho_-}{3}A+\frac{2\pi(\rho_-+p_c)}{(h_c-h)^{1/2}}B, \qquad (A19)$$

where A and B are arbitrary constants. The values of these constants are determined by demanding continuity at $h=h_t$ of the functions describing the inner core, u_+ and v_+, with the functions describing the outer envelope, u_- and v_-. These continuity conditions are satisfied for the following values of A and B:

$$A=-\frac{9(\rho_c-\rho_-)}{2\pi(\rho_c+3p_c)(\rho_-+3p_c)}, \qquad (A20)$$

$$B=\frac{3(\rho_c-\rho_-)(h_c-h_t)^{1/2}}{\pi(\rho_c+3p_c)(\rho_-+3p_c)}. \qquad (A21)$$

The complete expressions then for the inner portions of the structure functions in the low-density envelope of the star are

$$u_-(h,\lambda)=\frac{3(h_c-h)}{2\pi(\rho_-+3p_c)}-\frac{9(\rho_c-\rho_-)(h_c-h_t)}{2\pi(\rho_c+3p_c)(\rho_-+3p_c)}\left[1-\frac{2}{3}\left(\frac{h_c-h_t}{h_c-h}\right)^{1/2}\right]+\mathcal{O}((h_c-h)^2), \qquad (A22)$$

$$v_-(h,\lambda)=\frac{2\rho_-(h_c-h)}{\rho_-+3p_c}-\frac{6(\rho_c-\rho_-)(h_c-h_t)}{(\rho_c+3p_c)(\rho_-+3p_c)}\times\left[\rho_--(\rho_-+p_c)\left(\frac{h_c-h_t}{h_c-h}\right)^{1/2}\right]+\mathcal{O}((h_c-h)^2). \qquad (A23)$$

The match of u_- to the inner-core function u_+ is C^1 at $h=h_t$, as required by Eq. (A2). The match of v_- to v_+ is C^0 at $h=h_t$. The slope of v_- differs from that of v_+ at $h=h_t$ by the amount required by Eq. (A3).

Now consider the region in these models where $|h_c-h_t|\ll|h_c-h|\ll 1$. The approximate expressions given in Eqs. (A22),(A23) are valid for these models in this region. In addition the terms in these expressions proportional to $(h_c-h_t)^{1/2}$ can be neglected in this region as well. Next, evaluate the derivatives $\delta u_\downarrow=\partial u/\partial\lambda$, etc. in this region for these models having a very small core of high-density phase material:

$$\delta u_\downarrow(h,\lambda)=\frac{\partial u_-}{\partial\lambda}=\frac{3[-2\rho_c+3(\rho_-+p_c)]}{2\pi(\rho_-+3p_c)(\rho_c+3p_c)}\frac{dh_c}{d\lambda}+\mathcal{O}(h_c-h)+\mathcal{O}\left(\frac{h_c-h_t}{h_c-h}\right)^{1/2}, \qquad (A24)$$

$$\delta v_\downarrow(h,\lambda)=\frac{\partial v_-}{\partial\lambda}=\frac{2\rho_-[-2\rho_c+3(\rho_-+p_c)]}{(\rho_-+3p_c)(\rho_c+3p_c)}\frac{dh_c}{d\lambda}+\mathcal{O}(h_c-h)+\mathcal{O}\left(\frac{h_c-h_t}{h_c-h}\right)^{1/2}. \qquad (A25)$$

These expressions can now be compared with those that pertain to stars having no material at all in the high-density phase. Thus, define $\delta u_\uparrow=\partial u/\partial\lambda$, etc. for the models with no high density phase material at all using the expansions in Eqs. (A7),(A8) that are valid throughout the inner regions of these models:

$$\delta u_{\uparrow}(h,\lambda)=\frac{\partial u}{\partial\lambda}=\frac{3}{2\pi(\rho_{-}+3p_c)}\frac{dh_c}{d\lambda}+\mathcal{O}(h_c-h), \tag{A26}$$

$$\delta v_{\uparrow}(h,\lambda)=\frac{\partial v}{\partial\lambda}=\frac{2\rho_{-}}{\rho_{-}+3p_c}\frac{dh_c}{d\lambda}+\mathcal{O}(h_c-h). \tag{A27}$$

These expressions illustrate that the derivatives δu and δv are not continuous functions of λ near the critical model with $\lambda=\lambda_t$. However, the discontinuity is of a very special type. These expressions for $(\delta u_{\downarrow},\delta v_{\downarrow})$, are related to those for $(\delta u_{\uparrow},\delta v_{\uparrow})$ in the following simple way:

$$\begin{pmatrix}\delta u_{\downarrow}\\ \delta v_{\downarrow}\end{pmatrix}=2\frac{\Delta_c-\Delta}{2\Delta_c+\Delta}\begin{pmatrix}\delta u_{\uparrow}\\ \delta v_{\uparrow}\end{pmatrix}, \tag{A28}$$

where Δ and Δ_c are defined in Eqs. (4) and (5) respectively. Equation (A28) is exact in the limit that $\lambda\to\lambda_t$ from above and below respectively, and when the functions are evaluated at the center of the star $h=h_t$. The derivatives δu and δv satisfy the linear differential Eqs. (A14),(A15). Further, the continuity of u and v as functions of λ at the point $\lambda=\lambda_t$, implies that both $(\delta u_{\uparrow},\delta v_{\uparrow})$ and $(\delta u_{\downarrow},\delta v_{\downarrow})$ satisfy the *same* differential equation at $\lambda=\lambda_t$. (This fact is the reason that it was necessary to establish the continuity of u and v in some detail above.) Thus, it follows that the functions $(\delta u_{\uparrow},\delta v_{\uparrow})$ are proportional to $(\delta u_{\downarrow},\delta v_{\downarrow})$ throughout the critical model with $\lambda=\lambda_t$ since they are proportional to one another in a neighborhood of $h=h_t$.

At the surface of the star, $h=0$, the derivatives δu and δv are related to the total mass and radius of the star as a consequence of Eqs. (A5),(A6). In particular these functions must satisfy

$$\delta u(0,\lambda)=2R(\lambda)\frac{dR(\lambda)}{d\lambda}, \tag{A29}$$

$$\delta v(0,\lambda)=\frac{1}{R(\lambda)}\frac{dM(\lambda)}{d\lambda}-\frac{M(\lambda)}{R^2(\lambda)}\frac{dR(\lambda)}{d\lambda}. \tag{A30}$$

The functions $(\delta u_{\downarrow},\delta v_{\downarrow})$ (evaluated for models just above the critical one) are proportional to $(\delta u_{\uparrow},\delta v_{\uparrow})$ (evaluated for models just below the critical one) throughout the star. Thus, the surface values of these functions are proportional as well. This implies in particular that the tangent vectors to the mass radius curve evaluated above and below the critical model are related by:

$$\begin{pmatrix}\dfrac{dM_{\downarrow}}{d\lambda}\\[2mm] \dfrac{dR_{\downarrow}}{d\lambda}\end{pmatrix}=2\frac{\Delta_c-\Delta}{2\Delta_c+\Delta}\begin{pmatrix}\dfrac{dM_{\uparrow}}{d\lambda}\\[2mm] \dfrac{dR_{\uparrow}}{d\lambda}\end{pmatrix}. \tag{A31}$$

This expression has several interesting consequences. First, it shows that the tangent vector to the mass-radius curve is discontinuous at the critical model whenever one parameterizes the curve in a way that makes $h_c(\lambda)$ smooth. Second, this expression shows that the mass-radius curve in fact reverses direction at the critical model if the phase transition is sufficiently severe so that $\Delta>\Delta_c$. Third and finally, Eq. (A31) implies that the slope of the mass-radius curve, dM/dR, is continuous even at the critical model:

$$\frac{dM}{dR}=\frac{dM_{\downarrow}}{d\lambda}\left(\frac{dR_{\downarrow}}{d\lambda}\right)^{-1}=\frac{dM_{\uparrow}}{d\lambda}\left(\frac{dR_{\uparrow}}{d\lambda}\right)^{-1}. \tag{A32}$$

Continuity of the slope pertains even if the curve has a cusp and reverses direction at the critical model, unless $\Delta=\Delta_c$. In this special case Eq. (A31) merely implies that λ [chosen so that $h_c(\lambda)$ is smooth] is not a good affine parameter for the mass-radius curve. A higher order analysis is needed to understand the differentiability of the curve in this special case.

[1] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).

[2] A. D. Rendall and B. G. Schmidt, Class. Quantum Grav. **8**, 985 (1991).

[3] Necessary conditions on the equation of state to guarantee that stellar models have finite mass and radius are discussed by L. Lindblom and A. K. M. Masood-ul-Alam, in *Directions in General Relativity II: Papers in Honor of Dieter Brill*, edited by B. L. Hu and T. A. Jacobson (Cambridge University Press, Cambridge, England, 1993), p. 172.

[4] L. Lindblom, Astrophys. J. **398**, 569 (1992).

[5] A. B. Migdal, Zh. Eksp. Teor. Fiz. **61**, 2209 (1971); R. F. Sawyer and D. J. Scalapino, Phys. Rev. D **7**, 953 (1972).

[6] E. Witten, Phys. Rev. D **30**, 272 (1984); E. Farhi and R. L. Jaffe, *ibid.* **30**, 2379 (1984).

[7] W. H. Ramsey, Mon. Not. R. Astron. Soc. **110**, 325 (1950).

[8] M. J. Lighthill, Mon. Not. R. Astron. Soc. **110**, 339 (1950).

[9] Z. F. Seidov, Sov. Astron. **15**, 347 (1971).

[10] K. Schwarzschild, Sitzungsber. K. Preuss. Akad. Wiss., 424 (1916).

[11] See for example E. A. Coddington, and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955), Sec. 1.7.