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THE OSCILLATIONS OF SUPERFLUID NEUTRON STARS

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ABSTRACT

The effects of superfluid hydrodynamics on the oscillations of neutron stars are investigated. The equations describing the small-amplitude pulsations of a neutron star's neutron-proton superfluid mixture are reduced to a system of three second-order equations for three scalar potentials. A variational principle is developed from which the frequencies of the modes of these superfluid oscillations may be estimated. These pulsations are studied by finding analytical solutions to the equations for simple uniform models of nonrotating neutron stars and numerical solutions for realistic models (including rotation). These solutions reveal that the lowest frequency modes are almost indistinguishable from those based on ordinary-fluid hydrodynamics. The analytical solutions also reveal the existence of a new set of modes having no ordinary-fluid counterpart.

Subject headings: dense matter — stars: neutron — stars: oscillations

1. INTRODUCTION

Superfluidity in neutron stars was first suggested as a possibility by Migdal (1959) and has since become a part of the standard description of these stars (see, e.g., Baym, Bethe, & Pethick 1981 or Shapiro & Teukolsky 1983). Calculations predict for temperatures below about 10^9 K that the neutrons will form a superfluid condensate in the inner crust and outer core of a neutron star, and that protons in the outer core will condense into a type II superconducting state (see, e.g., Amundson & Østgaard 1985a, b and Epstein 1988 for a review). Since the temperatures of neutron stars are expected to fall well below 10^9 K within 1 yr of their birth (Tsuruta 1979), essentially all neutron stars are expected to contain superfluid matter. The consequences of superfluidity on the observable properties of neutron stars have been the subject of numerous studies. The effect of superfluidity on neutron star cooling has been investigated by several groups (e.g., Nomoto & Tsuruta 1987; Page & Baron 1990), and superfluidity plays a central role in the current theory of pulsar glitches (see Pines & Alpar 1985; Sauls 1989).

The primary motivation for the present study is the expectation that superfluidity will also have important effects on the oscillations and stability of neutron stars. This expectation is based on the fact that the dynamics and dissipative properties of a superfluid are quite different from those of an ordinary fluid. Superfluid effects on the oscillation frequencies of these stars could effect the interpretation of various periodicities observed in pulsars, X-ray binaries, and gamma-ray bursters (see Van Horn 1980; Epstein 1988 and references therein) and the theory of neutron star interactions with an accretion disk (Rappaport & Joss 1983 and references therein). Superfluidity should also effect the dissipation timescales of these oscillations (Cutler, Lindblom, & Splinter 1990; Mendell 1991b) which could change the secular stability of these stars. In particular, superfluidity may have important effects on the gravitational radiation instability believed to limit the maximum neutron star rotation rate (Lindblom & Mendell 1992; see Lindblom 1992 for a review). However, all previous studies of the global oscillations of neutron stars were based on ordinary-fluid hydrodynamics (see, e.g., Meltzer & Thorne 1966; Lindblom & Detweiler 1983; Carroll et al. 1986; Ipser & Lindblom 1990). The theory of superfluid hydrodynamics in neutron stars is only beginning to be investigated (Epstein 1988; Mendell 1991a, b). This paper transforms the theory of neutron star superfluid hydrodynamics into a form appropriate for the study of the global oscillations of these stars. The resulting equations are solved numerically for the $l = m$ f-modes that most resemble the ordinary-fluid oscillations of a rapidly rotating neutron star.

The superfluid hydrodynamics that we use is based on the two-fluid model proposed by Tisza (1938), and developed by Landau (1941) (see also Khalatnikov 1989 and Putterman 1974). This basic theory has been extended to incorporate a number of additional effects which are expected to play an important role in the neutron star superfluid. First, the theory has been extended by Khalatnikov (1957) and Andreev & Bashkin (1976) to describe superfluid mixtures such as the neutron-proton superfluid of neutron star matter. The most profound new feature of the equations for mixtures is that the superfluid velocities, defined in terms of the superfluid condensate momenta, are no longer parallel to the mass currents. Instead, a superfluid velocity flow of one species carries some mass current of the other species along with it; this property is referred to as the "drag effect" (Sauls 1989). Second, the two-fluid model has also been modified to accommodate the observation that the circulation and magnetic flux are carried by arrays of quantized vortices. In this case the macroscopic theory describes the behavior of quantities that are smooth-averaged over volumes containing many vortices. Such an approach has been developed by Hall & Vinen (1956), Bekarivich & Khalatnikov (1961), Baym & Chandler (1983), Chandler & Baym (1986), and others (see Sonin 1987 for a review). The most significant effect of the vortices is a new form of dissipation (known as mutual friction) caused by scattering of normal particles off the cores of the vortices. Our preliminary evaluation (Lindblom & Mendell 1992) suggests that this form of dissipation has a profound effect on the secular stability of rotating neutron stars. Third, the equations that describe the vortex and electromagnetic forces in mixtures of charged superfluids were derived by Mendell & Lindblom (1991), generalizing the earlier work of Holm & Kuperschmidt (1987).

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The explicit model of the neutron star superfluid which we use here was derived from the general superfluid mixture theory by Mendell (1991a, b). For temperatures well below the critical temperature, $T_c \approx 10^9$ K, this fluid consists of a mixture of superfluid neutrons, superconducting protons, and ordinary (nonsuperconducting) electrons (and muons). Our primary interest here is to obtain a description of the global oscillations of neutron stars. The equations that describe the dynamics of this material simplify considerably in this case because the oscillation frequencies of interest to us are much smaller than the plasma and cyclotron frequencies of the proton-electron plasma. In this case Mendell (1991a) showed (by taking the superfluid version of the magneto-hydrodynamic limit) that the electromagnetic forces merely enforce charge neutrality in this material. He also showed, using a version of the theory that is suitably averaged over many microscopic vortices, that the averaged effects of the vortices can be ignored when studying large-scale oscillations that have pressure as the dominant restoring force.

Our purpose here is to map out the main effects that superfluidity has on the global oscillations and stability of rotating neutron stars. We choose to study, therefore, a relatively simple model of these stars. We limit our discussion here to a strictly Newtonian description of the superfluid dynamics and the gravitational interaction. We consider stellar models whose interior is divided into only two distinct regions: a "core" consisting entirely of superfluid neutrons, superconducting protons, and normal electrons; and a "crust" consisting entirely of an ordinary fluid with barotropic equation of state. We choose as a model for the equation of state of the neutron star matter the "relativistic mean field" equation of state of Serot (1979) in the highest density regime, and the Harrison-Wheeler equation of state at lower densities. These (moderately realistic) equations of state were chosen because they have simple analytic representations which allow us to evaluate easily the complicated thermodynamic potentials that appear in the superfluid equations. We have also adopted an overly simple model of the neutron and proton vortices. This model ignores the intrinsic dynamics of the vortices, possible pinning effects, and other vortex-vortex interactions. Mendell (1991a) showed that vortex forces (based on this simple model) play no important role in the global oscillations of neutron stars. Thus we expect that for small amplitude global oscillations, the details of the vortex interactions will not change our results.

We begin our analysis, in § 2, by transforming Mendell's (1991a) equations for the dynamics of neutron star superfluid matter into an equivalent but analytically simpler form. This is achieved by solving analytically the equations for the velocity perturbations in terms of three scalar potentials. This reduction of the theory is the generalization to superfluid dynamics of the two-potential formalism developed by Ipser & Managan (1985), Managan (1985), and Ipser & Lindblom (1991) to describe the dynamics of ordinary fluids. In § 3 we derive the appropriate boundary conditions for the potentials which appear in our representation of the theory. Boundary conditions must be given at the interface between the superfluid and ordinary fluid portions of the star, at the surface between the interior and exterior of the star, and at spatial infinity. In § 4 we derive an action from which our representation of the superfluid equations may be derived as a variational principle. This action is an extremely useful tool for obtaining numerical estimates of the oscillation frequencies of these stars. In § 5 we discuss the equation of state which we adopt to describe the neutron star matter. We show in detail there how the various unusual thermodynamic potentials which appear in the superfluid equations may be determined. In § 6 we solve analytically the equations for the three scalar potentials in the very special case of spatially uniform and nonrotating superfluid matter. These elementary solutions reveal that the lowest frequency modes of the superfluid mixture in a neutron star are almost identical to those based on ordinary fluid hydrodynamics. These elementary solutions also reveal that the superfluid material admits a new set of modes having no ordinary-fluid counterparts. The frequencies of these new superfluid modes are somewhat higher than the ordinary-fluid modes. In § 7 we find numerical solutions to the three-potential equations for our "realistic" models of rotating neutron stars. These realistic solutions share the interesting feature of the elementary solutions studied in § 6, that the lowest frequency modes are almost indistinguishable from the analogous modes based on ordinary-fluid hydrodynamics. In § 8 we discuss the limitations of the present study and suggest future work. In the Appendix we give explicit spherical coordinate representations of the equations for the three scalar potentials. We also give there the simplified equations for the (possibly nonradial) perturbations of nonrotating neutron stars.

2. SUPERFLUID HYDRODYNAMICS

Our primary interest in this paper is to explore the effects of superfluidity on the dynamics of neutron star matter. Since these dynamical effects have not been thoroughly investigated to date, we limit our consideration here to a fairly simple model of this superfluid material. Our model is nevertheless sufficiently rich to include many interesting superfluid effects not present in an ordinary fluid. We consider the material in the core of our neutron star model to be a fluid composed of "free" neutrons, protons, and electrons (ignoring the few muons present). We assume that the temperature is sufficiently low (i.e., less than about 10^9 K) that the protons and neutrons have each completely condensed into a superfluid state. Mendell (1991a) has shown that two scalar and two vector fields determine the dynamics of this material for dynamical effects with timescales longer than the periods of the plasma and the cyclotron oscillations of the proton-electron plasma (i.e., longer than about 10^{-15} s), and shorter than the period of the oscillations of the superfluid vortices (i.e., shorter than about 10^{-1} s). In this limit (the charged-superfluid version of the magnitohy-drodynamic limit in an ordinary plasma) the electromagnetic forces merely enforce charge neutrality and the quantized superfluid vortices play no dynamical role. It is most natural to describe the perturbations of this material from an equilibrium state using as dynamical variables $\delta\rho_n$ and $\delta\rho_p$ (the perturbations in the neutron and proton densities), and δv_n^a and δu^a (the perturbations in the velocity of the neutrons and the average velocity of the combined proton-electron charged fluids). The equations for the evolution of these dynamical fields were derived by Mendell (1991a) and are given as follows:

$$0 = \partial_t \delta\rho_n + v^a \nabla_a \delta\rho_n + \nabla_a(\rho_{nn} \delta v_n^a + \rho_{np} \delta u^a) , \tag{1}$$

$$0 = \partial_t \delta\rho_p + v^a \nabla_a \delta\rho_p + \nabla_a(\rho_{pp} \delta u^a + \rho_{np} \delta v_n^a) , \tag{2}$$

$$0 = \partial_t \delta v_n^a + v^b \nabla_b \delta v_n^a + \delta v_n^b \nabla_b v^a + \nabla^a(\delta\mu_n + \delta\Phi) - 2\Omega \frac{\rho_{np}}{\rho_n} [(\delta \boldsymbol{u} - \delta \boldsymbol{v}_n) \times \hat{z}]^a , \tag{3}$$

$$0 = \partial_t \delta u^a + v^b \nabla_b \delta u^a + \delta u^b \nabla_b v^a + \nabla^a\left(\delta\mu_p + \frac{m_e}{m_p}\,\delta\mu_e + \delta\Phi\right) + 2\Omega\,\frac{\rho_{np}}{\rho_p}\,[(\delta \boldsymbol{u} - \delta \boldsymbol{v}_n) \times \hat{z}]^a\,, \tag{4}$$

$$\nabla^a \nabla_a \delta\Phi = 4\pi G\,\delta\rho\,. \tag{5}$$

In these equations the indices a, b, c, etc., take on values which correspond to the three spatial coordinates; summation on repeated indices is implied. The quantities in these equations with the prefix δ represent small Eulerian pertubations, while those without δ are to be evaluated in the equilibrium state. All the quantities have been smooth-averaged over a volume containing many vortices. In this case, the velocity of the fluids in the equilibrium state, denoted v^a, can be assumed to be that of rigid rotation. In cylindrical coordinates (ϖ, φ, z) for a velocity which represents a rotation with angular velocity Ω about the $\hat{z}$ direction, the equilibrium velocity is given by $v^a \equiv \Omega\varphi^a$, where $\varphi^a = \varpi\hat{\varphi}^a$. The total mass density of the fluid is given by $\rho = \rho_n + \rho_p$; we neglect terms of order m_e/m_p. The functions ρ_{nn}, ρ_{np}, and ρ_{pp} describe the "drag effect" between the protons and neutrons (see § 5); Galilean invariance of equations (1) and (2) requires that $\rho_{nn} + \rho_{np} = \rho_n$ and $\rho_{pp} + \rho_{np} = \rho_p$. The quantities $\delta\mu_n$, $\delta\mu_p$, and $\delta\mu_e$ represent perturbations in the chemical potentials (energy per unit mass) of the neutrons, protons, and electrons, respectively. An explicit model for all the thermodynamic variables is described in § 5. The perturbed gravitational potential is denoted $\delta\Phi$.

These dynamical equations (1)–(5) may be simplified by introducing new "velocity" fields

$$\delta v^a = \frac{\rho_n}{\rho}\,\delta v_n^a + \frac{\rho_p}{\rho}\,\delta u^a\,, \tag{6}$$

$$\delta w^a = \delta u^a - \delta v_n^a\,, \tag{7}$$

and new scalar potentials

$$\delta U = \frac{\delta p}{\rho} + \delta\Phi\,, \tag{8}$$

$$\delta\beta = \delta\mu_p - \delta\mu_n + \frac{m_e}{m_p}\,\delta\mu_e\,, \tag{9}$$

where p is the pressure defined in equation (16) below. In terms of these new variables the equations for the fluid velocities (3)–(4)

$$\partial_t \delta v^a + \Omega(\varphi^b \nabla_b \delta v^a + \delta v^b \nabla_b \varphi^a) = -\nabla^a \delta U + \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p \delta\beta \nabla^a p\,, \tag{10}$$

$$\partial_t \delta w^a + \Omega\left(\varphi^b \nabla_b \delta w^a - \delta w^b \nabla_b \varphi^a + 2\,\frac{\det\rho}{\rho_p \rho_n}\,\delta w^b \nabla_b \varphi^a\right) = -\nabla^a \delta\beta\,, \tag{11}$$

where $\det\rho = \rho_{nn}\rho_{pp} - \rho_{np}^2$. In equation (11) we have used the fact that

$$\beta = \mu_p - \mu_n + \frac{m_e}{m_p}\,\mu_e \tag{12}$$

vanishes (up to terms of order m_e/m_p) in the equilibrium state as a consequence of the β-equilibrium between the neutrons, protons, and electrons. The mass conservation laws (1)–(2) when written in terms of these new variables reduce to

$$0 = \partial_t \delta\rho + \Omega\varphi^a \nabla_a \delta\rho + \nabla_a(\rho\,\delta v^a)\,, \tag{13}$$

$$0 = (\partial_t + \Omega\varphi^a \nabla_a)\left[\left(\frac{\partial\rho}{\partial\beta}\right)_p (\delta U - \delta\Phi) + \frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p \delta\beta\right] + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p \delta v^a \nabla_a p + \nabla_a\left(\frac{\det\rho}{\rho}\,\delta w^a\right). \tag{14}$$

In order to put the equations into this form we have used the thermodynamic identity

$$\left(\frac{\partial\rho}{\partial\beta}\right)_p = \rho^2\,\frac{\partial}{\partial p}\left(\frac{\rho_p}{\rho}\right)_\beta\,. \tag{15}$$

Equation (15) is a Maxwell relation which may easily be derived from the following form of the first law of thermodynamics:

$$d\mu_n = \frac{dp}{\rho} - \frac{\rho_p}{\rho}\,d\beta\,. \tag{16}$$

Next, we assume that the perturbed quantities have time dependence $e^{-i\omega t}$ and angular dependence $e^{im\varphi}$ where the frequency of the mode ω is a constant, and m is an integer. For perturbations having this form, equations (10) and (11) reduce to algebraic equations for δv^a and $\delta\omega^a$:

$$iQ_{ab}^{-1}\,\delta v^b \equiv -i(\sigma g_{ab} + 2i\Omega\nabla_b \varphi_a)\delta v^b = -\nabla_a \delta U + \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p \delta\beta\nabla_a p\,, \tag{17}$$

$$i\tilde{Q}_{ab}^{-1}\,\delta w^b \equiv -i(\sigma g_{ab} + 2i\tilde{\Omega}\nabla_b \varphi_a)\delta w^b = -\nabla_a \delta\beta\,, \tag{18}$$

where $\sigma = \omega - m\Omega$; g_{ab} is the Euclidean metric tensor (i.e., the identity matrix in Cartesian coordinates) whose inverse is denoted g^{ab}; and where $\tilde{\Omega}$ is given by

$$\tilde{\Omega} = \frac{\det \rho}{\rho_p \rho_n} \Omega . \tag{19}$$

These equations can be solved simply for δv^a and δw^a in terms of the scalar potentials δU and $\delta\beta$,

$$\delta v^a = iQ^{ab}\left[\nabla_b \delta U - \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial \beta}\right)_p \delta\beta \nabla_b p\right], \tag{20}$$

$$\delta w^a = i\tilde{Q}^{ab}\nabla_b \delta\beta . \tag{21}$$

Equation (21) shows that differences between the velocities of the neutrons and protons in this superfluid mixture are driven by $\delta\beta$, the deviation of the fluid state from β-equilibrium. This equation only makes sense then for dynamical effects with timescales shorter than the weak interaction timescales. Estimates of the weak interaction timescale in neutron star matter by Epstein (1988) and Sawyer (1989) show that it is far longer than typical neutron star pulsation periods, so equation (21) does make sense for our purposes.

The tensors Q^{ab} and $\tilde{Q}^{ab}$, defined as the inverses of Q^{-1}_{ab} and $\tilde{Q}^{-1}_{ab}$ respectively, have fairly simple forms:

$$Q^{ab} = -\frac{\lambda}{\sigma} g^{ab} - \frac{1-\lambda}{\sigma} z^a z^b - 2i\frac{\Omega\lambda}{\sigma^2}\nabla^a \varphi^b , \tag{22}$$

$$\tilde{Q}^{ab} = -\frac{\tilde{\lambda}}{\sigma} g^{ab} - \frac{1-\tilde{\lambda}}{\sigma} z^a z^b - 2i\frac{\tilde{\Omega}\tilde{\lambda}}{\sigma^2}\nabla^a \varphi^b . \tag{23}$$

In these equations we have used the notation

$$\lambda = \frac{\sigma^2}{\sigma^2 - 4\Omega^2} , \tag{24}$$

$$\tilde{\lambda} = \frac{\sigma^2}{\sigma^2 - 4\tilde{\Omega}^2} . \tag{25}$$

We note that Q^{ab} and $\tilde{Q}^{ab}$ are Hermitian for real values of the frequency, that is, when σ is real $Q^{ab} = (Q^{ba})^*$ and $\tilde{Q}^{ab} = (\tilde{Q}^{ba})^*$ where * denotes complex conjugation.

The expressions for δv^a and δw^a may now be substituted into the mass conservation laws. The resulting equations together with the gravitational field equation form a system of equations for the three scalar potentials δU, $\delta\beta$, and $\delta\Phi$:

$$\nabla_a(\rho Q^{ab}\nabla_b \delta U) - \sigma\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta \delta U = \nabla_a\left[\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \beta}\right)_p \delta\beta Q^{ab}\nabla_b p\right] + \sigma\left(\frac{\partial\rho}{\partial\beta}\right)_p \delta\beta - \sigma\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta \delta\Phi , \tag{26}$$

$$\nabla_a(\tilde{\rho}\tilde{Q}^{ab}\nabla_b \delta\beta) - \left[\sigma\frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p + \frac{1}{\rho^3}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2 Q^{ab}\nabla_a p \nabla_b p\right]\delta\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p Q^{ab}\nabla_a p \nabla_b \delta U + \sigma\left(\frac{\partial\rho}{\partial\beta}\right)_p (\delta U - \delta\Phi) , \tag{27}$$

$$\nabla^a\nabla_a \delta\Phi + 4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta \delta\Phi = 4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta \delta U + 4\pi G\left(\frac{\partial\rho}{\partial\beta}\right)_p \delta\beta , \tag{28}$$

where we have defined $\tilde{\rho} = \det \rho/\rho$. These equations represent a significant reduction in complexity over the original system of dynamical equations (1)–(5). The original system was tenth order (consisting of two first-order scalar, two first-order vector, and one second-order scalar equations) while the reduced system (26)–(28) is sixth order. While the abstract covariant form of the equations used here is very convenient for derivations and manipulations, it is not very useful for finding explicit solutions. For that purpose it is necessary to represent the equations in some explicit coordinate system. The spherical coordinate representations of these equations are given in the Appendix. For the special case of nonrotating neutron stars these equations simplify considerably. They reduce to a system of second-order ordinary differential equations for the radial parts of δU, $\delta\beta$, and $\delta\Phi$. These ordinary differential equations are also given explicitly in the Appendix.

3. BOUNDARY CONDITIONS

The matter in the core of our neutron-star model is taken to be a superfluid mixture of "free" neutrons, protons, and electrons whose dynamics is described by equations (26)–(28). In the lower density outer regions of a neutron star the matter consists almost entirely of heavy nuclei. In our simple model we assume that all of the matter below some critical density ρ_s is an ordinary perfect fluid (as has been done in all previous studies of neutron star pulsations), while above this critical density all of the matter is a superfluid mixture satisfying equations (26)–(28). The equations that describe the pulsations of a perfect fluid (Ipser & Lindblom 1990) are given by

$$\nabla_a(\rho Q^{ab}\nabla_b \delta U) - \sigma\rho\left(\frac{\partial\rho}{\partial p}\right)_s \delta U = -\sigma\rho\left(\frac{\partial\rho}{\partial p}\right)_s \delta\Phi , \tag{29}$$

$$\nabla^a\nabla_a \delta\Phi + 4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_s \delta\Phi = 4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_s \delta U . \tag{30}$$

These equations are a degenerate special case of equations (26)–(28) (obtained formally by setting $\delta\beta = 0$ in equations [26] and [28]). Note that the partial derivatives in equations (29) and (30) are evaluated at constant s, the entropy per particle. In the very low temperature equilibrium states of interest to us $s = 0$, and because of β-equilibrium $\beta = 0$. In our model we treat the partial derivatives, at constant s, that appear in equations (29)–(30) as identical to the partial derivatives, at constant β, that appear in equations (26)–(28).

To find the global physical solutions to the equations which describe the dynamics of the entire neutron star, (26)–(28) together with (29)–(30), we must specify the boundary conditions to be satisfied by δU, $\delta\beta$, and $\delta\Phi$ on three different boundary surfaces. Boundary conditions must be given on the surface between the superfluid core and the outer region containing ordinary fluid, on the actual surface of the star where the ordinary fluid meets the vacuum exterior region, and on the surface at spatial infinity.

We consider first the boundary conditions at the surface which joins the superfluid core material with the ordinary-fluid outer layers of the star. These boundary conditions are obtained by insisting that the macroscopic conservation laws (e.g., mass and momentum conservation) are satisfied at this boundary. The mass conservation law will be satisfied at the boundary between the superfluid and the ordinary fluid whenever the perpendicular component of the total mass current is continuous. Thus,

$$n_a[\rho\,\delta v^a + \delta\rho v^a]_s = n_a[\rho\,\delta v^a + \delta\rho v^a]_o\,, \tag{31}$$

where n_a is the unit normal to the boundary surface; and the brackets $[\ ]_s$ and $[\ ]_o$ indicate that the quantity is to be evaluated on the superfluid side or the ordinary-fluid side of the boundary surface, respectively. We assume that the boundary between the superfluid and ordinary-fluid regions is a surface of constant density: $\rho = \rho_s$. In this case the velocity of the fluid in the equilibrium state, v^a, is orthogonal to n_a: $n_a v^a = 0$. Since the equilibrium mass density ρ is continuous across this boundary surface, the continuity of the total mass current, equation (31), implies

$$n_a[\delta v^a]_s = n_a[\delta v^a]_o\,. \tag{32}$$

The conservation of the total momentum of the fluid at the boundary surface requires the continuity of the normal components of the stress tensor at this surface:

$$n_a[\delta\pi^{ab}]_s = n_a[\delta\pi^{ab}]_o\,. \tag{33}$$

Using the expressions for the superfluid stress tensor (Mendell & Lindblom 1991; Mendell 1991a), and the continuity of the mass currents, equation (31), it follows from equation (33) that the pressure perturbation must be continuous across the boundary:

$$[\delta p]_s = [\delta p]_o\,. \tag{34}$$

The physical nonsingular solutions to equations (26)–(28) and (29)–(30) have bounded perturbations in all of the physical quantities. In particular the mass density perturbation $\delta\rho$ must be bounded. Thus, the perturbed gravitational potential $\delta\Phi$ must be C^1 in order for equation (5) to be satisfied. This implies in particular that

$$[\delta\Phi]_s = [\delta\Phi]_o\,, \tag{35}$$

$$n^a[\nabla_a\delta\Phi]_s = n^a[\nabla_a\delta\Phi]_o\,, \tag{36}$$

at the boundary between the superfluid and the orindary-fluid regions. From the definition of δU in equation (8) and the boundary conditions (34) and (35) it follows then that δU must be continuous across the boundary between the superfluid and the ordinary fluid:

$$[\delta U]_s = [\delta U]_o\,. \tag{37}$$

This condition when combined with equations (32), (20) and its ordinary-fluid analogue, determines the discontinuity in the normal derivative of δU at the boundary surface:

$$n^a[\nabla_a\delta U]_s - \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p n^a\nabla_a p[\delta\beta]_s = n^a[\nabla_a\delta U]_o\,. \tag{38}$$

We note that the continuity of δU implies that the tangential components of the gradient of δU must be continuous across the boundary surface.

These equations are a degenerate special case of equations (26)–(28) (obtained formally by setting $\delta\beta = 0$ in equations [26] and [28]). Note that the partial derivatives in equations (29) and (30) are evaluated at constant s, the entropy per particle. In the very low temperature equilibrium states of interest to us $s = 0$, and because of β-equilibrium $\beta = 0$. In our model we treat the partial derivatives, at constant s, that appear in equations (29)–(30) as identical to the partial derivatives, at constant β, that appear in equations (26)–(28).

$$n_a[\rho_e\,\delta v_e^a]_s = n_a[\rho_e\,\delta v^a]_o\,. \tag{39}$$

The velocity of the electrons in the superfluid mixture can be expressed in terms of the variables used in this paper using equations from Mendell (1991a), in particular

$$\delta v_e^a = \delta v^a + \frac{\det\rho}{\rho\rho_p}\,\delta w^a\,. \tag{40}$$

Thus, the continuity of the mass density of electrons in the equilibrium state, and the continuity of the component of the electron mass current that is normal to the boundary surface implies that

$$n_a[\delta w^a]_s = 0 \ . \tag{41}$$

This condition and equation (21) determine in turn a condition on the normal derivative of $\delta\beta$ on the boundary surface:

$$n^a[\nabla_a \delta\beta]_s - \frac{4\tilde{\Omega}^2}{\sigma^2} z^b n_b z^a[\nabla_a \delta\beta]_s - \frac{2m\tilde{\Omega}}{\sigma\varpi} \varpi^b n_b[\delta\beta]_s = 0 \ . \tag{42}$$

These boundary conditions imply that the proton and neutron mass currents also have continuous normal components, thus ensuring the conservation of these species as well.

The final boundary conditions constrain the perturbations at the surface of the star and at spatial infinity. To ensure that the perturbation in the total mass density $\delta\rho$ is bounded, we insist that $\delta\Phi$ and its gradient must be continuous on the surface of the star. This is the exact analogue of equations (35) and (36) applied at the actual surface of the star. We also impose the condition that the time-dependent pressure remain zero on the perturbed surface of the star. This requires in particular that

$$0 = \delta U - \delta\Phi - \frac{1}{\sigma} Q^{ab}[\varpi\Omega^2\nabla_a \varpi - \nabla_a \Phi]\nabla_b \delta U \ , \tag{43}$$

be satisfied at the surface of the star (see Ipser & Lindblom 1990 for more details). Finally, we demand that the perturbed gravitational potential falls to zero faster than $1/r$ near spatial infinity:

$$\lim_{r\to\infty} r\,\delta\Phi = 0 \ . \tag{44}$$

This condition ensures that the perturbed star has the same total mass as the unperturbed equilibrium state.

4. A VARIATIONAL PRINCIPLE

It is often helpful to have, as an aid in solving equations (26)–(28), a variational principle from which the frequencies of the modes may be estimated. Let us define a quadratic form for superfluid perturbation fields $\delta X \equiv (\delta U, \delta\beta, \delta\Phi)$ having temporal and angular dependence $e^{-i\omega t + im\varphi}$. Let δX and $\delta\hat{X}$ denote independent perturbation fields, then we define the form

$$\begin{aligned} S_s(\delta\hat{X}, \delta X) = \int_{\Sigma_s} \Bigg\{ & \frac{\rho}{\sigma} Q^{ab}\nabla_a \delta\hat{U}^*\nabla_b \delta U + \frac{\tilde{\rho}}{\sigma} \tilde{Q}^{ab}\nabla_a \delta\hat{\beta}^*\nabla_b \delta\beta - \frac{1}{4\pi G} \nabla^a\delta\hat{\Phi}^*\nabla_a \delta\Phi \\ & + \rho\left(\frac{\partial\rho}{\partial p}\right)_\beta (\delta\hat{U}^* - \delta\hat{\Phi}^*)(\delta U - \delta\Phi) + \left[\frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p + \frac{1}{\sigma\rho^3}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2 Q^{ab}\nabla_a p\nabla_b p\right]\delta\hat{\beta}^*\,\delta\beta \\ & - \frac{1}{\rho\sigma}\left(\frac{\partial\rho}{\partial\beta}\right)_p \nabla_a p(\delta\beta Q^{ba}\nabla_b \delta\hat{U}^* + \delta\hat{\beta}^* Q^{ab}\nabla_b \delta U) + \left(\frac{\partial\rho}{\partial\beta}\right)_p [(\delta\hat{U}^* - \delta\hat{\Phi}^*)\delta\beta + \delta\hat{\beta}^*(\delta U - \delta\Phi)]\Bigg\} d^3x \ . \end{aligned} \tag{45}$$

This integral is to be performed over the region Σ_s containing superfluid material within the neutron star, that is where, $\rho \geq \rho_s$. In the Appendix we give an explicit representation of S_s is spherical coordinates. Note that this quadratic form is Hermitian for real values of the the frequency σ:

$$S_s(\delta\hat{X}, \delta X) = S_s^*(\delta X, \delta\hat{X}) \ . \tag{46}$$

This quadratic form reduces to a surface integral if δX is a solution to the superfluid perturbation equations, (26)–(28), with the given value of the frequency:

$$S_s(\delta\hat{X}, \delta X) = -\int_{\partial\Sigma_s} \left(i\frac{\rho}{\sigma}\,\delta\hat{U}^*\,\delta v^a + i\frac{\tilde{\rho}}{\sigma}\,\delta\hat{\beta}^*\,\delta w^a + \frac{1}{4\pi G}\,\delta\hat{\Phi}^*\nabla^a\,\delta\Phi\right) d^2S_a \ , \tag{47}$$

where the integral on the right is performed on the surface $\partial\Sigma_s$, the boundary of Σ_s; and d^2S_a is the two-dimensional volume element on this boundary surface. An analogous quadratic form may be defined (see Ipser & Lindblom 1990) for perfect-fluid perturbation fields:

$$S_o(\delta\hat{X}, \delta X) \equiv \int_{\Sigma_o} \left\{\frac{\rho}{\sigma} Q^{ab}\nabla_a \delta\hat{U}^*\nabla_b \delta U - \frac{1}{4\pi G}\nabla^a\,\delta\hat{\Phi}^*\nabla_a \delta\Phi + \rho\left(\frac{\partial\rho}{\partial p}\right)_s (\delta\hat{U}^* - \delta\hat{\Phi}^*)(\delta U - \delta\Phi)\right\} d^3x \ . \tag{48}$$

This integral is to be performed on the region Σ_o consisting of the outer portions of the stellar model (composed of ordinary fluid where $\rho \leq \rho_s$), plus the entire vacuum exterior of the star. This quadratic form is also Hermitian for real values of the frequency,

$$S_o(\delta\hat{X}, \delta X) = S_o^*(\delta X, \delta\hat{X}) \ , \tag{49}$$

and also reduces to a surface integral when δX solves the ordinary-fluid pulsation equations, (29) and (30), for a given value of the frequency σ:

$$S_o(\delta\hat{X}, \delta X) = -\int_{\partial\Sigma_o} \left(i\frac{\rho}{\sigma}\,\delta\hat{U}^*\,\delta v^a + \frac{1}{4\pi G}\,\delta\hat{\Phi}^*\nabla^a\,\delta\Phi\right) d^2S_a \ . \tag{50}$$

The boundary $\partial\Sigma_o$ has two disconnected components: one component at the superfluid ordinary-fluid interface, the second component at spatial infinity.

We may define a global quadratic form by combining the superfluid form S_s from equation (45) with the ordinary-fluid form S_o from equation (48),

$$S(\delta\hat{X}, \delta X) = S_s(\delta\hat{X}, \delta X) + S_o(\delta\hat{X}, \delta X) . \tag{51}$$

When δX (or $\delta\hat{X}$) satisfies the pulsation equations then S_s and S_o reduce to boundary integrals, as we have seen. The boundary integral over $\partial\Sigma_s$ cancels the portion of the $\partial\Sigma_o$ integral at the superfluid interface whenever the boundary conditions (32), (35)–(37), and (41) are satisfied. The portion of the $\partial\Sigma_o$ boundary integral at spatial infinity vanishes whenever the boundary condition (44) is satisfied. Thus, whenever δX (or $\delta\hat{X}$) satisfies the pulsation equations, the combined quadratic form vanishes,

$$S(\delta\hat{X}, \delta X) = 0 . \tag{52}$$

It can be shown using these properties that the functional

$$S_\sigma(\delta X) \equiv S(\delta X, \delta X) \tag{53}$$

is an action from which the pulsation equations may be derived as a variational principle.

We may use S_σ to estimate the frequencies of the modes. For a given estimate of the perturbation eigenfunction δX we may set

$$S_\sigma(\delta X) = 0 , \tag{54}$$

and solve for σ. Since S_σ is an action, the frequency determined in this way will be more accurate than the initial estimate of δX. Equation (54) may also be used as a powerful test of the accuracy of any direct solution of the pulsation equations. After the eigenfunctions δX are determined by solving equations (26)–(28) and (29)–(30) directly, their accuracy can be tested by solving equation (54) for a "variational frequency" associated with δX. This "variational frequency" should be identical to the frequency determined as part of the direct solution of the equations. Its deviation from the directly determined frequency is one measure, then, of the accuracy of the solution.

5. THE EQUATION OF STATE

Before we can solve the equations for the pulsations of a superfluid neutron star, we must determine the equation of state of neutron star matter, including all of the thermodynamic quantities that appear in the pulsation equations. Our primary purpose in this paper is to investigate the role that superfluidity plays in the pulsations of neutron stars. Therefore, we choose a model for the equation of state that balances our interest in having an accurate realistic description of a neutron star, with our interest in having a simple description from which all of the needed thermodynamic quantities can easily be determined with sufficient accuracy. We model the lowest density neutron star matter, $\rho \leq 2 \times 10^{12}$ g cm^{-3}, with the Harrison-Wheeler equation of state. This model consists of neutron-rich nuclei in β-equilibrium with free electrons and at sufficiently high density with free neutrons. The energy states of the nuclei in this model are determined from the semiempirical mass formula of Green (1955). Shapiro & Teukolsky (1983) give in § 2.6 a succinct discussion of the equations that determine this model. For the highest density portion of the neutron-star matter, $\rho \geq 3 \times 10^{13}$ g cm^{-3}, we use a variation of the Serot (1979) equation of state. This model, described in more detail below, consists of strongly interacting neutrons and protons in β-equilibrium with free electrons. For intermediate densities, $2 \times 10^{12} \leq \rho \leq 3 \times 10^{13}$ g cm^{-3}, we simply interpolate between the Harrison-Wheeler and the Serot models with the polytropic equation of state,

$$p = 8.49 \times 10^{16} \rho^{1.09} , \tag{55}$$

where the density and pressure are in cgs units. Figure 1 illustrates the pressure-density relationship for our amalgam equation of state. The dots are located at the endpoints of the domain of the interpolated equation of state. The pressure determined from equation (55) differs for a given density by no more than 30% from either the Harrison-Wheeler or the Serot equation of state in this intermediate density region.

The most important portion of the equation of state from our perspective is the highest density region: $\rho \geq 3 \times 10^{13}$ g cm^{-3}. In a 1.4 $M_\odot$ neutron star 99.6% of the matter has densities above this value, and all of the superfluid material has densities above this value. In this density region we have chosen to use the Serot (1979) equation of state because of its relative analytic simplicity. This simplicity allows us to compute the various thermodynamic quantities needed in the superfluid pulsation equations in a straightforward manner. The version of the Serot equation of state that we use here differs somewhat from the original: we determine the ratio of neutrons to protons in the material by imposing β-equilibrium rather than assuming pure neutron matter. Since our version of the equation of state differs somewhat from Serot (1979) and since we require from the equation of state several thermodynamic quantities not discussed by Serot, we now review the equations that determine this equation of state in some detail.

It is convenient to parameterize the state of the nuclear-matter fluid in terms of the Fermi momenta of the protons and neutrons, k_p and k_n, respectively. These determine the mass densities by the standard relationships:

$$\rho_p = \frac{m_p k_p^3}{3\pi^2\hbar^3} , \tag{56}$$

$$\rho_n = \frac{m_n k_n^3}{3\pi^2\hbar^3} . \tag{57}$$

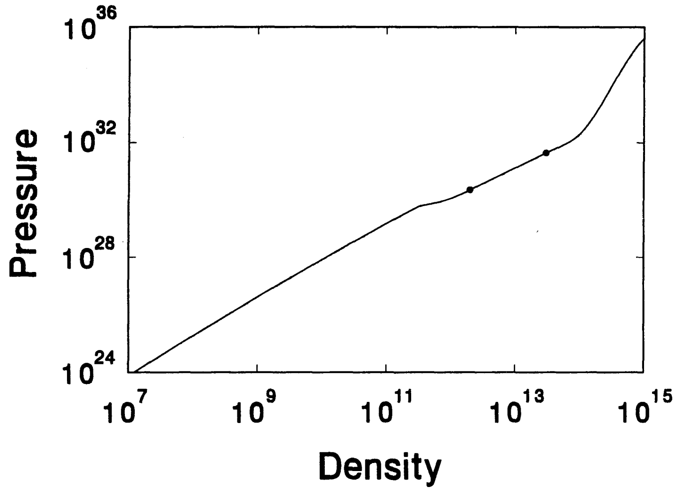

FIG. 1.—Equation of state of neutron star matter. The lowest density matter is described by the Harrison-Wheeler equation of state; the highest density matter, by the Serot equation of state. The pressures and densities are displayed in cgs units.

The condition of charge neutrality then determines the mass density of the electrons in terms of k_p,

$$\rho_e = \frac{m_e k_p^3}{3\pi^2 \hbar^3} . \tag{58}$$

For any given values of k_p and k_n the Serot equation of state provides an expression for the energy density and pressure that is based on a relativistic mean-field theory including π- and ρ-meson interactions. The equilibrium state of the matter is determined in four steps. First, an effective mass m_* is determined for given k_p and k_n by solving the equation

$$m_* = m_b - 534.2 \frac{m_*^3}{m_b^2} \left[\psi\left(\frac{k_n}{m_* c}\right) + \psi\left(\frac{k_p}{m_* c}\right) \right] , \tag{59}$$

where $m_b \approx m_p \approx m_n = 1.675 \times 10^{-24}$ g and the function $\psi(x)$ is defined by

$$\psi(x) = \frac{1}{4\pi^2} [x\sqrt{1 + x^2} - \ln(x + \sqrt{1 + x^2})] . \tag{60}$$

While equation (59) is intractable analytically, it can be solved numerically for m_* in a straightforward manner. The second step determines the energy density of the nuclear matter for given k_p and k_n from the expression

$$\epsilon = \frac{m_*^4 c^5}{\hbar^3} \left[\chi\left(\frac{k_n}{m_* c}\right) + \chi\left(\frac{k_p}{m_* c}\right) \right] + \frac{(k_n^3 + k_p^3)^2}{8.950 m_b^2 \hbar^3 c} + \frac{m_b^2 (m_b - m_*)^2 c^5}{534.2 \hbar^3} + \frac{(k_n^3 - k_p^3)^2}{128.2 m_b^2 \hbar^3 c} , \tag{61}$$

where $\chi(x)$ is the function that determines the energy density of an ideal Fermi gas,

$$\chi(x) = \frac{1}{8\pi^2} [x\sqrt{1 + x^2}(1 + 2x^2) - \ln(x + \sqrt{1 + x^2})] . \tag{62}$$

Serot (1979) fixed the values of the numerical coupling constants in the interaction terms in the energy by fitting observed properties of symmetric nuclear matter. The third step imposes β-equilibrium to determine the relative number of neutrons and protons in the nuclear matter at a given density. This is equivalent to determining the value of k_p for a given value k_n. We do this by determining from equation (61) the chemical potentials of the neutrons, protons, and electrons:

$$\mu_n = \left(\frac{\partial \epsilon}{\partial \rho_n}\right)_{\rho_p} , \tag{63}$$

$$\mu_p = \left(\frac{\partial \epsilon}{\partial \rho_p}\right)_{\rho_n} , \tag{64}$$

$$\mu_e = c^2 \sqrt{1 + \frac{k_p^2}{m_e^2 c^2}} . \tag{65}$$

It is straightforward to evaluate the partial derivatives expressed in equations (63) and (64) either numerically or analytically (although the analytic expressions are rather lengthy). We impose β-equilibrium by requiring that these chemical potentials satisfy

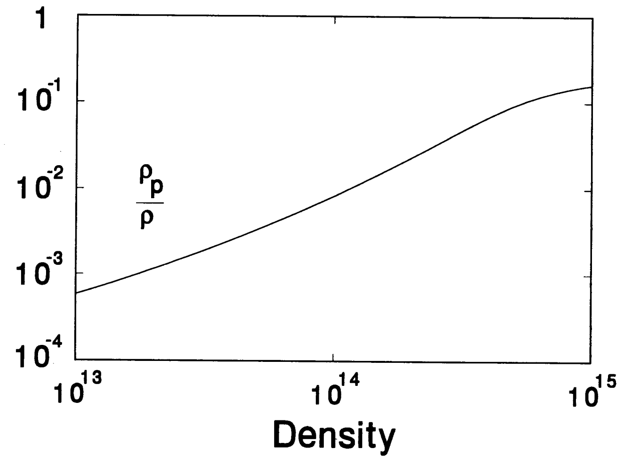

FIG. 2.—The fractional density of protons which are in beta equilibrium with neutrons and electrons, using the Serot equation of state

the condition

$$m_e \mu_e + m_p \mu_p = m_n \mu_n . \tag{66}$$

While equation (66) is too complicated to be solved analytically, it is easy to solve it numerically for k_p for any given value of k_n. Figure 2 illustrates the fractional density of protons as a function of the total mass density for the nuclear matter determined in this way. The fourth and final step in determining the basic equation of state using Serot's prescription is to evaluate the pressure of the combined neutron, proton, and electron fluid:

$$p = \frac{m_*^4 c^5}{\hbar^3}\left[\phi\left(\frac{k_n}{m_* c}\right) + \phi\left(\frac{k_p}{m_* c}\right)\right] + \frac{m_e^4 c^5}{\hbar^3}\phi\left(\frac{k_p}{m_e c}\right) + \frac{(k_n^3 + k_p^3)^2}{8.950 m_b^2 \hbar^3 c} - \frac{m_b^2(m_b - m_*)^2 c^5}{534.2\hbar^3} + \frac{(k_n^3 - k_p^3)^2}{128.2 m_b^2 \hbar^3 c}, \tag{67}$$

where $\phi(x)$ is the function that determines the pressure in an ideal Fermi gas,

$$\phi(x) = \frac{1}{8\pi^2}\left[x\sqrt{1+x^2}\left(\frac{2x^2}{3} - 1\right) + \ln(x + \sqrt{1+x^2})\right]. \tag{68}$$

Figure 1 illustrates the pressure-density relationship obtained in this way for densities in the range $3 \times 10^{13} \leq \rho \leq 10^{15}$ g cm^{-3}. We note that the Fermi energies of the nucleons are sufficiently low and the binding energies sufficiently small that there is no significant difference between the total mass density and the total energy density (divided by c^2) of this material for densities below 10^{15} g cm^{-3}.

The equations that describe the pulsations of superfluid neutron star matter involve a number of thermodynamic derivatives which must be determined in addition to the basic equation of state. These partial derivatives, for example, $(\partial\rho/\partial\beta)_p$, may be expressed in terms of the chemical potentials defined in equations (63)–(65) and their derivatives, which are straightforward to evaluate. The relationships between these derivatives are obtained by changing thermodynamic variables from the set (ρ_n, ρ_p) to (p, β), using the first law of thermodynamics equation (16), and by employing the usual arsenal of partial-derivative identities. The resulting relationships are

$$\left(\frac{\partial\rho}{\partial p}\right)_\beta = \frac{\Delta}{\rho}\left[\left(\frac{\partial\mu_n}{\partial\rho_n}\right)_{\rho_p} - 2\left(\frac{\partial\mu_n}{\partial\rho_p}\right)_{\rho_n} + \left(\frac{\partial\mu_p}{\partial\rho_p}\right)_{\rho_n} + \frac{m_e^2}{m_p^2}\frac{d\mu_e}{d\rho_e}\right], \tag{69}$$

$$\left(\frac{\partial\rho}{\partial\beta}\right)_p = \frac{\Delta}{\rho}\left[\rho_n\left(\frac{\partial\mu_n}{\partial\rho_n}\right)_{\rho_p} - \rho_p\left(\frac{\partial\mu_p}{\partial\rho_p}\right)_{\rho_n} + (\rho_p - \rho_n)\left(\frac{\partial\mu_n}{\partial\rho_p}\right)_{\rho_n} - \rho_p\frac{m_e^2}{m_p^2}\frac{d\mu_e}{d\rho_e}\right], \tag{70}$$

$$\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p = \frac{\Delta}{\rho}\left[\left(\frac{\partial\mu_n}{\partial\rho_n}\right)_{\rho_p} + 2\frac{\rho_p}{\rho_n}\left(\frac{\partial\mu_n}{\partial\rho_p}\right)_{\rho_n} + \frac{\rho_p^2}{\rho_n^2}\left(\frac{\partial\mu_p}{\partial\rho_p}\right)_{\rho_n} + \frac{\rho_p^2 m_e^2}{\rho_n^2 m_p^2}\frac{d\mu_e}{d\rho_e}\right], \tag{71}$$

where Δ is define by the expression

$$\frac{1}{\Delta} = \left(\frac{\partial\mu_n}{\partial\rho_n}\right)_{\rho_p}\left[\left(\frac{\partial\mu_p}{\partial\rho_p}\right)_{\rho_n} + \frac{m_e^2}{m_p^2}\frac{d\mu_e}{d\rho_e}\right] - \left(\frac{\partial\mu_n}{\partial\rho_p}\right)_{\rho_n}^2. \tag{72}$$

Figures 3 and 4 illustrate the values of these thermodynamic derivatives as evaluated in this way for the Serot equation of state.

Finally, to complete the set of thermodynamic functions needed to evaluate the pulsations of superfluid neutron stars, we must determine the function det ρ that controls the "drag effect" between the neutron and proton superfluids. The analysis of Andreev & Bashkin (1976) can be used to relate the coefficients ρ_{nn}, ρ_{np}, and ρ_{pp} to the effective masses of the neutrons and protons (see also Alpar, Langer, & Sauls 1984; Sauls 1989). These expressions can be used in turn to determine det ρ in terms of the effective mass of

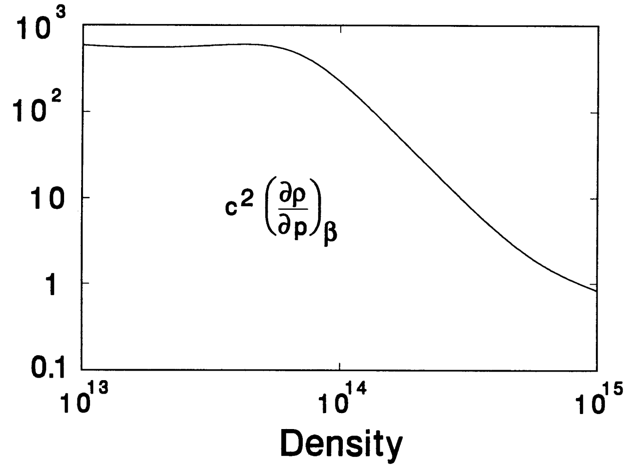

FIG. 3

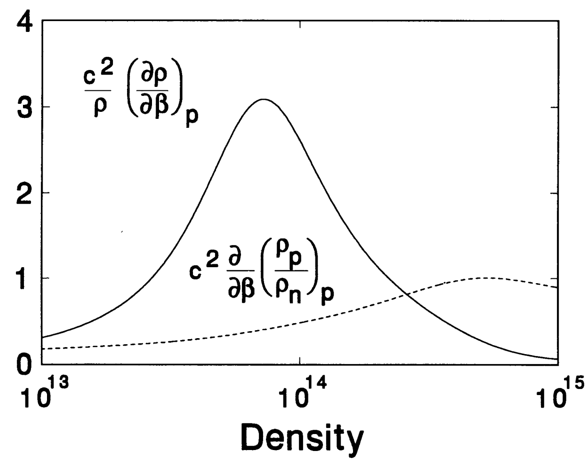

FIG. 4

FIG. 3.—The thermodynamic derivative $(\partial\rho/\partial p)_\beta$ based on the Serot equation of state. This quantity is dimensionless when multiplied by c^2, the speed of light squared.

FIG. 4.—Two thermodynamic derivatives which determine the dependence of the density functions on the parameter β using the Serot equation of state. The graphed quantities are dimensionless.

the protons:

$$\frac{\det \rho}{\rho_p \rho_n} = \frac{m_p}{m_p^*} + \left(\frac{m_p}{m_p^*} - 1\right)\frac{\rho_p}{\rho_n} . \tag{73}$$

The Serot equation of state does not provide expressions for individual effective masses of the neutrons and protons separately; rather it provides a single effective mass for the nucleons as an aggregate. This model is too simple to provide a realistic description of the protons for our purpose. Therefore, we have adopted the values for the proton effective mass computed in a more complete picture of nuclear matter by Sjöberg (1976). We find that the following simple analytic expressions fit his values for the proton effective mass to within about 1%:

$$\frac{m_p^*}{m_p} = \begin{cases} 0.588 - 0.447y + 0.210y^2 & \text{model A ,} \\ 0.528 - 0.475y + 0.228y^2 & \text{model B ,} \end{cases} \tag{74}$$

where $y = \log_{10}(\rho/10^{14})$. Figure 5 illustrates the function det ρ that results from these values of the effective mass.

6. ELEMENTARY SOLUTIONS

The full superfluid oscillation equations, (26)–(28), are extremely complicated. It is helpful to gain some insight into the nature of their solutions by exploring a very simple special case. In this section we find solutions to equations (26)–(28) for the special case of a

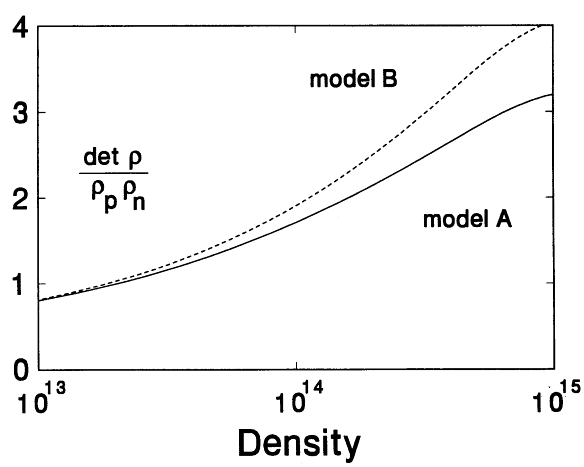

FIG. 5.—The dimensionless quantity det $\rho/\rho_p\rho_n$ which governs the "drag effect" between the proton and neutron superfluids. The quantity depicted here is based on two models for the effective mass of the proton in superfluid neutron star matter as computed by Sjöberg.

spatially uniform nonrotating equilibrium state. In this case equations (26)–(28) simplify considerably:

$$\nabla^a\nabla_a \delta U = -\omega^2\left(\frac{\partial\rho}{\partial p}\right)_\beta (\delta U - \delta\Phi) - \frac{\omega^2}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p \delta\beta\,, \tag{75}$$

$$\nabla^a\nabla_a \delta\beta = -\omega^2 \frac{\rho_n^2}{\det\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p \delta\beta - \frac{\omega^2\rho}{\det\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p (\delta U - \delta\Phi)\,, \tag{76}$$

$$\nabla^a\nabla_a \delta\Phi = 4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta (\delta U - \delta\Phi) + 4\pi G\left(\frac{\partial\rho}{\partial\beta}\right)_p \delta\beta\,. \tag{77}$$

Note that the thermodynamic functions appearing on the right sides of equations (75)–(77) are constants in this simple model. Analytical solutions to this system of equations can easily be found. Separating in spherical coordinates we find the general solution (which is bounded at $r = 0$) to be a sum of terms of the form

$$\delta U = [\delta A r^l + \delta B j_l(k_+ r) + \delta C j_l(k_- r)] Y_{lm} e^{-i\omega t}\,, \tag{78}$$

$$\delta\beta = [D_+(\omega)\delta B j_l(k_+ r) + D_-(\omega)\delta C j_l(k_- r)] Y_{lm} e^{-i\omega t}\,, \tag{79}$$

$$\delta\Phi = \left\{\delta A r^l - \frac{4\pi G\rho}{\omega^2}[\delta B j_l(k_+ r) + \delta C j_l(k_- r)]\right\} Y_{lm} e^{-i\omega t}\,, \tag{80}$$

where δA, δB, and δC are arbitrary constants; the quantities $D_\pm(\omega)$ are defined by

$$D_\pm(\omega) = \rho\left(\frac{\partial\beta}{\partial p}\right)_p\left[\frac{k_\pm^2}{\omega^2} - \left(\frac{\partial\rho}{\partial p}\right)_\beta\left(1 + \frac{4\pi G\rho}{\omega^2}\right)\right]; \tag{81}$$

and $k_\pm$ are the two roots of the "dispersion relation,"

$$0 = \omega^2(\omega^2 + 4\pi G\rho)\left[\left(\frac{\partial\rho}{\partial\beta}\right)_p^2 - \rho_n^2\left(\frac{\partial\rho}{\partial p}\right)_\beta \frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\right] + \omega^2\left[\det\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta + \rho_n^2\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\right]k^2 + \det\rho\left[4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta - k^2\right]k^2\,, \tag{82}$$

for a given value of the frequency ω. The functions j_l are spherical Bessel functions, and Y_{lm} are the spherical harmonics.

Appropriate boundary conditions must be imposed in order to determine the "physical" oscillations in this simple model. Thus, we consider solutions limited to the interior of a sphere, $r \le R$, which satisfy the analogs of the physical boundary conditions discussed in § 3 at $r = R$:

$$\delta U - \delta\Phi - \frac{4\pi G\rho}{3\omega^2} R \frac{d\delta U}{dr} = 0\,. \tag{83}$$

$$\frac{d\delta\Phi}{dr} + \frac{l+1}{R}\delta\Phi = 0\,, \tag{84}$$

$$\frac{d\delta\beta}{dr} = 0\,. \tag{85}$$

When these boundary conditions are imposed on the solutions in equations (78)–(80), a linear homogeneous system of equations for the constants δA, δB, and δC is obtained. These equations have nontrivial solutions only for the particular values of the frequency that are the roots of the equation,

$$0 = f_s(\omega) = \frac{R}{3}\left(2l + 1 + l\frac{4\pi G\rho}{\omega^2}\right)(k_+^2 - k_-^2)k_+ k_- j_l'(k_+ R) j_l'(k_- R) - \left[(2l+1)\left(1 + \frac{\omega^2}{4\pi G\rho}\right) - l(l+1)\frac{4\pi G\rho}{3\omega^2}\right]$$
$$\times\left\{\left[k_+^2 - \left(\frac{\partial\rho}{\partial p}\right)_\beta(\omega^2 + 4\pi G\rho)\right]k_+ j_l(k_- R) j_l'(k_+ R) - \left[k_-^2 - \left(\frac{\partial\rho}{\partial p}\right)_\beta(\omega^2 + 4\pi G\rho)\right]k_- j_l(k_+ R) j_l'(k_- R)\right\}. \tag{86}$$

The roots of $f_s(\omega) = 0$ are easily determined numerically. We present in Table 1 the roots of this equation using values of the parameters that are typical of neutron stars: $\rho = 4 \times 10^{14}$ g cm^{-3}, and $R = 15$ km. The thermodynamic derivatives needed in these equations are obtained, for the selected value of the density, from the equation of state described in § 5. For comparison, we also give in Table 1 the roots of the function, $f_o(\omega)$, that describe the oscillations of an ordinary-fluid star in this same simple approximation:

$$0 = f_o(\omega) = \frac{R}{3}\left(2l + 1 + l\frac{4\pi G\rho}{\omega^2}\right)k j_l'(kR) - \left[(2l+1)\left(1 + \frac{\omega^2}{4\pi G\rho}\right) - l(l+1)\frac{4\pi G\rho}{3\omega^2}\right]j_l(kR)\,, \tag{87}$$

where $k^2 = (\omega^2 + 4\pi G\rho)(\partial\rho/\partial p)_s$. Listed under each value of the spherical harmonic index l in Table 1 are two frequencies: the left column contains the three smallest roots of $f_s(\omega) = 0$, while the right column contains the two smallest roots of $f_o(\omega) = 0$. Each frequency in the table is presented as the dimensionless ratio $\omega/(\pi G\rho)^{1/2}$.

These elementary solutions reveal several interesting things about the oscillations of this superfluid material. The most striking

TABLE 1

ELEMENTARY FREQUENCIES

Mode	$l = 2$		$l = 3$		$l = 4$		$l = 5$	
Model A								
f	1.406	1.408	1.808	1.810	2.141	2.142	2.430	2.431
s_0	3.267		4.396		5.484		6.546	
p_1	5.871	5.930	7.142	7.235	8.367	8.503	9.557	9.743
Model B								
f	1.406	1.408	1.809	1.810	2.141	2.142	2.430	2.431
s_0	3.603		4.844		6.036		7.198	
p_1	5.944	5.930	7.260	7.235	8.537	8.503	9.787	9.743

feature is that one set of modes has frequencies which are nearly identical to those based on ordinary-fluid hydrodynamics. We refer to this set of modes as "ordinary fluid"–like and label them using the usual f and p_n notations. There is in addition a second set of modes in the superfluid case that we refer to as "superfluid"–like. We label these using the notation s_n. Unlike the second-sound modes in superfluid helium, the frequencies of these "superfluid"–like modes in neutron star matter are somewhat higher than the frequencies of the ordinary-fluid acoustical modes. The eigenfunctions in the lowest frequency "ordinary fluid"–like modes have $|\delta\beta| \ll |\delta U|$. Conversely, the lowest frequency "superfluid"–like modes have $|\delta U| \ll |\delta\beta|$. The potential $\delta\beta$ drives the relative velocity between the neutrons and protons in the superfluid mixture, equations (7) and (21). Thus, the neutrons and the protons move together (for the most part) in the "ordinary fluid"–like modes, but move opposite each other in the "superfluid"–like modes. The particular choice of superfluid model has little effect ($< 10\%$) on the values of the frequencies of these modes, but a rather larger effect on the form of the eigenfunctions. This dependence on the superfluid model is much more pronounced in the "superfluid"–like modes and the higher order "ordinary fluid"–like modes.

7. REALISTIC SOLUTIONS

We have also found more realistic numerical solutions to the superfluid pulsation equations. We have solved equations (26)–(28) for the perturbations of the $M = 1.400\ M_\odot$ equilibrium stellar model constructed from the combined Serot and Harrison-Wheeler equations of state. The radius of this equilibrium model is $R_o = 14.91$ km, and its central density is 4.310×10^{14} g cm^{-3}. The density which marks the transition between the superfluid and the ordinary-fluid state has been taken to be $\rho_s = 2.8 \times 10^{14}$ g cm^{-3}, the density at which the nuclei completely dissolve into a homogeneous fluid of neutrons, protons, and electrons (see, e.g., Baym, Bethe, & Pethick 1981). In our stellar model the superfluid transition occurs, therefore, at the radius $R_s = 10.84$ km with 66.6% of the material (by mass) in the star being in the superfluid state.

We have solved equations (26)–(28) using two different numerical techniques. For nonrotating stars equations (26)–(28) become an almost standard eigenvalue problem for a system of ordinary differential equations (see the Appendix) which can be solved using standard "shooting" techniques (see, e.g., Press et al. 1986). We have also solved the partial differential equations directly on a two-dimensional grid using the numerical techniques developed in Ipser & Lindblom (1990) for ordinary-fluid oscillations. This second method is more general in that it allows us to investigate the modes of rapidly rotating neutron stars. The frequencies of the modes of nonrotating neutron stars computed using these two different techniques agree to within about 0.1%. We believe the differences between the two methods are caused by subtle differences in the way we handle the equation of state in our two separate codes.

The frequencies of the fundamental $l = 2$ through $l = 6$ modes are given in Table 2 for the nonrotating $1.400\ M_\odot$ neutron star. The radial parts of the eigenfunctions δU, $\delta\Phi$, and $\delta\beta$ are depicted in Figures 6–8 for these modes. Each of these radial functions must be multiplied by the spherical harmonic Y_{lm} to obtain the complete eigenfunction. For a given value of the radial coordinate r in these graphs the eigenfunctions of the modes having successively larger values of l have successively smaller amplitudes. These eigenfunctions have been normalized by setting $\delta U(R) = 1$. The eigenfunctions δU and $\delta\Phi$ computed with the two different models of the proton effective mass differ from one another by no more than 0.02% for these modes. Such differences are not distinguishable in these figures. The eigenfunction $\delta\beta$ computed with model B for the proton effective mass is smaller than its values computed using model A by about 20%. This is illustrated in Figure 8 for the $l = 2$ and $l = 6$ modes. Note that the amplitude of $\delta\beta$ is smaller than that of δU and $\delta\Phi$ for these modes by about a factor of 1000.

TABLE 2

REALISTIC FREQUENCIES

l	ω/Ω_o
2	1.359
3	1.954
4	2.408
5	2.787
6	3.117

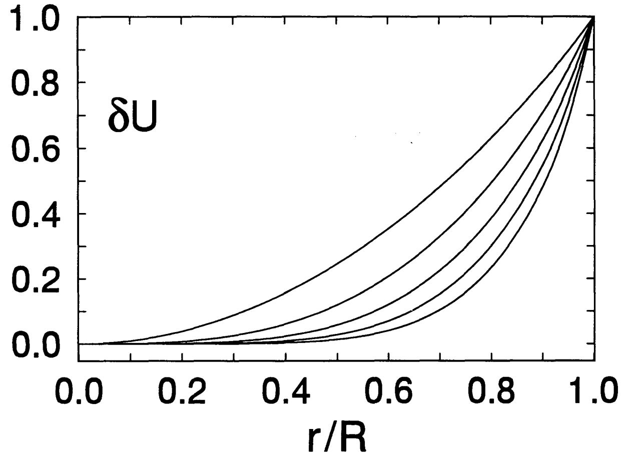

FIG. 6

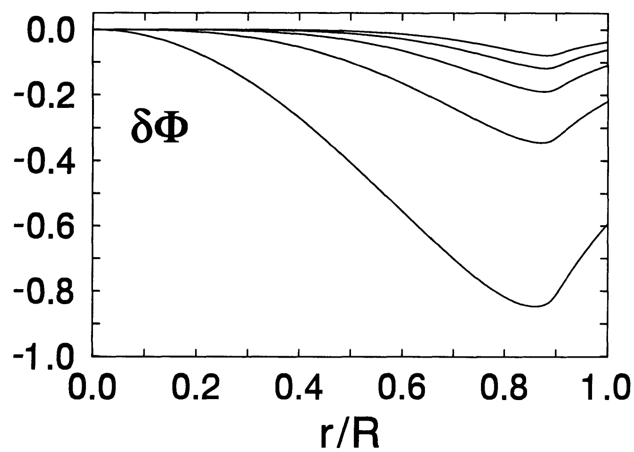

FIG. 7

FIG. 6.—The radial dependence of δU for the $l = 2$ through $l = 6$ modes of a nonrotating 1.4 $M_\odot$ neutron star

FIG. 7.—The radial dependence of $\delta\Phi$ for the $l = 2$ through $l = 6$ modes of a nonrotating 1.4 $M_\odot$ neutron star

The differences between the frequencies computed with the different models for the proton effective mass are less than 0.01%. Like the elementary solutions found in § 6, these "ordinary fluid" *f*-modes do not excite the superfluid degrees of freedom of this material very much. It is not surprising, therefore that the frequencies of these modes also agree (to within 0.01%) with those computed using the ordinary-fluid equations, (29) and (30), throughout the entire star. And, the eigenfunctions δU and $\delta\Phi$ computed using the ordinary-fluid equations differ by at most 0.1% from those computed with the full superfluid equations. We have also computed these modes for models in which the superfluid transition density was set to the unphysically low value $\rho_s = 5 \times 10^{13}$ g cm^{-3}. In these models the superfluid transition occurs at the radius 13.31 km and the material in the superfluid state makes up 99.4% (by mass) of the star. Superfluidity is more important in determining the dynamics in this case and the magnitude of $\delta\beta$ is larger by about a factor of 3 over those depicted in Figure 8. Even so, decreasing the superfluid transition density in this way makes no significant change (i.e., less than about 0.01%) on the frequencies of these modes.

We have also examined the properties of a one parameter family of "rigidly rotating" (in the smooth-averaged sense) neutron star models. We have kept the mass of the stars in this family fixed at the value 1.400 $M_\odot$ and use the same combined Serot & Harrison-Wheeler equation of state discussed above. The angular velocity dependence of some of the macroscopic physical properties of these stars are depicted in Figure 9. The angular velocities in this figure are given in terms of $\Omega_o = (\pi G\bar{\rho}_o)^{1/2}$ where $\bar{\rho}_o$ is the average density of the nonrotating star in the family. Figure 9 contains graphs of the moment of inertia I, the central density ρ_c, the ratio of equatorial to polar radii R_e/R_p, and the ratio of the rotational kinetic to the gravitational potential energy $K/|W|$ for the stars in this sequence. Figure 10 shows the angular velocity dependence of the frequencies of the "ordinary fluid"-like *f*-modes, where as before $\sigma = \omega - m\Omega$ and ω_o is the frequency of this mode in the nonrotating star in the sequence. The curve having the largest values of σ/ω_o is the $l = m = 2$ mode and successively lower curves correspond to the modes $l = m = 3$ through $l = m = 6$. The dashed portion of the $l = m = 2$ curve was computed using the ordinary-fluid equations (29) and (30), throughout the star. For small angular velocities equations (26) and (27) are elliptic, and the techniques developed by Ipser & Lindblom (1990) are sufficient to solve them. However, in the angular velocity range indicated by the dashed portion of the $l = m = 2$ curve the $\delta\beta$ equation, (27), is

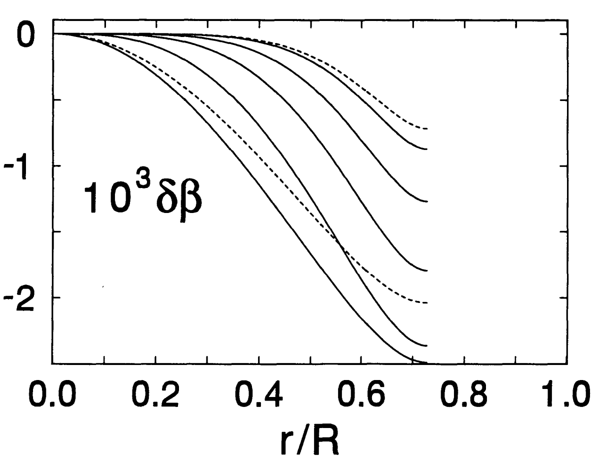

FIG. 8.—The radial dependence of $\delta\beta$ for the modes of a nonrotating 1.4 $M_\odot$ neutron star. The solid curves are for the $l = 2$ through $l = 6$ modes using model A for the proton effective mass, while the dashed curves are for the $l = 2$ and $l = 6$ modes using model B.

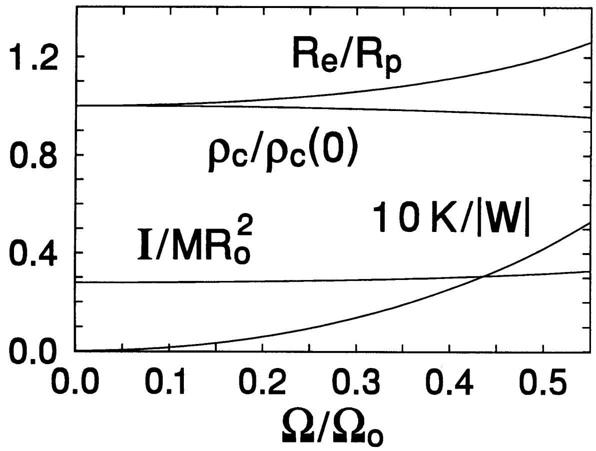

FIG. 9

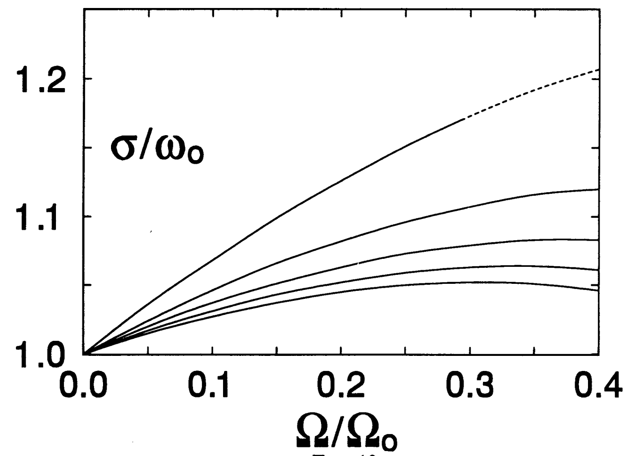

FIG. 10

FIG. 9.—The angular velocity dependence of four functions that characterize these rotating stellar models: I/MR_o^2, the ratio of the moment of inertia and MR_o^2; $K/|W|$, the ratio of the rotational kinetic energy to the gravitational potential energy; R_e/R_p, the ratio of the equatorial to the polar radius; and $\rho_c/\rho_c(0)$, the ratio of the central density of the star to the central density of the nonrotating model with the same mass.

FIG. 10.—The angular velocity dependence of the frequencies of the $l = 2$ through $l = 6$ "ordinary fluid"–like *f*-modes

hyperbolic (in the r and θ coordinates) for this mode. Our current numerical techniques are not adequate to solve the equations when they become hyperbolic. We do not expect that this signature change in the equation signals any instability in this mode, however. Rather we expect the form of the eigenfunction to change somewhat, but that the frequencies are not likely to differ significantly from their ordinary-fluid counterparts shown by the dashed curve.

8. CONCLUDING REMARKS

In this paper we have developed a theoretical framework in which to study the effects of superfluidity on the oscillations of neutron stars. We have shown that three scalar potentials are sufficient to determine the superfluid dynamics of neutron star matter, and have derived a system of three second-order equations that determine these basic potentials. A variational principle has been developed from which the frequencies of the modes of superfluid neutron stars can be estimated. Numerical techniques have been developed which allow us to find the "ordinary fluid"–like *f*-modes of rotating neutron stars. Perhaps the most striking result of our numerical analysis is that one class of solutions to the full superfluid hydrodynamics equations is essentially indistinguishable from the solutions to the ordinary-fluid equations. Thus, all the work that has been done in the past on the pulsations of neutron stars using ordinary-fluid hydrodynamics is relevant even in very cold neutron stars. We do not yet understand any simple physical mechanism responsible for the similarity of these modes. We are confident that this feature is real, however, since it was found in the elementary analytical solutions to the pulsation equations described in § 6 as well as in each of our separate numerical solutions to the realistic equations described in § 7.

A number of interesting questions remain to be investigated. We intend to study the effects of dissipation (internal fluid dissipation, superfluid mutual friction, and gravitational radiation) on these superfluid neutron star pulsations. We are particularly interested in checking the "back of the envelope" calculation of Lindblom & Mendell (1992) which suggested that the gravitational-radiation induced secular instability in rapidly rotating neutron stars is completely suppressed by dissipative mutual friction effects. Improvements in the numerical techniques used here to evaluate the modes of rapidly rotating superfluid stars will have to be made before this is possible, however. Techniques must be developed for solving equation (27) in rapidly rotating stars when it becomes a hyperbolic equation. We are also interested in determining for realistic neutron stars the "superfluid"–like modes that we found in our elementary solutions to the pulsation equations. For some reason (not yet understood) the numerical techniques that we developed to find the "ordinary fluid"–like modes, were not capable of finding the "superfluid"–like modes even in nonrotating stars.

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APPENDIX

While the abstract covariant representations of the superfluid pulsation equations are very good for derivations and manipulations, they are not very useful for finding actual solutions to the equations. For this we must express the equations in some particular coordinate system. Our experience from studies of the ordinary-fluid oscillations of rapidly rotating neutron stars indicates that spherical coordinates are the most useful choice. Therefore, in this appendix we give explicit representations of the superfluid pulsation equations in a spherical coordinate system. We let r and $\mu = \cos\theta$ denote the standard spherical coordinates.

Equations (26)–(28) are given by

$$
\begin{aligned}
&[\lambda + \mu^2(1-\lambda)]\partial_r^2 \delta U + (1-\mu^2)(1-\lambda)\frac{2\mu}{r}\partial_\mu \partial_r \delta U + [1-\mu^2(1-\lambda)]\frac{1-\mu^2}{r^2}\partial_\mu^2 \delta U \\
&\quad + \left\{\frac{1}{r}[1-\mu^2+\lambda(1+\mu^2)] + \frac{\partial_r \rho}{\rho}[\lambda+\mu^2(1-\lambda)] + \frac{\partial_\mu \rho}{\rho}\frac{\mu}{r}(1-\mu^2)(1-\lambda)\right\}\partial_r \delta U \\
&\quad + \left\{\frac{\mu}{r^2}[\lambda - 3 + 3\mu^2(1-\lambda)] + \frac{\partial_r \rho}{\rho}\frac{\mu}{r}(1-\mu^2)(1-\lambda) + \frac{\partial_\mu \rho}{\rho}\frac{1-\mu^2}{r^2}[1-\mu^2(1-\lambda)]\right\}\partial_\mu \delta U \\
&\quad + \left[\sigma^2\left(\frac{\partial \rho}{\partial p}\right)_\beta - \frac{2m\lambda\Omega}{r\sigma\rho}\left(\partial_r \rho - \frac{\mu}{r}\partial_\mu \rho\right) - \frac{m^2\lambda}{r^2(1-\mu^2)}\right]\delta U \\
&= \frac{1}{r^2\rho}\partial_r\left[\!\left[\frac{r^2}{\rho}\left(\frac{\partial \rho}{\partial \beta}\right)_p\left\{\partial_r p[\mu^2+\lambda(1-\mu^2)] + \partial_\mu p\frac{\mu}{r}(1-\mu^2)(1-\lambda)\right\}\delta\beta\right]\!\right] \\
&\quad + \frac{1}{r\rho}\partial_\mu\left[\!\left[\frac{1-\mu^2}{\rho}\left(\frac{\partial \rho}{\partial \beta}\right)_p\left\{\frac{\partial_\mu p}{r}[1-\mu^2(1-\lambda)] + \partial_r p\,\mu(1-\lambda)\right\}\delta\beta\right]\!\right] \\
&\quad + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial \beta}\right)_p\left[\frac{2m\lambda\Omega}{r\sigma\rho}\left(\partial_r p - \frac{\mu}{r}\partial_\mu p\right) - \sigma^2\right]\delta\beta + \sigma^2\left(\frac{\partial \rho}{\partial p}\right)_\beta \delta\Phi\,,
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
&[\tilde\lambda + \mu^2(1-\tilde\lambda)]\partial_r^2 \delta\beta + [1-\mu^2(1-\tilde\lambda)]\frac{1-\mu^2}{r^2}\partial_\mu^2 \delta\beta + (1-\mu^2)(1-\tilde\lambda)\frac{2\mu}{r}\partial_\mu\partial_r \delta\beta \\
&\quad + \left\{\frac{1}{r}[1-\mu^2+\tilde\lambda(1+\mu^2)] + \frac{\partial_r\tilde\rho}{\tilde\rho}[\tilde\lambda+\mu^2(1-\tilde\lambda)] + \frac{\partial_\mu\tilde\rho}{\tilde\rho}\frac{\mu}{r}(1-\mu^2)(1-\tilde\lambda) + (1-\mu^2)\left(\partial_r\tilde\lambda - \frac{\mu}{r}\partial_\mu\tilde\lambda\right)\right\}\partial_r\delta\beta \\
&\quad + \left\{\frac{\mu}{r^2}[\tilde\lambda - 3 + 3\mu^2(1-\tilde\lambda)] + \frac{\partial_r\tilde\rho}{\tilde\rho}\frac{\mu}{r}(1-\mu^2)(1-\tilde\lambda) + \frac{\partial_\mu\tilde\rho}{\tilde\rho}\frac{1-\mu^2}{r^2}[1-\mu^2(1-\tilde\lambda)] - \frac{\mu}{r}(1-\mu^2)\left(\partial_r\tilde\lambda - \frac{\mu}{r}\partial_\mu\tilde\lambda\right)\right\}\partial_\mu\delta\beta \\
&\quad - \left[\!\left[\frac{2m}{r\sigma\tilde\rho}\left[\partial_r(\tilde\rho\tilde\lambda\tilde\Omega) - \frac{\mu}{r}\partial_\mu(\tilde\rho\tilde\lambda\tilde\Omega)\right] + \frac{m^2\tilde\lambda}{r^2(1-\mu^2)} - \sigma^2\frac{\rho_n^2}{\tilde\rho\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p\right.\right. \\
&\quad \left.\left. + \frac{1}{\tilde\rho\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2\left\{\frac{1-\mu^2}{r^2}[1-\mu^2(1-\lambda)]\left(\frac{\partial_\mu p}{\rho}\right)^2 + [\mu^2+\lambda(1-\mu^2)]\left(\frac{\partial_r p}{\rho}\right)^2 + (1-\mu^2)(1-\lambda)\frac{2\mu}{r\rho^2}\partial_\mu p\,\partial_r p\right\}\right]\!\right]\delta\beta \\
&= -\frac{1}{\tilde\rho\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p\left\{[\mu^2+\lambda(1-\mu^2)]\partial_r p\,\partial_r\delta U + \frac{1-\mu^2}{r^2}[1-\mu^2(1-\lambda)]\partial_\mu p\,\partial_\mu\delta U\right. \\
&\quad \left. + \frac{\mu}{r}(1-\mu^2)(1-\lambda)(\partial_r p\,\partial_\mu\delta U + \partial_\mu p\,\partial_r\delta U) - \frac{2m\lambda\Omega}{r\sigma}\left(\partial_r p - \frac{\mu}{r}\partial_\mu p\right)\delta U + \sigma^2\rho(\delta U - \delta\Phi)\right\}
\end{aligned}
\tag{89}
$$

$$
\partial_r^2\delta\Phi + \frac{1-\mu^2}{r^2}\partial_\mu^2\delta\Phi + \frac{2}{r}\partial_r\delta\Phi - \frac{2\mu}{r^2}\partial_\mu\delta\Phi - \left[\frac{m^2}{r^2(1-\mu^2)} - 4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta\right]\delta\Phi = 4\pi G\left[\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta\delta U + \left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta\right]. \tag{90}
$$

The quadratic form S_s (eq. [45]) used to construct a variational principle has the following representation as a two-dimensional integral in spherical coordinates:

$$
\begin{aligned}
S_s(\delta\hat X, \delta X) = -2\pi\int_{-1}^{+1}\int_0^{R_s(\mu)}&\left[\!\left[\frac{\rho}{\sigma}\left\{\frac{\lambda+\mu^2(1-\lambda)}{\sigma}\partial_r\delta\hat U^*\partial_r\delta U + \frac{1-\mu^2(1-\lambda)}{\sigma r^2}(1-\mu^2)\partial_\mu\delta\hat U^*\partial_\mu\delta U\right.\right.\right. \\
&\left. + \frac{m^2\lambda}{\sigma r^2(1-\mu^2)}\delta\hat U^*\delta U - \frac{2m\Omega\lambda}{\sigma^2 r}\left[\partial_r(\delta\hat U^*\delta U) - \frac{\mu}{r}\partial_\mu(\delta\hat U^*\delta U)\right] + \frac{\mu(1-\mu^2)(1-\lambda)}{\sigma r}(\partial_r\delta\hat U^*\partial_\mu\delta U + \partial_\mu\delta\hat U^*\partial_r\delta U)\right\} \\
&+ \frac{\tilde\rho}{\sigma}\left\{\frac{\tilde\lambda+\mu^2(1-\tilde\lambda)}{\sigma}\partial_r\delta\hat\beta^*\partial_r\delta\beta + \frac{1-\mu^2(1-\tilde\lambda)}{\sigma r^2}(1-\mu^2)\partial_\mu\delta\hat\beta^*\partial_\mu\delta\beta + \frac{m^2\tilde\lambda}{\sigma r^2(1-\mu^2)}\delta\hat\beta^*\delta\beta\right. \\
&\left. - \frac{2m\tilde\Omega\tilde\lambda}{\sigma^2 r}\left[\partial_r(\delta\hat\beta^*\delta\beta) - \frac{\mu}{r}\partial_\mu(\delta\hat\beta^*\delta\beta)\right] + \frac{\mu(1-\mu^2)1(1-\tilde\lambda)}{\sigma r}(\partial_r\delta\hat\beta^*\partial_\mu\delta\beta + \partial_\mu\delta\hat\beta^*\partial_r\delta\beta)\right\} \\
&+ \frac{1}{4\pi G}\left[\partial_r\delta\hat\Phi^*\partial_r\delta\Phi + \frac{1-\mu^2}{r^2}\partial_\mu\delta\hat\Phi^*\partial_\mu\delta\Phi + \frac{m^2}{r^2(1-\mu^2)}\delta\hat\Phi^*\delta\Phi\right] - \rho\left(\frac{\partial\rho}{\partial p}\right)_\beta(\delta\hat U^* - \delta\hat\Phi^*)(\delta U - \delta\Phi) \\
&- \left\{\frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p - \frac{1}{\sigma\rho^3}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2\left[\frac{1-\mu^2(1-\lambda)}{\sigma r^2}(1-\mu^2)(\partial_\mu p)^2 + \frac{\lambda+\mu^2(1-\lambda)}{\sigma}(\partial_r p)^2 + \frac{2\mu(1-\mu^2)(1-\lambda)}{\sigma r}\partial_r p\,\partial_\mu p\right]\right\}\delta\hat\beta^*\delta\beta
\end{aligned}
$$

$$-\frac{1}{\rho\sigma}\left(\frac{\partial\rho}{\partial\beta}\right)_p\left\{\frac{\lambda+\mu^2(1-\lambda)}{\sigma}(\delta\beta\,\partial_r\delta\hat{U}^*+\delta\hat{\beta}^*\,\partial_r\delta U)\partial_r p+\frac{1-\mu^2(1-\lambda)}{\sigma r^2}(1-\mu^2)(\delta\beta\,\partial_\mu\delta\hat{U}^*+\delta\hat{\beta}^*\,\partial_\mu\delta U)\partial_\mu p\right.$$

$$+\frac{\mu(1-\mu^2)(1-\lambda)}{\sigma r}[(\partial_r\delta\hat{U}^*\,\partial_\mu p+\partial_\mu\delta\hat{U}^*\,\partial_r p)\delta\beta+\delta\hat{\beta}^*(\partial_r\delta U\,\partial_\mu p+\partial_\mu\delta U\,\partial_r p)]$$

$$\left.+\frac{2m\Omega\lambda}{\sigma r}\left(\frac{\mu}{r}\partial_\mu p-\partial_r p\right)(\delta\beta\,\delta\hat{U}^*+\delta\hat{\beta}^*\,\delta U)\right\}-\left(\frac{\partial\rho}{\partial\beta}\right)_p[(\delta\hat{U}^*-\delta\hat{\Phi}^*)\delta\beta+\delta\hat{\beta}^*(\delta U-\delta\Phi)]\Bigg]r^2\,dr\,d\mu\,. \tag{91}$$

The analogous expression for S_o is obtained from equation (91) simply by setting $\delta\beta = 0$ and changing the domain of integration from $0 \le r \le R_s(\mu)$, the portion of the star containing superfluid, to $R_s(\mu) \le r$ the region containing ordinary fluid plus the vacuum exterior region. The striking difference in complexity between these equations, (88)–(91), and their abstract covariant counterparts, (26)–(27) and (45), is an illustration of the power of the covariant approach in this subject.

We note that for non-rotating stars equations (88)–(91) simplify considerably. In particular in this limit $\Omega = \tilde{\Omega} = 0$ and $\lambda = \tilde{\lambda} = 1$. Furthermore, the three scalar potentials δU, $\delta\beta$, and $\delta\Phi$ can be expressed as functions of r multiplied by a spherical harmonic Y_{lm}. Thus equations (88)–(90) reduce to the following much simpler system of ordinary differential equations (where the angular dependence has been factored out after separation of variables):

$$\frac{1}{\rho r^2}\frac{d}{dr}\left(\rho r^2\frac{d\delta U}{dr}\right)+\left[\sigma^2\left(\frac{\partial\rho}{\partial p}\right)_\beta-\frac{l(l+1)}{r^2}\right]\delta U=\frac{1}{r^2\rho}\frac{d}{dr}\left[\frac{r^2}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p\frac{dp}{dr}\delta\beta\right]-\frac{\sigma^2}{\rho}\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta+\sigma^2\left(\frac{\partial\rho}{\partial p}\right)_\beta\delta\Phi\,, \tag{92}$$

$$\frac{1}{\tilde{\rho}r^2}\frac{d}{dr}\left(\tilde{\rho}r^2\frac{d\delta\beta}{dr}\right)+\left[\sigma^2\frac{\rho_n^2}{\rho\tilde{\rho}}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p-\frac{l(l+1)}{r^2}-\frac{1}{\rho^3\tilde{\rho}}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2\left(\frac{dp}{dr}\right)^2\right]\delta\beta=-\frac{1}{\rho\tilde{\rho}}\left(\frac{\partial\rho}{\partial\beta}\right)_p\left[\frac{dp}{dr}\frac{d\delta U}{dr}+\sigma^2\rho(\delta U-\delta\Phi)\right], \tag{93}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\delta\Phi}{dr}\right)+\left[4\pi G\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta-\frac{l(l+1)}{r^2}\right]\delta\Phi=4\pi G\left[\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta\delta U+\left(\frac{\partial\rho}{\partial\beta}\right)_p\delta\beta\right]. \tag{94}$$

And, the quadratic form, equation (91), reduces to the one-dimensional integral

$$S_s(\delta\hat{X},\delta X)=-\int_0^{R_s}\left\{\frac{\rho}{\sigma^2}\left[\frac{d\delta\hat{U}^*}{dr}\frac{d\delta U}{dr}+\frac{l(l+1)}{r^2}\delta\hat{U}^*\delta U\right]+\frac{\tilde{\rho}}{\sigma^2}\left[\frac{d\delta\hat{\beta}^*}{dr}\frac{d\delta\beta}{dr}+\frac{l(l+1)}{r^2}\delta\hat{\beta}^*\delta\beta\right]+\frac{1}{4\pi G}\left[\frac{d\delta\hat{\Phi}^*}{dr}\frac{d\delta\Phi}{dr}+\frac{l(l+1)}{r^2}\delta\hat{\Phi}^*\delta\Phi\right]\right.$$

$$-\rho\left(\frac{\partial\rho}{\partial p}\right)_\beta(\delta\hat{U}^*-\delta\hat{\Phi}^*)(\delta U-\delta\Phi)-\left[\frac{\rho_n^2}{\rho}\frac{\partial}{\partial\beta}\left(\frac{\rho_p}{\rho_n}\right)_p-\frac{1}{\sigma^2\rho^3}\left(\frac{\partial\rho}{\partial\beta}\right)_p^2\left(\frac{dp}{dr}\right)^2\right]\delta\hat{\beta}^*\delta\beta$$

$$\left.-\frac{1}{\rho\sigma^2}\left(\frac{\partial\rho}{\partial\beta}\right)_p\frac{dp}{dr}\left(\delta\beta\frac{d\delta\hat{U}^*}{dr}+\delta\hat{\beta}^*\frac{d\delta U}{dr}\right)-\left(\frac{\partial\rho}{\partial\beta}\right)_p[(\delta\hat{U}^*-\delta\hat{\Phi}^*)\delta\beta+\delta\hat{\beta}^*(\delta U-\delta\Phi)]\right\}r^2\,dr\,. \tag{95}$$

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