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The Relaxation Effect in Dissipative Relativistic Fluid Theories

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The dynamics of the fluid fields in a large class of causal dissipative fluid theories is studied. It is shown that the physical fluid states in these theories must relax (on a time scale that is characteristic of the microscopic particle interactions) to ones that are essentially indistinguishable from the simple relativistic Navier–Stokes descriptions of these states. Thus, for example, in the relaxed form of a physical fluid state the stress energy tensor is in effect indistinguishable from a perfect fluid stress tensor plus small dissipative corrections proportional to the shear of the fluid velocity, the gradient of the temperature, etc.

I. INTRODUCTION

A simple mathematical model provides an elegant and accurate description of the common materials called fluids. The effects of internal dissipation in these materials—viscosity and thermal conductivity—are also well modeled by a simple generalization of the basic theory called the Navier–Stokes equations. Unfortunately, the most straightforward approaches to constructing relativistic generalizations of the Navier–Stokes equations result in rather pathological theories (Eckart [1], Landau and Lifschitz [2]). These theories are non-causal, unstable, and without a well posed initial value formulation (see for example Hiscock and Lindblom [3]). Less straightforward approaches have succeeded more recently in producing a class of causal dissipative fluid theories (e.g., Israel and Stewart [4], Carter [5], Liu, Müller, and Ruggeri [6], Geroch and Lindblom [7], etc.). These theories have eliminated the pathologies of the straightforward relativistic generalizations of the Navier–Stokes equations, but they do so at the expense of increasing significantly the number of dynamical fields needed to describe the fluid. Unfortunately the additional dynamical degrees of freedom associated with these extra fields have never been directly observed in real fluids. This is probably why these new theories have not found widespread acceptance.

In this paper the dynamics associated with these additional fluid fields are analyzed in a very large class of causal dissipative fluid theories. It is shown that the physical fluid states relax (on a time scale characteristic of the inter-particle

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interactions) to ones that are also well described by the simple relativistic Navier–Stokes theory. For example, the stress-energy tensor in such a relaxed fluid state is well described by the usual perfect fluid stress-energy tensor plus the Navier–Stokes expressions for the dissipative corrections involving the shear of the fluid velocity, the gradient of the temperature, etc. This result suggests that meaningful differences between the causal theories and the non-causal Navier–Stokes theory can not be observed. The complicated dynamical structure of the causal theories is necessary to insure that the fluid evolves in a causal and stable way. But this rich dynamical structure is unobservable, since the physical states of a fluid always evolve in a way that is also well described by the Navier–Stokes expressions for the stress-energy tensor, etc. The arguments which lead to these conclusions are extremely general: they are based on a fully non-linear analysis of the equations and do not assume that the fluid state is close to equilibrium. This analysis generalizes significantly the studies of the analogous relaxation effect in the solutions of the hyperbolic heat equation (see Nagy, Ortiz, and Reula [8]), and the studies of the relationship between the relativistic Navier–Stokes and the causal fluid theories in the near equilibrium fluid states (see Hiscock and Lindblom [9]).

Let us begin by recalling the theory of a perfect fluid: the mathematical description of a fluid having negligible internal dissipation. The state of such a fluid is determined by three fields on spacetime: a future directed unit timelike vector field, u^a, and two scalar fields n and ρ. These fields are assumed to be solutions of the differential equations

$$\nabla_m N^m = 0, \tag{1}$$

$$\nabla_m T^{ma} = 0, \tag{2}$$

where N^a and T^{ab} are given in terms of the fluid fields by

$$N^a = n u^a, \tag{3}$$

$$T^{ab} = (\rho + p)\, u^a u^b + p\, g^{ab}. \tag{4}$$

Here p is a smooth function of n and ρ (the equation of state), that is fixed once and for all for a given type of fluid. The conserved vector N^a is the particle current of the fluid, and thus u^a may be identified as the four-velocity and n as the number density as measured by an observer co-moving with the fluid. The conserved tensor T^{ab} is the stress energy of fluid. Thus from Eq. (4), ρ is identified as the mass-energy density and p as the pressure of the fluid, both as measured by a co-moving observer. These quantities are all directly observable because the particle current N^a and the stress energy T^{ab} are themselves directly observable.

The theory of a perfect fluid, Eqs. (1)–(4), has a number of attractive mathematical properties. One of the most important of these is that Eqs. (1)–(2) form a symmetric-hyperbolic and causal system when suitable restrictions are placed on the equation of state. Let $\xi^\alpha = (n, \rho, u^a)$ denote the dynamical fluid fields. Then Eqs. (1)–(2) are equivalent to

$$M^m{}_{\alpha\beta}\nabla_m \xi^\beta = 0, \tag{5}$$

where

$$M^m{}_{\alpha\beta} = P_\alpha \frac{\partial N^m}{\partial \xi^\beta} + P_{\alpha a}\frac{\partial T^{am}}{\partial \xi^\beta}. \tag{6}$$

The quantities P_α and $P_{\alpha a}$ (functions of ξ^α and the spacetime metric g_{ab}) may be chosen so that these equations are symmetric in the sense that $M^m{}_{\alpha\beta} = M^m{}_{\beta\alpha}$ (see Ruggeri and Strumia [10], Geroch and Lindblom [7], and §III below). When suitable restrictions are placed on the equation of state these equations are also hyperbolic and causal, because $\lambda^m = M^m{}_{\alpha\beta}Z^\alpha Z^\beta$ is past directed timelike for every choice of $Z^\alpha \neq 0$ in these theories.

Next, let us turn to the main subject of this paper: theories of dissipative fluids. Since there is as yet no universally accepted theory for such fluids, a rather broad class of theories has been included in this discussion. The state of the fluid in these theories is determined by two sets of fields, ξ^α and φ^A, each representing some collection of tensor fields (possibly subject to certain algebraic constraints) on spacetime. The ξ^α are to represent, as in the perfect fluid case, the dynamical fluid fields. The φ^A are to represent additional 'dissipation' fields that are needed to complete an acceptable causal fluid theory. It seems reasonable to restrict the dimension of the combined ξ^α and φ^A spaces to be equal to the number of independent observable fields in the theory.[1] In the case of a simple dissipative fluid—the case that is of primary interest here—the particle current N^a and the stress-energy tensor T^{ab} are the independent observable fields. Hence the most appropriate choice for the dimension of these combined spaces is fourteen in this case. For now, however, neither the structures nor the dimensions of these spaces will be restricted.

The fields, ξ^α and φ^A, are assumed to be solutions of the system of equations

$$M^m{}_{\alpha\beta}\nabla_m \xi^\beta + M^m{}_{\alpha A}\nabla_m \varphi^A = 0, \tag{7}$$

$$M^m{}_{AB}\nabla \varphi^B + M^m{}_{\alpha A}\nabla_m \xi^\alpha = -I_{AB}\varphi^B. \tag{8}$$

The quantities $M^m{}_{\alpha\beta}$, $M^m{}_{\alpha A}$, $M^m{}_{AB}$, and I_{AB} are assumed to be smooth functions, fixed once and for all for a given theory, of the fields ξ^α, φ^A, and the spacetime metric g_{ab}. Thus Eqs. (7)–(8) form a first-order system of partial differential equations for the fluid fields ξ^α and φ^A. Three conditions are now imposed on this

[1] This restriction is not required in the analysis presented here however. If the number of fluid fields were taken to be larger than the number of observables then some of the fluid fields would not be observable. The results derived here would still apply, but some of them would change character from experimentally testable predictions to mathematical identities.

system of equations. These conditions are very general and should apply to essentially any theory of fluids (including those describing superfluids, mixtures of different kinds of fluids, etc.).

Condition (i). The first condition is on the M's that appear on the left sides of Eqs. (7)–(8). Assume that the M's are symmetric, $M^m{}_{\alpha\beta} = M^m{}_{(\alpha\beta)}$ and $M^m{}_{AB} = M^m{}_{(AB)}$; and assume that every vector λ^m given by

$$\lambda^m = M^m{}_{\alpha\beta} Z^\alpha Z^\beta + 2M^m{}_{\alpha A} Z^\alpha Z^A + M^m{}_{AB} Z^A Z^B, \tag{9}$$

for some $(Z^\alpha, Z^A) \neq 0$ is past-directed timelike. This is just the condition needed to insure that the system (7)–(8) is symmetric, hyperbolic, and causal (see for example Geroch and Lindblom [11] or Müller and Ruggeri [12]).

Condition (ii). The second condition involves the tensor I_{AB} that appears on the right side of Eq. (8). Assume that $I_{AB} Z^A Z^B > 0$ for every $Z^A \neq 0$.[2] This condition is adopted to insure, as will be seen more clearly below, that this fluid theory is strictly dissipative. It is precisely analogous to requiring that the viscosity coefficients and the thermal conductivity not vanish in the Navier–Stokes equation.

Condition (iii). The third condition concerns the conservation laws. Assume that there exist specific smooth functions N^a and T^{ab} of the fields ξ^α, φ^A, and g_{ab}, such that Eq. (7) implies the conservation laws, Eqs. (1) and (2). This condition merely insures that the theory possesses a conserved stress energy tensor and particle current.

The main result of this paper is derived in Sect. II. It is shown that physical states of the fluid relax—on a time scale τ that is characteristic of the inter-particle interactions—to ones in which the dissipation field is determined in effect by the dynamical field ξ^α and its derivative. In particular, a bound is derived for the quantity $\Delta\varphi^A$, defined by

$$\varphi^A = -[(I^{-1})^{AB} M^m{}_{\alpha B} \nabla_m \xi^\alpha]_{\varphi^C=0} + \Delta\varphi^A, \tag{10}$$

in the physical fluid states of any fluid theory which satisfies Conditions (i)–(iii). This bound on $\Delta\varphi^A$ is smaller by the factor $b^{5/2}(\tau v/L)^2$ than it is expected to be. The constant v is a characteristic sound speed, L is a macroscopic length scale that characterizes the particular state of the fluid, and the constant b is a dimensionless bound on $M^m{}_{AB}$, $M^m{}_{\alpha A}$, $(I^{-1})^{AB}$ and their derivatives (which will be defined precisely in §II). The constant b is expected to be of order unity for 'reasonable'

[2] The somewhat more general function $-I_A(\xi^\alpha, \varphi^B, g_{ab})$ could have been adopted for the right side of Eq. (8) if it satisfied a few additional constraints. This more general form is equivalent to that given in Eq. (8) if and only if I_A satisfies the following three conditions: (a) $I_A = 0$ when $\varphi^B = 0$, (b) $I_A \varphi^A > 0$ when $\varphi^B \neq 0$, and (c) $\partial I_A / \partial \varphi^B$ is not degenerate when evaluated at $\varphi^C = 0$.

fluid theories. Thus the factor $b^{5/2}(\tau v/L)^2$ should be extremely small for real fluids. For example in water $(\tau v/L)^2 \approx 10^{-12}$ for fluid states with $L \approx 0.1$cm. Since the dissipation field in a relaxed fluid state is determined in effect by the dynamical field ξ^α and its derivative, then so are all other functions of the fluid fields. In particular, the particle current N^a and stress energy tensor T^{ab} are given by

$$N^a = \left[N^a - \frac{\partial N^a}{\partial \varphi^A} (I^{-1})^{AB} M^m{}_{\alpha B} \nabla_m \xi^\alpha \right]_{\varphi^C = 0} + \Delta N^a, \tag{11}$$

$$T^{ab} = \left[T^{ab} - \frac{\partial T^{ab}}{\partial \varphi^A} (I^{-1})^{AB} M^m{}_{\alpha B} \nabla_m \xi^\alpha \right]_{\varphi^C = 0} + \Delta T^{ab}. \tag{12}$$

It is shown that the quantities ΔN^a and ΔT^{ab} are also smaller than their expected values by the factor $b^3(\tau v/L)^2$. These results apply to any dissipative fluid theory that satisfies Conditions (i)–(iii) above, and to any physical fluid state (i.e., as defined more precisely below, a state in which the spatial and temporal variations of the fluid fields are larger than the microscopic scales). This result explains why the independent dynamics of the dissipation field φ^A is never observed: its value is determined in effect by the dynamical field ξ^α and its derivative, via Eq. (10), on any time scale over which a macroscopic measurement of the system can be made. Although measurements could in principle be carried out on fluid systems over very short time and distance scales, it is not required or even expected that such measurements will be modeled in detail by any macroscopic fluid theory.

The results of §II show that a dissipative fluid quickly relaxes to a state in which the particle current and stress-energy tensor are determined (in effect) by the dynamical fluid field ξ^α and its derivative $\nabla_m \xi^\alpha$. Such relationships are quite familiar to us; for these are precisely the forms that the expressions for these quantities take in the Navier–Stokes theory. Recall that in the relativistic Navier–Stokes theory (as formulated by Eckart [1]) the particle current and stress-energy tensor are given in terms of the fields $\xi^\alpha = (n, \rho, u^a)$ by

$$N^a = n u^a, \tag{13}$$

$$T^{ab} = (\rho + p)\, u^a u^b + p g^{ab} + \tau^{ab} + \tau q^{ab} + 2u^{(a} q^{b)}, \tag{14}$$

where

$$\tau^{ab} = 2\eta_1 [q^{am} q^{bc} - \tfrac{1}{3} q^{ab} q^{cm}]\, \nabla_{(m} u_{c)}, \tag{15}$$

$$\tau = \eta_2 \nabla_m u^m, \tag{16}$$

$$q^a = -\kappa (q^{am} \nabla_m T + T u^m \nabla_m u^a). \tag{17}$$

The quantities η_1, η_2, and κ (positive functions of n and ρ) are the viscosities and thermal conductivity respectively; and, the quantity T (a function of n and ρ) is the thermodynamic temperature which satisfies the first law of thermodynamics,

$$d\rho = nT\, ds + \frac{\rho + p}{n}\, dn. \tag{18}$$

In this theory the conservation laws, Eqs. (1)–(2), and Eqs. (15)–(17) are the differential equations that determine the field ξ^α. If the dissipation fields are defined as, $\varphi^A = (\tau,\ q^a,\ \tau^{ab})$, then this theory is of the same general form as those being considered here. The conservation laws are precisely in the form of Eq. (7), while Eqs. (15)–(17) have the form of Eq. (8). This theory fails to be an acceptable theory because $M^m{}_{AB} = 0$ and thus it fails to satisfy Condition (i). Note that for this relativistic Navier–Stokes theory, the quantities $\Delta\varphi^A$, ΔN^a, and ΔT^{ab} as defined in Eqs. (10)–(12) vanish identically.

The vanishing (effectively) of ΔN^a and ΔT^{ab} for the general dissipative fluid theories considered here implies that the particle current and stress-energy tensor depend (in effect) only on ξ^α and its derivative $\nabla_m \xi^\alpha$ in any physical fluid state. In the relativistic Navier–Stokes theory, however, only certain components of $\nabla_m \xi^\alpha$ appear in these expressions. For example, in the Navier–Stokes theory $\nabla_m T$ appears in these expressions but not the gradient of any other thermodynamic scalar. It is natural to ask then, what class of fluid theories have the property that their fluid states always relax to ones in which the particle current and stress-energy tensor are in effect indistinguishable from those of the relativistic Navier–Stokes theory? Or in particular, in which theories do N^a and T^{ab} depend on ξ^α and $\nabla_m \xi^\alpha$ in precisely the same way as in the Navier–Stokes theory? The following two additional conditions are necessary and sufficient to guarantee that a theory will be indistinguishable from Navier–Stokes in this way:

Condition (iv). The fourth condition concerns the space of dynamical fields ξ^α. Assume that Eq. (7) is precisely equivalent to the conservation laws, Eqs. (1)–(2). This implies that the space of the ξ^α consists of one vector and one scalar field which may, without loss of generality (as shown in Sect. III), be taken to be $\xi^\alpha = (n,\ \rho,\ u^a)$, where $nu^a = N^a$ (with $u^a u_a = -1$) and $\rho = u_a u_b T^{ab}$.

Condition (v). The fifth condition concerns the tensor $M^m{}_{\alpha A}$ that appears in Eqs. (7)–(8). Assume that $M^m{}_{\alpha A} \nabla_m \xi^\alpha$ depends on $\nabla_m n$ and $\nabla_m \rho$ only in the combination $\nabla_m T = (\partial T/\partial n)_\rho\, \nabla_m n + (\partial T/\partial \rho)_n\, \nabla_m \rho$ in the $\varphi^A = 0$ states of the fluid. This condition is required to insure that heat flow is generated by the gradient of the thermodynamic temperature T and not the gradient of some other thermodynamic scalar. This condition is equivalent to the requirement that the equilibrium states of the fluid be 'isothermal.'

The theories that satisfy these two additional conditions are the natural causal generalizations of the Navier–Stokes theory: the causal theories of a simple dissipative fluid.

In Section III the expressions for the relaxed forms of the particle current and stress-energy tensor are evaluated for the theories of simple dissipative fluids, i.e., those satisfying Conditions (i)–(v). With the convenient choice of dynamical fields, $\xi^\alpha = (n,\ \rho,\ u^a)$, Eqs. (11)–(12) reduce to

$$N^a = nu^a, \tag{19}$$

$$T^{ab} = (\rho + p)\, u^a u^b + p g^{ab} + 2\eta_1 [q^{am} q^{bc} - \tfrac{1}{3} q^{ab} q^{cm}]\, \nabla_{(m} u_{c)} + \eta_2 \nabla_m u^m q^{ab}$$
$$- 2\kappa u^{(a} [q^{b)m} \nabla_m T + T u^{|m|} \nabla_m u^{b)}] + \Delta T^{ab}, \tag{20}$$

for suitably chosen functions (of n and ρ) p, η_1, η_2, and κ. Since ΔT^{ab} is extremely small in the physical fluid states of these theories, this shows that the particle current and stress energy tensor are (in effect) indistinguishable from those of the relativistic Navier–Stokes theory.[3]

II. The Relaxation Effect

The key result in this paper is that the physical states of the fluid relax to ones in which the dissipation field φ^A is determined (in effect) by the dynamical fluid field ξ^α and its derivative $\nabla_m \xi^\alpha$. That some form of relaxation should occur in the solutions of Eqs. (7)–(8) can be seen fairly easily. Consider the quantity $I_{AB} \varphi^B + M^m{}_{\alpha A} \nabla_m \xi^\alpha$. If this quantity does not vanish at some point, then the first term in Eq. (8) causes φ^A to evolve in the direction that tends to make it vanish. The rate at which this evolution occurs is determined by the time scale that is encoded in the tensor I_{AB}. For fluids this time scale will be determined by the viscosity and thermal conductivity coefficients contained in I_{AB}, and therefore will be characteristic of the inter-particle interaction times for the fluid. The demonstration that the quantity $\Delta\varphi^A$ defined in Eq. (10) is small will be done in two steps. First, it is shown that a related quantity $\varphi^A + \Lambda^A$, defined below, is small using a fairly simple and straightforward argument. Second, a slightly more elaborate argument shows that quantities $\Delta\varphi^A$, ΔN^a, and ΔT^{ab} of Eqs. (10)–(12) are also small.

Begin by obtaining the following equation for $\varphi^A + \Lambda^A$ from Eq. (8):

$$\nabla_m [M^m{}_{AB} (\varphi^A + \Lambda^A)(\varphi^B + \Lambda^B)]$$
$$= -2 I_{AB} (\varphi^A + \Lambda^A)(\varphi^B + \Lambda^B) + (\varphi^A + \Lambda^A)\, \mathscr{A}_A, \tag{21}$$

where Λ^A and $\mathscr{A}_A$ are defined by

$$\Lambda^A = (I^{-1})^{AB}\, (M^m{}_{\alpha B} \nabla_m \xi^\alpha - \tfrac{1}{2} \varphi^C \nabla_m M^m{}_{BC}), \tag{22}$$

$$\mathscr{A}_A = \Lambda^B \nabla_m M^m{}_{AB} + 2 M^m{}_{AB} \nabla_m \Lambda^B. \tag{23}$$

[3] Note that ΔN^m and $u_a u_b \Delta T^{ab}$ vanish identically as a consequence of the particular choice of ξ^α made here. Had a different choice been made, such as the one traditionally used in the Landau–Lifschitz theory [2], then other components of these quantities would have vanished identically instead.

Next consider $S(0)$, a bounded open subset of some Cauchy surface. Use the timelike vector field whose divergence appears on the left side of Eq. (21) to define a map between the points on successive Cauchy surfaces. Let $S(\kappa)$ denote the image of $S(0)$ under this map into the Cauchy surface labeled by the time function κ. Choose this time function κ so that it satisfies

$$I_{AB}Z^A Z^B \geqslant -Z^A Z^B M^m{}_{AB}\nabla_m \kappa \tag{24}$$

(for every Z^A), in the spacetime region $(0, \kappa_o)\times S(\kappa_o)$. Next, define the following $\mathscr{L}^2$ norm of $\varphi^A + \Lambda^A$,

$$\alpha^2(\kappa) = \int_{S(\kappa)} \mathscr{G}_{AB}(\varphi^A + \Lambda^A)(\varphi^B + \Lambda^B)\, dV, \tag{25}$$

where $\mathscr{G}_{AB} = n_m M^m{}_{AB}$, and n_m is the future directed unit vector proportional to $\nabla_m \kappa$. The evolution of this norm is determined by integrating Eq. (21) over the spacetime region consisting of points in $S(\kappa)$ that lie between two nearby $\kappa =$ constant slices. The integral along the timelike boundary of this region vanishes because of the choice of $S(\kappa)$. The integral of the terms on the right in Eq. (21) may be transformed using Eq. (24) for the first term and the Schwartz inequality for the second. Taking the limit as the difference between κ on these two slices goes to zero, the following differential inequality is obtained for α,

$$\frac{d\alpha}{d\kappa} \leqslant -\alpha + \frac{1}{2}\,\|\mathscr{A}\|, \tag{26}$$

where

$$\|\mathscr{A}\|\,(\kappa) = \left[\int_{S(\kappa)} \frac{\mathscr{G}^{AB}\mathscr{A}_A\mathscr{A}_B\, dV}{-\nabla_m\kappa\nabla^m\kappa}\right]^{1/2}, \tag{27}$$

and $\mathscr{G}^{AB}$ denotes the inverse of $\mathscr{G}_{AB}$. This ordinary differential inequality, Eq. (26), can be integrated to obtain the following bound on α,

$$\alpha(\kappa_o) \leqslant \alpha(0)e^{-\kappa_o} + \tfrac{1}{2}\int_0^{\kappa_o} e^{-(\kappa_o-\kappa)}\,\|\mathscr{A}\|\,(\kappa)\, d\kappa. \tag{28}$$

To proceed further a bound must be obtained for the quantity $\|\mathscr{A}\|$ that appears in Eq. (28). To this end a norm is introduced on tensors: The positive definite $\mathscr{G}_{\alpha\beta} = n_m M^m{}_{\alpha\beta}$ and its inverse $\mathscr{G}^{\alpha\beta}$ are used for indices associated with the dynamical field, ξ^α; and the positive definite definite $\mathscr{G}_{AB} = n_m M^m{}_{AB}$ and its inverse $\mathscr{G}^{AB}$ are used for indices associated with the dissipation field φ^A. For spacetime indices the positive definite metric $\mathscr{G}_{ab} = n_a n_b + v^{-2}(g_{ab} + n_a n_b)$ and its inverse $\hat{\mathscr{G}}^{ab} = n^a n^b + v^2(g^{ab} + n^a n^b)$ are used. The constant v, with $0 < v < 1$, is chosen to be an upper bond on the speed (relative to n_a) of signal propagation, i.e., a number such that $(n_a\lambda^a)^2 \geqslant v^{-2}(g_{ab} + n_a n_b)\,\lambda^a\lambda^b$ for every λ^a given in Eq. (9). As examples of this norm, the integrand in Eq. (25) can be written as $|\varphi^A + \Lambda^A|^2 = \mathcal{G}_{AB}(\varphi_A + \Lambda^A)(\varphi^B + \Lambda^B)$, while $|\mathcal{A}_A|^2 = \mathcal{G}^{AB}\mathcal{A}_A\mathcal{A}_B$ and $|\nabla_m\kappa|^2 = \hat{\mathcal{G}}^{ab}\nabla_a\kappa\nabla_b\kappa = -\nabla_a\kappa\nabla^a\kappa$. Note that $|M^m{}_{AB}| \leqslant 4d$ where d is the dimension of the space of dissipation fields. Since d will be some relatively small integer, say $d=9$, the norm of the M's will be of order unity in these fluid theories.

In the fluid theories considered here the quantities $M^m{}_{AB}$, $M^m{}_{\alpha A}$ and I_{AB} are assumed to be smooth functions of the fluid fields. Therefore, these quantities and their derivatives with respect to the fluid fields are bounded. It is convenient to quantify these bounds in terms of three constants b, τ and ζ. Consider fluid fields ξ^α and φ^A that are bounded by the constant ζ:

$$|\varphi^A| \leqslant \zeta, \qquad |\xi^\alpha| \leqslant \zeta. \tag{29}$$

Next define the dimensionless constant b to be a bound on the M's and their derivatives. In particular assume that

$$\begin{aligned} &|M^m{}_{AB}| \leqslant b, && \left|\frac{\partial M^m{}_{AB}}{\partial \xi^\alpha}\right| \leqslant \frac{b}{\zeta}, && \left|\frac{\partial M^m{}_{AB}}{\partial \varphi^C}\right| \leqslant \frac{b}{\zeta}, \\ &\left|\frac{\partial^2 M^m{}_{AB}}{\partial \xi^\alpha \partial \xi^\beta}\right| \leqslant \frac{b}{\zeta^2}, && \left|\frac{\partial^2 M^m{}_{AB}}{\partial \xi^\alpha \partial \varphi^C}\right| \leqslant \frac{b}{\zeta^2}, && \left|\frac{\partial^2 M^m{}_{AB}}{\partial \varphi^C \partial \varphi^D}\right| \leqslant \frac{b}{\zeta^2}, \end{aligned} \tag{30}$$

and

$$|M^m{}_{\alpha A}| \leqslant b, \qquad \left|\frac{\partial M^m{}_{\alpha A}}{\partial \xi^\beta}\right| \leqslant \frac{b}{\zeta}, \qquad \left|\frac{\partial M^m{}_{\alpha A}}{\partial \varphi^B}\right| \leqslant \frac{b}{\zeta}. \tag{31}$$

Finally, the constant τ is defined as a bound on $(I^{-1})^{AB}$ and its derivatives

$$|(I^{-1})^{AB}| \leqslant b\tau, \qquad \left|\frac{\partial (I^{-1})^{AB}}{\partial \xi^\alpha}\right| \leqslant \frac{b\tau}{\zeta}, \qquad \left|\frac{\partial (I^{-1})^{AB}}{\partial \varphi^C}\right| \leqslant \frac{b\tau}{\zeta}, \tag{32}$$

for b given above. The constant τ that appears in these bounds is the characteristic time scale on which the dissipative term I_{AB} influences the evolution of the fluid in Eq. (8). This constant also fixes the relationship between physical time and the time function κ because of Eq. (24). The time function κ can be chosen so that

$$|\nabla_m\kappa| \geqslant \frac{1}{\tau}. \tag{33}$$

This κ in effect measures time in units of τ.

To proceed further bounds must now be placed on the spatial derivatives of the fluid fields. Assume that there exists a constant L such that

$$|\nabla_m\xi^\alpha| \leqslant \frac{v\zeta}{L}, \quad |\nabla_m\varphi^A| \leqslant \frac{v\zeta}{L}, \quad |\nabla_m\nabla_n\xi^\alpha| \leqslant \frac{v^2\zeta}{L^2}, \quad |\nabla_m\nabla_n\varphi^A| \leqslant \frac{v^2\zeta}{L^2}. \tag{34}$$

These inequalities restrict the solutions to the fluid equations[4] to those which do not vary appreciably on length scales shorter than L and on time scales shorter than L/v. These inequalities select, therefore, the set of solutions that represent real physical fluid states. Fluid states in which rapid variations of the fluid fields occur on time and length scales smaller than the microscopic particle interaction scales probably can not be adequately modeled by any macroscopic fluid theory. Thus, solutions to the fluid equations having these properties are not considered physical. The inequalities in Eq. (34) therefor select out the physical solutions of the fluid equations when L is larger than the microscopic interaction length scale. For these solutions the quantity $\|\mathscr{A}\|$ can be bounded by using Eqs. (30)–(34) in Eq. (27):

$$\|\mathscr{A}\|\,(\kappa) \leqslant 26b^3\zeta\left(\frac{\tau v}{L}\right)^2 V^{1/2}(\kappa), \tag{35}$$

where $V(\kappa)$ is the volume of the region $S(\kappa)$:

$$V(\kappa) = \int_{S(\kappa)} dV. \tag{36}$$

Including this bound into Eq. (28) the following bound is then obtained for α:

$$\alpha(\kappa_o) \leqslant \alpha(0)e^{-\kappa_o} + 13b^3\zeta\left(\frac{\tau v}{L}\right)^2 \int_0^{\kappa_o} e^{-(\kappa_o-\kappa)}\, V^{1/2}(\kappa)\, d\kappa. \tag{37}$$

This bound consists of two pieces. The first is simply the initial value of α multiplied by $e^{-\kappa_o}$. This term falls exponentially to zero on the characteristic time scale τ. The second term is a constant multiplied by the time average of the spatial volume over which the norm α is defined. This second term, the asymptotic bound on the norm α, is smaller than the *a priori* expectation of its value, $\zeta(bV)^{1/2}$, by the factor $b^{5/2}(\tau v/L)^2$. This factor will be extremely small, being proportional to the square of the ratio of the characteristic dissipation time scale τ to the characteristic dynamical time scale L/v, as long as the constant b is of order unity and the constant v is comparable to the sound speed in the material. The constant b is a measure of the M's and I and their derivatives with respect to the fluid fields. This constant will be of order unity unless these quantities depend on the fluid fields in a very perverse way (e.g. if the M's dependence on the fields were highly oscillatory). The constant v will be comparable to the sound speed of the material as long the foliation of Cauchy surfaces is chosen so that the fluid motion is not highly supersonic, and as long as the characteristic speeds associated with the dissipation fields are comparable to the usual sound speed. In this case the bound on α derived in Eq. (37) implies that the

[4] It is expected that large numbers of solutions to the fluid equations exist which satisfy these conditions. In particular, it is expected that initial data satisfying these conditions on a Cauchy surface will evolve (for some macroscopic time) as a solution that satisfies these conditions everywhere in the development of these data. There do not exist theorems at present, however, which prove the existence of solutions having these properties.

dissipation field φ^A relaxes in the physical states of these fluid theories in such a way that the quantity $\varphi^A + \Lambda^A$ becomes extremely small.

To complete the argument that the quantities $\Delta\varphi^A$, ΔN^a, and ΔT^{ab} of Eqs. (10)–(12) are small, additional $\mathscr{L}^2$ and $\mathscr{L}^4$ bounds are needed on the dissipation field φ^A. To obtain these bounds the following identities are derived from Eq. (8),

$$\nabla_m[M^m{}_{AB}\varphi^A\varphi^B] = -2I_{AB}\varphi^A\varphi^B + \varphi^A\mathscr{B}_A, \tag{38}$$

$$\nabla_m[M^m{}_{AB}\mathscr{G}_{CD}\varphi^A\varphi^B\varphi^C\varphi^D] = -2I_{AB}\mathscr{G}_{CD}\varphi^A\varphi^B\varphi^C\varphi^D + \varphi^A\varphi^B\varphi^C\mathscr{C}_{ABC}, \tag{39}$$

where $\mathscr{B}_A$ and $\mathscr{C}_{ABC}$ are defined by

$$\mathscr{B}_A = \varphi^B\nabla_m M^m{}_{AB} - 2M^m{}_{\alpha A}\nabla_m\xi^\alpha, \tag{40}$$

$$\mathscr{C}_{ABC} = \mathscr{G}_{AB}\mathscr{B}_C + 2M^m{}_{AB}\mathscr{G}_{CD}\nabla_m\varphi^D + M^m{}_{AB}\varphi^D\nabla_m\mathscr{G}_{CD}. \tag{41}$$

Next an integral norm, analogous to α above, is defined for the field φ^A:

$$\beta^2(\kappa) = \int_{\hat{S}(\kappa)} [K_1^2\,|\varphi^A + \Lambda^A|^2 + K_2^2\,|\varphi^A|^2 + K_3^2\,|\varphi^A|^4]\,dV, \tag{42}$$

where $K_1 > 0$, $K_2 > 0$, and $K_3 > 0$ are constants whose values will be specified later. The time evolution of β is determined in analogy with Eq. (26) by integrating K_1^2 multiplied by Eq. (21), plus K_2^2 multiplied by Eq. (38) plus K_3^2 multiplied by Eq. (39), over the spacetime region consisting of points in $\hat{S}(\kappa)$ that lie between two nearby $\kappa = $ constant slices. The sequence of spatial sections $\hat{S}(\kappa)$ is chosen in this case so that the timelike boundary integral vanishes identically here as well. The integrations on the right side of this equation may be simplified, again in analogy with Eq. (26), by using Eq. (24) and the Schwartz inequality. The result is the following differential inequality on β,

$$\frac{d\beta}{d\kappa} \leqslant -\beta + \frac{1}{2}(K_1\,\|\mathscr{A}\| + K_2\,\|\mathscr{B}\|) + \frac{1}{2}K_3\left[\int_{\hat{S}(\kappa)} \frac{|\mathscr{C}_{ABC}|^2\,|\varphi^D|^2\,dV}{|\nabla_m\kappa|^2}\right]^{1/2}, \tag{43}$$

where $\|\mathscr{A}\|$ is given by Eq. (27) and $\|\mathscr{B}\|$ is

$$\|\mathscr{B}\|\,(\kappa) = \left[\int_{\hat{S}(\kappa)} \frac{|\mathscr{B}_A|^2\,dV}{|\nabla_m\kappa|^2}\right]^{1/2}. \tag{44}$$

The quantities that appear on the right side of Eq. (43) can be bounded if one additional restriction is made on the physical solutions of the fluid theory. Assume that the extrinsic curvature and acceleration of the $\kappa = $ constant surfaces are bounded by

$$|\nabla_a n_b| \leqslant \frac{v}{L}. \tag{45}$$

Using the bounds given in Eqs. (29)–(33), (34), and (45), the following bounds can be obtained for the quantities that appear on the right side of Eq. (43):

$$\|\mathscr{B}\| \leqslant 4b\zeta \frac{\tau v}{L} \hat{V}^{1/2}(\kappa), \tag{46}$$

$$\left[\int_{\hat{S}(\kappa)} \frac{|\mathscr{C}_{ABC}|^2 \, |\varphi^D|^2 \, dV}{|\nabla_m \kappa|^2}\right]^{1/2} \leqslant 9b^2\zeta \frac{\tau v}{L} \frac{\beta}{K_2}. \tag{47}$$

Combining these bounds with Eq. (35), the differential inequality for β can be simplified to the following

$$\frac{d\beta}{d\kappa} \leqslant -\left[1 - \frac{9}{2} b^2\zeta \frac{\tau v}{L} \frac{K_3}{K_2}\right]\beta + \left[13b^2\left(\frac{\tau v}{L}\right)K_1 + 2K_2\right] b\zeta\left(\frac{\tau v}{L}\right)\hat{V}^{1/2}(\kappa). \tag{48}$$

The constants K_1, K_2 and K_3 are now chosen to be

$$K_1 = \frac{1}{13b^2}, \qquad K_2 = \frac{1}{2}\left(\frac{\tau v}{L}\right), \qquad K_3 = \frac{1}{18b^2\zeta}. \tag{49}$$

With these choices Eq. (48) becomes

$$\frac{d\beta}{d\kappa} \leqslant -\frac{\beta}{2} + 2b\zeta\left(\frac{\tau v}{L}\right)^2 \hat{V}^{1/2}(\kappa). \tag{50}$$

Integrating this inequality, the desired bound on the norm β is obtained:

$$\beta(\kappa_o) \leqslant \beta(0)\, e^{-\kappa_o/2} + 4b\zeta\left(\frac{\tau v}{L}\right)^2 \langle \hat{V}^{1/2}\rangle, \tag{51}$$

where $\langle \hat{V}^{1/2}\rangle$ denotes the time average of the spatial volume,

$$\langle \hat{V}^{1/2}\rangle = \tfrac{1}{2}\int_0^{\kappa_o} e^{-(\kappa_o - \kappa)/2}\, \hat{V}^{1/2}(\kappa)\, d\kappa. \tag{52}$$

This bound on β implies an $\mathscr{L}^2$ bound on $\varphi^A + \Lambda^A$, and simultaneously $\mathscr{L}^2$ and $\mathscr{L}^4$ bounds on φ^A. The asymptotic values of these bounds are given by

$$\left[\int_{\hat{S}(\kappa)} |\varphi^A + \Lambda^A|^2 \, dV\right]^{1/2} \leqslant 52b^3\zeta\left(\frac{\tau v}{L}\right)^2 \langle \hat{V}^{1/2}\rangle, \tag{53}$$

$$\left[\int_{\hat{S}(\kappa)} |\varphi^A|^2 \, dV\right]^{1/2} \leqslant 8b\zeta\left(\frac{\tau v}{L}\right) \langle \hat{V}^{1/2}\rangle, \tag{54}$$

$$\left[\int_{\hat{S}(\kappa)} |\varphi^A|^4 \, dV\right]^{1/2} \leqslant 72b^3\zeta^2\left(\frac{\tau v}{L}\right)^2 \langle \hat{V}^{1/2}\rangle. \tag{55}$$

These bounds on φ^A are smaller by the factor $\tau v/L$ than their *a priori* expected values. The bound on $\varphi^A+\Lambda^A$ is even smaller, however, being reduced from its *a priori* expected value by the factor $(\tau v/L)^2$. Thus $\varphi^A+\Lambda^A$ becomes small not simply because φ^A and Λ^A get small individually. Rather, this quantity becomes small because φ^A approaches Λ^A asymptotically. Note that the region $\hat{S}(\kappa)$ over which these norms are computed may be chosen arbitrarily on any particular slice.[5] Also note that $\langle \hat{V}^{1/2}(\kappa)\rangle \approx \hat{V}^{1/2}(\kappa)$ if $\hat{V}(\kappa) \gg (\tau v)^3$. The time average used here is exponentially weighted and hence only those slices within about one microscopic interaction time τ of κ contribute significantly.

The main results of this section are bounds on the quantities $\Delta\varphi^A$, ΔN^a, and ΔT^{ab} defined in Eqs. (10)–(12). These bounds are obtained beginning with the quantity $\Delta\varphi^A$,

$$\Delta\varphi^A = \varphi^A + [(I^{-1})^{AB} M^m{}_{\alpha B}\nabla_m \xi^\alpha]_{\varphi^C=0}. \tag{56}$$

This quantity can be re-written as the sum of $\varphi^A+\Lambda^A$, a quantity whose bound was established above, plus ε^A:

$$\Delta\varphi^A = \varphi^A + \Lambda^A + \varepsilon^A, \tag{57}$$

where ε^A may be written (using the standard expression for the remainder in a Taylor expansion) as

$$\begin{aligned}\varepsilon^A = &\frac{1}{2}(I^{-1})^{AB}\varphi^C\nabla_m M^m{}_{BC}\\ &-\varphi^C\nabla_m\xi^\alpha\int_0^1\left\{\frac{\partial}{\partial\varphi^C}[(I^{-1})^{AB}M^m{}_{\alpha B}](\xi^\beta,\lambda\varphi^D)\right\}d\lambda.\end{aligned} \tag{58}$$

Now, using Eqs. (30)–(32) and (34) it is straightforward to obtain the following bound on ε^A,

$$|\varepsilon^A| \leqslant 3b^2\left(\frac{\tau v}{L}\right)|\varphi^A|. \tag{59}$$

Using the triangle inequality for $\mathscr{L}^2$ norms, the norm of $\Delta\varphi^A$ can be expressed as the sum of the norms for $\varphi^A+\Lambda^A$, from (53), and the norm of ε^A, using (54) and (59):

$$\left[\int_{\hat{S}(\kappa)}|\Delta\varphi^A|^2\,dV\right]^{1/2} \leqslant 38b^3\zeta\left(\frac{\tau v}{L}\right)^2\langle\hat{V}^{1/2}\rangle. \tag{60}$$

Thus, the norm of $\Delta\varphi^A$ is smaller than its *a priori* expected value by the factor $b^{5/2}(\tau v/L)^2$.

[5] The regions $\hat{S}(\kappa)$ on the other slices in the foliation are then fixed, however, in order to eliminate the spatial boundary terms from the integration.

Turn next to the quantity ΔT^{ab},

$$\Delta T^{ab} = T^{ab} - \left[T^{ab} - \frac{\partial T^{ab}}{\partial \varphi^A} (I^{-1})^{AB} M^m{}_{\alpha B} \nabla_m \xi^\alpha \right]_{\varphi^C=0}. \tag{61}$$

This quantity may be re-written (using Eq. [56] and again the standard expression for the remainder in a Taylor expansion) as

$$\Delta T^{ab} = \Delta\varphi^A \left[\frac{\partial T^{ab}}{\partial \varphi^A} \right]_{\varphi^C=0} + \varphi^A \varphi^B \int_0^1 \left\{ (1-\lambda) \frac{\partial^2 T^{ab}}{\partial \varphi^A \partial \varphi^B} (\xi^\alpha, \lambda\varphi^C) \right\} d\lambda. \tag{62}$$

The norm of this quantity can easily be bounded by

$$|\Delta T^{ab}| \leqslant \frac{\varepsilon}{\zeta} |\Delta\varphi^A| + \frac{\varepsilon}{2\zeta^2} |\varphi^A|^2, \tag{63}$$

if the field derivatives of T^{ab} satisfy the following bounds

$$\left| \frac{\partial T^{ab}}{\partial \varphi^A} \right| \leqslant \frac{\varepsilon}{\zeta}, \qquad \left| \frac{\partial^2 T^{ab}}{\partial \varphi^A \partial \varphi^B} \right| \leqslant \frac{\varepsilon}{\zeta^2}. \tag{64}$$

The constant ε is a characteristic internal energy density. Using the expressions for the bound on $\Delta\varphi^A$ from Eq. (60) and the bound on $|\varphi^A|^2$ from Eq. (55), the following bound on ΔT^{ab} is obtained,

$$\left[\int_{\hat{S}(\kappa)} |\Delta T^{ab}|^2 \, dV \right]^{1/2} \leqslant 112 b^3 \left(\frac{\tau v}{L} \right)^2 \varepsilon \langle \hat{V}^{1/2} \rangle. \tag{65}$$

This equation provides a bound on ΔT^{ab} that is smaller than its *a priori* expected value, $\varepsilon \langle \hat{V}^{1/2} \rangle$, by the factor $b^3(\tau v/L)^2$.

An exactly analogous bound can be obtained for ΔN^a if the field derivatives of N^a are bounded by

$$\left| \frac{\partial N^a}{\partial \varphi^A} \right| \leqslant \frac{\nu}{\zeta}, \qquad \left| \frac{\partial^2 N^a}{\partial \varphi^A \partial \varphi^B} \right| \leqslant \frac{\nu}{\zeta^2}, \tag{66}$$

where ν is a characteristic number density. The bound on ΔN^a is obtained in precisely the same way as the bound on ΔT^{ab}, with the result

$$\left[\int_{\hat{S}(\kappa)} |\Delta N^a|^2 \, dV \right]^{1/2} \leqslant 112 b^3 \left(\frac{\tau v}{L} \right)^2 \nu \langle \hat{V}^{1/2} \rangle. \tag{67}$$

Thus the bound on ΔN^a is also smaller than its *a priori* expected value by the factor $b^3(\tau v/L)^2$.

III. SIMPLE DISSIPATIVE FLUIDS

In this section the relaxed expressions for the particle current and stress energy tensor, Eqs. (11)–(12), are evaluated for the theories of a simple dissipative fluid. Condition (iv) guarantees that Eq. (7) is equivalent to the conservation laws in this case. This implies that the space of the ξ^α must consist on one vector and one scalar field. The form of Eqs. (7)–(8) is unchanged if the fluid fields are transformed in the following way: $\hat{\xi}^\alpha = \hat{\xi}^\alpha(\xi^\beta, \varphi^B)$ and $\hat{\varphi}^A = \hat{\varphi}^A(\varphi^B)$. The choice $\hat{\xi}^\alpha = (n, \rho, u^a)$ and $\hat{\varphi}^A = \varphi^A$, where

$$nu^a = N^a(\xi^\beta, \varphi^B, g_{bc}), \tag{68}$$

$$\rho = u_a u_b T^{ab}(\xi^\beta, \varphi^B, g_{cd}), \tag{69}$$

is a transformation of this form. Thus ξ^α may be chosen to be $\xi^\alpha = (n, \rho, u^a)$, without loss of generality. With this choice $\partial N^a/\partial\varphi^A = 0$ and $u_a u_b \partial T^{ab}/\partial\varphi^A = 0$. Evaluating Eq. (11) for this case we obtain $N^a = nu^a + \Delta N^a$, hence Eq. (19). The quantity ΔN^a vanishes identically as a consequence of the choice of ξ^α used here. Condition (iv) also implies that the tensors $M^m{}_{\alpha\beta}$ and $M^m{}_{\alpha A}$ of Eq. (7) must be given by

$$M^m{}_{\alpha\beta} = P_\alpha \frac{\partial N^m}{\partial \xi^\beta} + P_{\alpha a} \frac{\partial T^{am}}{\partial \xi^\beta}, \tag{70}$$

$$M^m{}_{\alpha A} = P_{\alpha a} \frac{\partial T^{am}}{\partial \varphi^A}, \tag{71}$$

where P_α and $P_{\alpha a}$ are suitably chosen functions of ξ^α, φ^A and g_{ab}. Note that the term proportional to P_α is missing from Eq. (71) because $\partial N^a/\partial\varphi^A = 0$ for our choice of ξ^α. Using this expression for $M^m{}_{\alpha a}$, the general expression for T^{ab} in Eq. (12) reduces to

$$T^{ab} = \left[T^{ab} - \frac{\partial T^{ab}}{\partial \varphi^A} (I^{-1})^{AB} \frac{\partial T^{cm}}{\partial \varphi^B} P_{\alpha c} \nabla_m \xi^\alpha \right]_{\varphi^C = 0} + \Delta T^{ab}. \tag{72}$$

Condition (v) places restrictions on the allowed forms of $M^m{}_{\alpha A} \nabla_m \xi^\alpha$ in the fluid states where $\varphi^A = 0$. From Eq. (71) it follows that this quantity is determined by $P_{\alpha a}$. In the $\varphi^A = 0$ fluid state the tensor $M^m{}_{\alpha\beta}$ is identical to the tensor that governs the evolution of a perfect fluid via Eqs. (5)–(6). The most general P's that make $M^m{}_{\alpha\beta}$ symmetric in this case are given by

$$P_\alpha d\xi^\alpha = -Q_1 \left[\left(\frac{\partial p}{\partial n} \right)_\rho^2 + Q_2 \left(\frac{\rho + p}{n} \right)^2 \right] dn$$
$$- Q_1 \left[\left(\frac{\partial p}{\partial \rho} \right)_n \left(\frac{\partial p}{\partial n} \right)_\rho - Q_2 \frac{\rho + p}{n} \right] d\rho, \tag{73}$$

$$P_{\alpha a}\,d\xi^{\alpha}=u_a Q_1\left[\left(\frac{\partial p}{\partial\rho}\right)_n\left(\frac{\partial p}{\partial n}\right)_\rho-Q_2\,\frac{\rho+p}{n}\right]dn+u_a Q_1\left[\left(\frac{\partial p}{\partial\rho}\right)_\rho^2+Q_2\right]d\rho$$
$$-Q_1(\rho+p)\left(\frac{\partial p}{\partial\rho}\right)_s q_{ab}\,du^b, \tag{74}$$

where Q_1 and Q_2 are arbitrary functions of n and ρ, and $q_{ab}=g_{ab}+u_a u_b$ (see Geroch and Lindblom [7]). The hyperbolicity and causality conditions for $M^m{}_{\alpha\beta}$ in this case are simply, $Q_1>0$ and $Q_2>0$, and the equation of state must satisfy

$$0<\left(\frac{\partial p}{\partial\rho}\right)_s\leqslant 1, \tag{75}$$

with $n>0$ and $\rho+p>0$. Thus the tensors $M^m{}_{\alpha\beta}$ and $M^m{}_{\alpha A}$ are determined completely (up to the arbitrary overall factor Q_1) in these fluid states by the function Q_2.

Condition (v) fixes Q_2 by demanding that $M^m{}_{\alpha A}\nabla_m\xi^\alpha$ and hence $P_{\alpha(a}\nabla_{m)}\xi^\alpha$ depend on $\nabla_m n$ and $\nabla_m\rho$ only in the combination $\nabla_m T=(\partial T/\partial n)_\rho\nabla_m n+(\partial T/\partial\rho)_n\nabla_m\rho$. The unique Q_2 which insures this is

$$Q_2=\frac{1}{nT}\left(\frac{\partial p}{\partial\rho}\right)_n\left(\frac{\partial T}{\partial s}\right)_p\left(\frac{\partial p}{\partial T}\right)_s=\frac{1}{n^2T^2}\left(\frac{\partial p}{\partial\rho}\right)_s\left(\frac{\partial\rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_\Theta, \tag{76}$$

where T and s are the temperature and entropy that satisfy the first law of thermodynamics, Eq. (18), and $\Theta=(\rho+p)/nT-s$. The second equality in Eq. (76) shows that the condition $Q_2>0$, needed to insure hyperbolicity of the equations, is equivalent to a well known condition for thermodynamic stability (see Hiscock and Lindblom [13]). With this choice of Q_2 the quantity $P_{\alpha(a}\nabla_{m)}\xi^\alpha$ reduces to

$$P_{\alpha(a}\nabla_{m)}\xi^\alpha=Q_1\,\frac{\rho+p}{T}\left(\frac{\partial p}{\partial\rho}\right)_s[u_{(a}\nabla_{m)}T-T\nabla_{(m}u_{a)}]. \tag{77}$$

The tensor $\partial T^{ab}/\partial\varphi^A(I^{-1})^{AB}\,\partial T^{cd}/\partial\varphi^B$ that appears in Eq. (72) depends only on ξ^α and g_{ab}. The most general such tensor (having the appropriate symmetries, etc.) depending only on ξ^α and g_{ab} is given by

$$\frac{\partial T^{ab}}{\partial\varphi^A}(I^{-1})^{AB}\frac{\partial T^{cd}}{\partial\varphi^B}$$
$$=\frac{1}{Q_1(\rho+p)}\left(\frac{\partial\rho}{\partial p}\right)_s\left\{2\eta_1\left[q^{a(c}q^{d)b}-\frac{1}{3}\,q^{ab}q^{cd}\right]+\eta_2\,q^{ab}q^{cd}-2\kappa T u^{(a}q^{b)(c}u^{d)}\right\}. \tag{78}$$

The arbitrary functions η_1, η_2, and κ (of n and ρ) that appear in Eq. (78) must be positive as a consequence of the positivity of I_{AB}, from Condition (ii), and the positivity of Q_1 and $(\partial p/\partial\rho)_s$, from the hyperbolicity and causality of the equations.[6] Combining this expression, Eq. (78), with Eq. (77) in Eq. (72) results in the desired form, Eq. (20). Thus, the relaxed form of the stress energy tensor is indistinguishable in these general causal theories from that of the relativistic Navier–Stokes theory.

IV. Concluding Remarks

The argument presented here demonstrates that a relaxation effect takes place in virtually every causal theory of dissipative fluids. In the relaxed fluid states the stress energy tensor and particle current are well described by expressions that depend only on a subset of the fluid fields (referred to here as dynamical fluid fields) and their derivatives. For those theories that represent simple dissipative fluids, these expressions are identical to the ones given by the relativistic Navier–Stokes theory. This implies that any measurement of the stress-energy tensor or particle current in these theories (made on any time and length scale that exceeds the microscopic particle interaction scales) will give results that are in effect indistinguishable from those of the Navier–Stokes theory. Of course the Navier–Stokes theory is not really a proper physical theory at all since it is non-causal, unstable, etc. It is incapable of predicting the future evolution of initial fluid states. The argument presented here shows, nevertheless, that the evolution of any physical fluid state according to any causal theory results in stress-energy tensors and particle currents that are experimentally indistinguishable from the Navier–Stokes expressions for these quantities. Further, this argument shows that the independent dynamics associated with the dissipation fields of the fluid (i.e., those additional fluid fields that are added to the theory to make it causal) is not directly observable in the physical fluid states. On a time scale that is characteristic of the inter-particle interaction times, these dissipation fields evolve to a relaxed state in which they are determined in effect by the dynamical fields and their derivatives.

A number of technical improvements could be made to strengthen the arguments presented here. The physical fluid states for which this result applies are those whose gradients are bounded locally to insure that they are not rapidly changing on microscopic scales. These local constraints are much stronger than are actually needed to complete the proof. All that is really needed are the $\mathscr{L}^2$ bounds on the fluid fields and their derivatives implicit in Eqs. (35), (46), and (47). These bounds could undoubtedly be derived using far weaker $\mathscr{L}^2$ conditions on the fluid fields and their derivatives than the local conditions used here. A more serious limitation of the present work is its failure to demonstrate the existence of any solutions at all

[6] The only requirement on the dissipation fields needed to obtain Eq. (78) is that the space of the φ^A be large enough to insure that none of the coefficients η_1, η_2, or κ vanishes identically. This requires in particular that this space be at least as large as the nine-dimensional space of symmetric trace-free tensors. This is precisely the dimension that is appropriate for a theory in which the particle current N^a and stress energy tensor T^{ab} are the only independent observable fields.

of the fluid equations which satisfy these conditions. The expectation is that essentially every 'physically relevant' solution to the fluid equations does satisfy these conditions. In particular it is expected that 'almost all' initial data which are suitably slowly varying on the relevant microscopic length and time scales will evolve in such a way that these conditions are preserved for some amount (large on microscopic scales) of time. At present, however, theorems of this sort do not exist for these theories.

Shock waves are one class of physical phenomena that do violate the conditions imposed on the fluid states in this work. Significant differences probably do exist in the descriptions of this type of fluid phenomenon among the various causal theories and the non-causal Navier–Stokes equations. Can meaningful experimental differentiation among the various theories be found by observing shock waves? Or, do the predictions of all macroscopic fluid theories become meaningless when applied to shocks, since these fluid states all contain rapid variations on microscopic particle interaction scales?

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