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DOES GRAVITATIONAL RADIATION LIMIT THE ANGULAR VELOCITIES OF SUPERFLUID NEUTRON STARS?

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ABSTRACT

The gravitational radiation instability which limits the rotation rates of normal neutron stars is studied using superfluid hydrodynamics and dissipation mechanisms. The influences of these superfluid effects on this instability are evaluated using simple analytical solutions to the superfluid pulsation equations. The superfluid dissipation mechanism "mutual friction" overwhelmingly suppresses the gravitational radiation instability according to these estimates. Thus, gravitational radiation does not limit the angular velocities of superfluid neutron stars.

Subject headings: gravitation — hydrodynamics — radiation mechanisms: nonthermal — relativity — stars: neutron — stars: rotation

1. INTRODUCTION

Gravitational radiation tends to make all rotating stars unstable (Chandrasekhar 1970a, b; Friedman & Schutz 1978; Friedman 1978). Internal fluid dissipation (e.g., viscosity) also influences the stability of rotating stars, and it tends to suppress the gravitational radiation instability (Lindblom & Detweiler 1977; Lindblom & Hiscock 1983). In stars having very little internal dissipation all but the most slowly rotating are unstable to the gravitational radiation instability, while in stars with extremely large internal dissipation the instability may be suppressed completely. This secular instability may have important astrophysical consequences: it has been proposed as the mechanism that limits the rotation rates of neutron stars (Friedman 1983; Wagoner 1984). The current ordinary-fluid models of neutron stars show that the gravitational radiation instability can occur in sufficiently rapidly rotating stars only if the internal temperatures are between about 10^7 and 2×10^{10} K (Ipser & Lindblom 1989, 1990, 1991; Lindblom 1992, 1995). Thus, gravitational radiation could limit the angular velocities of ordinary-fluid neutron stars with internal temperatures in this range.

However, below $\sim 10^9$ K the bulk of the material in the interior of a neutron star is expected to undergo a transition into a superconducting superfluid state (Migdal 1959; and for reviews of superfluidity in neutron stars see Epstein 1988; Sauls 1989; Pines & Alpar 1992). Since the dynamics and the dissipation mechanisms present in a superfluid are qualitatively different from those of an ordinary fluid, it is necessary to reinvestigate completely the gravitational radiation instability in these cool neutron stars. For the past several years we have been developing the fundamental superfluid hydrodynamics needed to study this problem (Mendell & Lindblom 1991; Mendell 1991a, b; Lindblom & Mendell 1994). In summary, Mendell (1991a) derived the equations for the superfluid hydrodynamics (averaged over vortices) needed to describe the macroscopic pulsations of neutron stars. The superfluid version of the magnetohydrodynamic limit of these equations was then taken so that the low-frequency (e.g., relative to the $\sim 10^{22}$ s^{-1} plasma frequency) and long-wavelength oscillations that participate in the gravitational radiation instability could be studied. Mendell (1991b) has also developed the theory of dissipation for the complicated superfluid mixture that occurs in neutron star cores. Lindblom & Mendell (1994) have transformed the basic hydrodynamic equations of Mendell (1991a) into a simplified form in which the global pulsations of superfluid neutron stars can more easily be studied. Numerical methods are also developed there for determining the modes of rapidly rotating superfluid neutron stars, neglecting the effects of dissipation.

In this paper we extend these studies to obtain the first realistic estimates of the effects of the superfluid dissipation mechanisms on the gravitational radiation instability. The most important dissipation mechanism in these stars is a superfluid effect called "mutual friction." In rotating neutron star matter mutual friction is caused by the scattering of electrons off the cores of the neutron vortices. This scattering is greatly enhanced by nuclear interactions between the neutrons and protons that induce proton supercurrents and hence strong magnetic fields within the neutron vortices (see Alpar, Langer, & Sauls 1984; Alpar & Sauls 1988). In this paper we estimate the effects of this (and other) dissipation mechanisms on the pulsations of neutron stars using simple analytical models for these pulsations. These estimates indicate that mutual friction completely suppresses the gravitational radiation instability in all neutron stars cooler than the superfluid transition temperature. Furthermore, the suppression is so strong in these estimates that we believe that more realistic calculations will agree qualitatively with these results. Thus, we conclude that gravitational radiation does not limit the angular velocities of superfluid neutron stars.

In § 2 we present the simple analytical approach used here to estimate the frequencies and eigenfunctions of the modes of superfluid neutron stars. We present in § 3 the methods used to determine the various dissipative timescales that influence the pulsations of superfluid neutron stars. And in § 4 we use these results to analyze the gravitational radiation instability in rotating neutron stars. We also present there our conclusions and a few caveats.

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2. SIMPLE SUPERFLUID PULSATIONS

The bulk of the matter in a cool neutron star will exist in a superfluid state in the core of the star where the density exceeds nuclear density (assuming that the equation of state is fairly stiff). This core superfluid material is a mixture composed primarily of superfluid neutrons, superconducting protons, and ordinary electrons. The global oscillations of such a neutron star are completely described by just three scalar potentials: δU, $\delta\beta$, and $\delta\Phi$. The first two potentials are thermodynamic scalars: δU is related to the pressure perturbation δp (and the equilibrium mass density ρ) by $\delta U \equiv \delta p/\rho + \delta\Phi$, while $\delta\beta$ is the deviation of the perturbed state from β-equilibrium. The third scalar, $\delta\Phi$, is the Newtonian gravitational potential. The effects of superfluidity on the dynamics and dissipation are governed by the potential $\delta\beta$. These scalars are determined by solving a system of three coupled second-order (typically elliptic) differential equations with appropriate boundary conditions. The derivation of this representation of the superfluid pulsation problem for neutron star matter is given in Lindblom & Mendell (1994). These equations reduce to those of an ordinary perfect fluid when the potential $\delta\beta$ vanishes.

Here we limit our consideration to a simple idealized situation. We assume that the equilibrium configuration is nonrotating and spatially uniform. In this case the equations that determine δU, $\delta\beta$, and $\delta\Phi$ reduce to the following system of coupled Helmholtz equations:

$$\nabla^2\delta U = -\omega^2\left(\frac{\partial\rho}{\partial p}\right)_\beta(\delta U - \delta\Phi) - \frac{\omega^2}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta\,, \tag{1}$$

$$\nabla^2\delta\beta = -\omega^2\frac{\rho_n^2}{\det\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\delta\beta - \frac{\omega^2\rho}{\det\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p(\delta U - \delta\Phi)\,, \tag{2}$$

$$\nabla^2\delta\Phi = 4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta(\delta U - \delta\Phi) + 4\pi G\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta\,, \tag{3}$$

where ω is the frequency of the oscillation; ρ_n and ρ_p are the mass densities of the neutrons and protons, respectively; and G is Newton's gravitational constant. The various thermodynamic quantities that appear in equations (1)–(3) are to be evaluated in the spatially uniform equilibrium state. The values of these quantities for realistic superfluid matter are given in Lindblom & Mendell (1994). Finally, the quantity $\det\rho$ in equation (2) describes the nuclear interaction between the superfluid neutrons and protons. This interaction plays a crucial role in determining how superfluidity influences the dynamics and the dissipation in neutron star matter. In the absence of any nuclear interactions $\det\rho = \rho_n\rho_p$, while most models of the nuclear coupling between the neutron and protons yield a value several times this. The values for $\det\rho$ as a function of the total mass density are given in Lindblom & Mendell (1994) for two models of the nuclear interaction.

The general solution to equations (1)–(3) is an arbitrary linear combination (over the allowed values of the integers l and m, and the frequency ω) of the following elementary solutions (see Lindblom & Mendell 1994):

$$\delta U = [\delta A r^l + \delta B j_l(k_+ r) + \delta C j_l(k_- r)]Y_{lm}(\theta, \phi)e^{-i\omega t}\,, \tag{4}$$

$$\delta\beta = [\delta B D_+(\omega) j_l(k_+ r) + \delta C D_-(\omega) j_l(k_- r)]Y_{lm}(\theta, \phi)e^{-i\omega t}\,, \tag{5}$$

$$\delta\Phi = \left\{\delta A r^l - \frac{4\pi G\rho}{\omega^2}[\delta B j_l(k_+ r) + \delta C j_l(k_- r)]\right\}Y_{lm}(\theta, \phi)e^{-i\omega t}\,, \tag{6}$$

where the functions j_l are spherical Bessel functions; the Y_{lm} are spherical harmonics; r, θ, and ϕ are the standard spherical coordinates; and $k_\pm$ are the two roots of the "dispersion relation,"

$$0 = \omega^2(\omega^2 + 4\pi G\rho)\left[\left(\frac{\partial\rho}{\partial\beta}\right)_p^2 - \rho_n^2\left(\frac{\partial\rho}{\partial p}\right)_\beta\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\right] + \omega^2\left[\det\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta + \rho_n^2\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\right]k^2 + \det\rho\left[4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta - k^2\right]k^2\,. \tag{7}$$

The quantities δA, δB, and δC in equations (4)–(6) are arbitrary constants, and $D_\pm(\omega)$ are defined by

$$D_\pm(\omega) = \rho\left(\frac{\partial\beta}{\partial\rho}\right)_p\left[\frac{k_\pm^2}{\omega^2} - \left(\frac{\partial\rho}{\partial p}\right)_\beta\left(1 + \frac{4\pi G\rho}{\omega^2}\right)\right]. \tag{8}$$

Appropriate boundary conditions must be imposed on equations (4)–(6) in order to determine the "physical" oscillations in this simple model. We consider solutions in the interior of a sphere, $r \le R$, that satisfy the following boundary conditions at $r = R$:

$$\delta U - \delta\Phi - \frac{4\pi G\rho}{3\omega^2}R\frac{\partial\,\delta U}{\partial r} = 0\,, \tag{9}$$

$$\frac{\partial\,\delta\Phi}{\partial r} + \frac{l(l+1)}{R}\delta\Phi = 0\,, \tag{10}$$

$$\frac{\partial\,\delta\beta}{\partial r} = 0\,. \tag{11}$$

These boundary conditions are the analogues of those derived for more realistic superfluid neutron stars in Lindblom & Mendell (1994). In the more realistic case these conditions enforce the appropriate continuity conditions on the pressure, gravitational potential, and mass currents at the stellar surface and at the boundary between the core superfluid and the "crust" of the neutron star.

When the boundary conditions, equations (9)–(11), are applied to the elementary solutions, equations (4)–(6), they become a system of linear homogeneous equations for the three constants δA, δB, and δC. These equations have nontrivial solutions only if the determinant of the matrix of coefficients of δA, δB, and δC vanishes. This condition is equivalent to the following equation that determines the allowed values of the frequency ω in this simple model:

$$0=\frac{R}{3}\left(2l+1+l\,\frac{4\pi G\rho}{\omega^2}\right)(k_+^2-k_-^2)k_+k_-j_l'(k_+R)j_l'(k_-R)-\left[(2l+1)\left(1+\frac{\omega^2}{4\pi G\rho}\right)-l(l+1)\,\frac{4\pi G\rho}{3\omega^2}\right]$$
$$\times\left\{\left[k_+^2-\left(\frac{\partial\rho}{\partial p}\right)_\beta(\omega^2+4\pi G\rho)\right]k_+j_l(k_-R)j_l'(k_+R)-\left[k_-^2-\left(\frac{\partial\rho}{\partial p}\right)_\beta(\omega^2+4\pi G\rho)\right]k_-j_l(k_+R)j_l'(k_-R)\right\}. \quad (12)$$

For these allowed values of the frequency, the constants δA, δB, and δC are given (up to an overall scale) by

$$\delta A=\frac{4\pi G\rho}{(2l+1)\omega^2R^l}\{D_+(\omega)k_+Rj_l'(k_+R)[k_-Rj_l'(k_-R)+(l+1)j_l(k_-R)]-D_-(\omega)k_-Rj_l'(k_-R)[k_+Rj_l'(k_+R)+(l+1)j_l(k_+R)]\}\,, \quad (13)$$

$$\delta B=-D_-(\omega)k_-Rj_l'(k_-R)\,, \quad (14)$$

$$\delta C=D_+(\omega)k_+Rj_l'(k_+R)\,. \quad (15)$$

It is straightforward to solve equation (12) to determine the frequencies and to evaluate equations (13)–(15) to determine the eigenfunctions which describe the oscillations of superfluid neutron stars in this simple approximation. We have evaluated these modes using typical neutron star values for the parameters: $\rho=4\times10^{14}$ g cm^{-3}, and $R=15$ km. For the thermodynamic quantities we use the Serot (1979) relativistic-mean-field equation of state, which for this value of the density gives: $\rho_n=3.726\times10^{14}$ g cm^{-3}, $(\partial\rho/\partial p)_\beta=4.819\times10^{-21}$ (s cm^{-1})^2, $(\partial\rho/\partial\beta)_p=1.911\times10^{-7}$ g s^2 cm^{-5}, and $\partial(\rho_p/\rho_n)/\partial\beta=1.076\times10^{-21}$ (s cm^{-1})^2. The parameter det ρ that describes the neutron-proton interactions is related to the parameter m_p^*/m_p (the ratio of the effective mass m_p^* to the bare mass m_p of the proton) by

$$\det\rho=\rho_n\rho_p\,\frac{m_p}{m_p^*}+\rho_p^2\left(\frac{m_p}{m_p^*}-1\right) \quad (16)$$

(see Lindblom & Mendell 1994). We evaluate the oscillations in our simple model using the minimum and maximum published values of this parameter (see Sjöberg 1976; Wambach, Ainsworth, & Pines 1991; and discussions in Pines & Alpar 1992; Alpar 1992): $m_p^*/m_p=0.3$ and $m_p^*/m_p=0.8$. Using equation (16) therefore we take det $\rho=3.581\times10^{28}$ (g cm^{-3})^2 for $m_p^*/m_p=0.3$ and det $\rho=1.296\times10^{28}$ (g cm^{-3})^2 for $m_p^*/m_p=0.8$.

The frequencies of the fundamental modes, for $2\le l\le 6$, in our simple model of the oscillations of superfluid neutron stars are presented in Table 1 in units of $\Omega_0=(\pi G\rho)^{1/2}$. The frequencies obtained in this way agree to within ~15% with the more realistic oscillation frequencies computed in Lindblom & Mendell (1994). In our simple model these modes have no nodes in the radial parts of the eigenfunctions δU or $\delta\beta$. The magnitude of $\delta\beta$ is smaller than δU in these modes: $|\delta\beta|/|\delta U|<\frac{1}{3}$ as shown in Table 1. This shows that superfluid effects are relatively unimportant in the dynamics of these particular modes.

3. DISSIPATION

The effects of dissipation on the evolution of superfluid neutron star matter can be studied conveniently with the use of the following energy functional (Mendell 1991b):

$$E=\frac{1}{2}\int\left\{\rho\,\delta\boldsymbol{v}\cdot\delta\boldsymbol{v}^\dagger+\frac{\det\rho}{\rho}\,\delta\boldsymbol{w}\cdot\delta\boldsymbol{w}^\dagger+\frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\delta\beta\,\delta\beta^\dagger+\left(\frac{\partial\rho}{\partial\beta}\right)_p\mathrm{Re}\,[(2\,\delta U-\delta\Phi)\delta\beta^\dagger]+\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta\mathrm{Re}\,[(\delta U-\delta\Phi)\delta U^\dagger]\right\}d^3x\,, \quad (17)$$

TABLE 1

FUNDAMENTAL FREQUENCIES AND DAMPING TIMES

$l=m$	$\omega(0)/\Omega_0$	$\tau_{GR}\Omega_0$	$\tilde{\tau}_{MF}\Omega_0$	$\tau_V\Omega_0$	$\lvert\delta\beta/\delta U\rvert$
		$m_p^*/m_p=0.3$			
2..........	1.407	2.685×10^3	1.071×10^4	7.371×10^{11}	0.082
3..........	1.809	1.303×10^5	1.638×10^4	2.887×10^{11}	0.064
4..........	2.141	5.711×10^6	2.347×10^4	1.551×10^{11}	0.052
5..........	2.430	2.614×10^8	3.195×10^4	9.677×10^{10}	0.044
6..........	2.689	1.291×10^{10}	4.182×10^4	6.609×10^{10}	0.038
		$m_p^*/m_p=0.8$			
2..........	1.401	2.660×10^3	1.127×10^5	6.407×10^{11}	0.315
3..........	1.805	1.287×10^5	1.854×10^5	2.656×10^{11}	0.235
4..........	2.139	5.637×10^6	2.861×10^5	1.469×10^{11}	0.183
5..........	2.428	2.582×10^8	4.143×10^5	9.326×10^{10}	0.149
6..........	2.687	1.277×10^{10}	5.700×10^5	6.435×10^{10}	0.125

where † denotes complex conjugation. The velocity perturbations that appear in equation (17) are determined by the scalars δU and $\delta\beta$. For the case of the simple pulsations described in § 2, these velocities are given by

$$\delta \boldsymbol{v} = -\frac{i}{\omega} \nabla \delta U , \tag{18}$$

$$\delta \boldsymbol{w} = -\frac{i}{\omega} \nabla \delta \beta , \tag{19}$$

where $\delta \boldsymbol{v}$ is the average velocity of the superfluid mixture and $\delta \boldsymbol{w}$ is the relative velocity between the neutron and proton superfluids. Thus the energy E defined in equation (17) is a real quadratic functional of the scalars δU, $\delta\beta$, and $\delta\Phi$.

While the energy functional E is conserved in the absence of dissipation, it will evolve with time if dissipation is present in any of its various forms. The resulting time dependence of E is very simple for the case of a mode: i.e., a solution in which δU, $\delta\beta$, and $\delta\Phi$ have the time dependence $e^{-i\omega t}$. Since E is a real quadratic functional of these quantities its time dependence will have the form $E \propto \exp(-2t/\tau)$, where τ is related to the imaginary part of the frequency of the mode by $\mathrm{Im}(\omega) \equiv -1/\tau$. The quantity τ is in effect the damping (or growth) time of the mode. Thus we may evaluate the damping time τ (and hence the imaginary part of the frequency of a mode) from a knowledge of E and its time derivative:

$$\frac{1}{\tau} = -\frac{1}{2E}\left(\frac{dE}{dt}\right). \tag{20}$$

Expressions for the evolution of the energy functional E under the influence of the various types of dissipation that are present in superfluid neutron star matter have been derived by Mendell (1991b). Of particular interest to us here is a dissipative effect called mutual friction. In rotating superfluid neutron star matter, mutual friction is caused by the scattering of the electrons off the cores of the neutron vortices. The damping time due to mutual friction for a superfluid neutron star rotating about the z-axis with angular velocity Ω is (Mendell 1991b)

$$\frac{1}{\tau_{\mathrm{MF}}} = -\frac{1}{2E}\left(\frac{dE}{dt}\right)_{\mathrm{MF}} = \frac{1}{2E}\int 2\Omega B_n \rho_n \left(\frac{\det \rho}{\rho_n \rho_p}\right)^2 (\delta \boldsymbol{w} \cdot \delta \boldsymbol{w}^\dagger - |\hat{z} \cdot \delta \boldsymbol{w}|^2)\, d^3x , \tag{21}$$

where the quantity B_n is the mutual friction coefficient. It characterizes the strength of the electron-vortex scattering process and is given by

$$B_n \approx 0.011\left(1 - \frac{m_p^*}{m_p}\right)^2 \left(\frac{m_p}{m_p^*}\right)^{1/2} \left(\frac{\rho_p}{\rho}\right)^{7/6} \left(1 - \frac{\rho_p}{\rho}\right)^{-1} \left(\frac{\rho}{10^{14}\ \mathrm{g\ cm^{-3}}}\right)^{1/6} . \tag{22}$$

Note that B_n is strongly dependent on the parameter m_p^*/m_p. This parameter measures the strength of the neutron-proton coupling in the superfluid. This coupling influences the magnitude of B_n because it induces proton supercurrents around the neutron vortices (see Alpar et al. 1984; Alpar & Sauls 1988). These supercurrents cause a substantial magnetic flux within the neutron vortices which greatly enhances their scattering with electrons. The values of B_n for the simple model pulsations studied here are $B_n = 5.84 \times 10^{-4}$ for $m_p^*/m_p = 0.3$ and $B_n = 2.92 \times 10^{-5}$ for $m_p^*/m_p = 0.8$. We also note that the mutual friction damping time τ_{MF} diverges when the angular velocity of star vanishes. It is convenient, therefore, to define a related quantity

$$\tilde{\tau}_{\mathrm{MF}}(\Omega) = \left|\frac{\Omega}{\Omega_0}\right| \tau_{\mathrm{MF}}(\Omega) , \tag{23}$$

which is finite even at $\Omega = 0$.

The evolution of the modes of superfluid neutron stars will also be influenced by shear viscosity. (The influences of bulk viscosity and normal thermal conductivity are negligible by comparison below the superfluid transition temperature; see Cutler & Lindblom 1987; Cutler, Lindblom, & Splinter 1990; Lindblom 1995). In superfluid neutron star matter shear viscosity is caused by electron-electron scattering. The damping time due to this process is given (see Mendell 1991b; Ipser & Lindblom 1991) by

$$\frac{1}{\tau_V} = -\frac{1}{2E}\left(\frac{dE}{dt}\right)_V = \frac{1}{2E}\int 2\eta_e \delta\Theta^{ab} \delta\Theta^\dagger_{ab}\, d^3x , \tag{24}$$

where the shear $\delta\Theta_{ab}$ of the electron velocity field, $\delta \boldsymbol{u}$, is given by

$$\delta\Theta_{ab} = \tfrac{1}{2}(\nabla_a \delta u_b + \nabla_b \delta u_a - \tfrac{2}{3}\delta_{ab}\nabla_c \delta u^c) , \tag{25}$$

and δ_{ab} is the Euclidean metric (the identity matrix in Cartesian coordinates). The velocity field of the electrons is also determined by the scalar potentials δU and $\delta\beta$. For the simple oscillations considered here $\delta \boldsymbol{u}$ is given by (Mendell 1991a; Lindblom & Mendell 1994)

$$\delta \boldsymbol{u} = -\frac{i}{\omega} \nabla\left(\delta U + \frac{\det \rho}{\rho \rho_p}\delta\beta\right). \tag{26}$$

The viscosity due to electron-electron scattering is given approximately by the expression (Cutler & Lindblom 1987)

$$\eta_e = 6.0 \times 10^{16}\ \mathrm{g\ cm^{-1}\ s^{-1}} \left(\frac{\rho}{10^{14}\ \mathrm{g\ cm^{-3}}}\right)^2 \left(\frac{10^9\ \mathrm{K}}{T}\right)^2 . \tag{27}$$

In the simple model considered here we take the temperature to have the value $T = 10^9$ K, the superfluid transition temperature. Thus the viscosity in this simple model is $\eta_e = 9.6 \times 10^{17}$ g cm^{-1} s^{-1}. We choose this value for the temperature because it minimizes the effect of viscosity on the gravitational radiation secular instability. For smaller values of the temperature, the shear viscosity will play a larger role in suppressing the instability.

Finally, the evolution of the superfluid state is influenced by the emission of gravitational radiation. The effect of this dissipative process on the damping time is given by (see Ipser & Lindblom 1991)

$$\frac{1}{\tau_{\mathrm{GR}}} = -\frac{1}{2E}\left(\frac{dE}{dt}\right)_{\mathrm{GR}} = \frac{\omega - m\Omega}{2E} \sum_{l=|m|}^{\infty} N_l \omega^{2l+1} \delta D_{lm} \delta D_{lm}^{\dagger} , \tag{28}$$

where

$$\delta D_{lm} = \int \left[\rho\left(\frac{\partial \rho}{\partial p}\right)_\beta (\delta U - \delta\Phi) + \left(\frac{\partial \rho}{\partial \beta}\right)_p \delta\beta\right] r^l Y_{lm}^{\dagger} d^3x , \tag{29}$$

and

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2} . \tag{30}$$

These equations are equivalent to those for the ordinary fluid case given in Ipser & Lindblom (1991), except the superfluid expression for $\delta\rho$ has been used in equation (29).

The damping times for the modes of superfluid neutron stars have been expressed as a set of integrals over the eigenfunctions of the modes, δU, $\delta\beta$, and $\delta\Phi$ in equations (17), (21), (24), and (29). In the general case, these eigenfunctions depend on the dissipation, and so these integrals cannot easily be evaluated. In neutron stars, however, the dissipation is relatively weak in the sense that the Q's of the modes are extremely large: τ Re $(\omega) \gg 1$. Thus, it is possible to obtain first approximations to the damping times of the modes by using in equations (17), (21), (24), and (29) the nondissipative eigenfunctions δU, $\delta\beta$, and $\delta\Phi$. We have evaluated these quantities using the simple oscillation eigenfunctions described in § 2. In Table 1 we list these damping times for the fundamental $2 \le l = m \le 6$ modes using the extreme values, $m_p^*/m_p = 0.3$ and $m_p^*/m_p = 0.8$, for the parameter that characterizes the interactions between the protons and the neutrons.

In a real neutron star all of the dissipative processes act together simultaneously. The time derivative of the energy E under this simultaneous action is merely the sum of the expressions given in equations (21), (24), and (28). Thus the total dissipative damping time is given by

$$\frac{1}{\tau} = \frac{1}{\tau_{\mathrm{GR}}} + \frac{1}{\tau_V} + \frac{1}{\tau_{\mathrm{MF}}} . \tag{31}$$

4. CRITICAL ANGULAR VELOCITIES

Although the various dissipative damping times defined in § 3 are positive for nonrotating stars, the gravitational radiation damping time, τ_{GR}, may change sign for larger values of the angular velocity. In fact in every rotating star there is some mode for which τ_{GR} is negative (see Friedman & Schutz 1978). Unless the internal fluid dissipation processes are sufficiently strong, therefore, $1/\tau$ may become negative and the star unstable. We are interested here in exploring the values of the angular velocity of the star where $1/\tau$ changes sign from positive (stable) to negative (unstable). These critical angular velocities are defined then by

$$\frac{1}{\tau(\Omega_c)} = 0 . \tag{32}$$

To begin the process of solving equation (32) we define three functions $\alpha_m(\Omega)$, $\gamma_m(\Omega)$, and $\mu_m(\Omega)$ that describe the angular velocity dependence of the real part of the frequency, ω, and the damping times of the modes

$$\alpha_m(\Omega) = \frac{\omega_m(\Omega) - m\Omega}{\omega_m(0)} , \tag{33}$$

$$\gamma_m(\Omega) = \frac{\omega_m(\Omega)}{\omega_m(0)} \left[\frac{\tau_V(0)}{\tau_{\mathrm{GR}}(0)} \frac{\tau_{\mathrm{GR}}(\Omega)}{\tau_V(\Omega)}\right]^{1/(2l+1)} , \tag{34}$$

$$\mu_m(\Omega) = \frac{\tilde{\tau}_{\mathrm{MF}}(0)}{\tau_V(0)} \frac{\tau_V(\Omega)}{\tilde{\tau}_{\mathrm{MF}}(\Omega)} , \tag{35}$$

where $\tilde{\tau}_{\mathrm{MF}}$ is defined in equation (23). These quantities are defined so that for nonrotating stars $\alpha_m(0) = \gamma_m(0) = \mu_m(0) = 1$.

With these definitions, equation (32) for the critical angular velocity can be rewritten in the form

$$\Omega_c = \frac{|\omega_m(0)|}{m} \left\{ \alpha_m(\Omega_c) + \gamma_m(\Omega_c) \left[\frac{\tau_{GR}(0)}{\tau_V(0)} + \mu_m(\Omega_c) \frac{\Omega_c}{\Omega_0} \frac{\tau_{GR}(0)}{\tilde{\tau}_{MF}(0)} \right]^{1/(2l+1)} \right\} . \tag{36}$$

This equation is easily solved once the frequencies and damping times of *nonrotating* stars are known, plus the functions α_m, γ_m, and μ_m. While these functions are not easy to determine, our experience with their ordinary fluid counterparts suggests that they will have values near one for all values of the angular velocity. Thus equation (36) may be approximated as

$$\Omega_c \approx \frac{|\omega_m(0)|}{m} \left\{ 1 + \left[\frac{\tau_{GR}(0)}{\tau_V(0)} + \frac{\Omega_c}{\Omega_0} \frac{\tau_{GR}(0)}{\tilde{\tau}_{MF}(0)} \right]^{1/(2l+1)} \right\} . \tag{37}$$

This equation can easily be solved numerically for the critical angular velocity of a particular model knowing only the frequency $\omega_m(0)$ and the damping times $\tau_{GR}(0)$, $\tau_V(0)$, and $\tilde{\tau}_{MF}(0)$ associated with the nonrotating model. In perfect-fluid stars the analogous approximation for the critical angular velocities agrees with the exact results to within $\sim 15\%$ (Lindblom 1986).

Critical angular velocities Ω_c can be found by solving equation (37) for any values of l, m, and ω. However, these solutions are not physically meaningful if the resulting Ω_c exceeds the maximum allowed angular velocity of the star. Above the maximum angular velocity, Ω_{MAX}, equilibrium stellar models do not exist. For the uniform-density stellar models $\Omega_{MAX} = 0.67\Omega_0$ (see, e.g., Chandrasekhar 1969). A number of studies (Friedman, Ipser, & Parker 1986; Ipser & Lindblom 1991) have shown that Ω_{MAX} is close to this value for a wide range of equations of state and stellar masses. We find no solutions to equation (37) with $\Omega_c < 0.68\Omega_0$. The smallest Ω_c that we find using the values of the damping times from Table 1 is $\Omega_c = 1.03\Omega_0$ for the $l = m = 2$ mode. The effects of mutual friction are probably overestimated, however, in the simple model of the oscillations used here. The ratio $|\delta\beta/\delta U|$, for example, is larger in these simple modes by a factor of ~ 100 over the more realistic modes studied by Lindblom & Mendell (1994). Reducing the effects of mutual friction (by increasing the values of $\tilde{\tau}_{MF}$) by a factor of 10^6 over those given in Table 1 still gives no $\Omega_c < 0.68\Omega_0$, however. Thus we find no physical solutions to equation (37): mutual friction (due to superfluidity) suppresses the gravitational radiation instability in all modes and for all temperatures below the superfluid transition temperature. Thus, we conclude that gravitational radiation will not limit the angular velocities of superfluid neutron stars. We note that this conclusion confirms the prediction of Lindblom & Mendell (1992).

Although the suppression of the gravitational radiation instability by mutual friction in the estimate given here is overwhelming, the physics of the superfluid interiors of neutron stars is only poorly understood. Thus we thought it would be appropriate to summarize here the basic superfluid effects that contribute crucially to our conclusion.

1. The mutual friction damping of the gravitational radiation instability depends strongly on m_p^*/m_p. If this parameter were significantly larger than 0.8, the maximum value used in our estimates, then the effects of mutual friction are reduced perhaps to the point where the gravitational radiation instability could again become a factor.

2. In our simple estimates we assume that the entire neutron star is in the superfluid state. More realistic estimates (Lindblom & Mendell 1994) have $\sim 66\%$ of the star by mass in the core superfluid. If real neutron stars had a substantially smaller superfluid core, then the effects of mutual friction would again be reduced perhaps to the point where the gravitational radiation instability could be a factor.

3. In the estimates given here we have assumed that the function μ_m that describes the angular velocity dependence of the mutual friction damping time is nearly constant: $\mu_m \approx 1$. If, contrary to our expectations, this function is extremely small for large angular velocities, then the gravitational radiation instability could again become a factor.

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