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ON THE PULSATIONS OF RELATIVISTIC ACCRETION DISKS AND ROTATING STARS: THE COWLING APPROXIMATION

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ABSTRACT

This paper shows that the hydrodynamic degrees of freedom of the adiabatic pulsations of relativistic fluids (e.g., accretion disks or rotating stars) can be described by a single scalar potential. When the gravitational perturbations are neglected—the Cowling approximation—this potential is determined by a second-order (typically elliptic) partial differential equation. A variational principle is developed from which the pulsation frequencies may be evaluated in this approximation. For objects like accretion disks in which self-gravitational effects are negligible, this approximation becomes an exact description of the pulsations.

Subject headings: accretion, accretion disks — relativity — stars: oscillations — stars: rotation

1. INTRODUCTION

Cowling (1941) found that the perturbations in the gravitational potential have little effect on the higher order nonradial modes in the Newtonian theory of stellar pulsations. Thus, setting the gravitational perturbations to zero—referred to as the Cowling approximation—has little effect on the frequencies of these modes. In the general-relativistic theory of stellar pulsations, the gravitational perturbations are described by perturbations in the space-time metric. Setting these metric perturbations to zero, therefore, is the most straightforward generalization of the Cowling approximation to general-relativistic stellar pulsations (McDermott, Van Horn, & Scholl 1983).

Numerical studies (Lindblom & Splinter 1990a) have shown that the relativistic Cowling approximation accurately predicts the frequencies of the higher order dipole p-modes in nonrotating relativistic stellar models. Table 1 shows that the relativistic Cowling approximation can be used to predict the frequencies of the fundamental nonradial modes of nonrotating stars with reasonable accuracy as well (Lindblom & Splinter 1990b). The frequencies given in Table 1 are for the f-modes of a nonrotating relativistic $n = 1$ polytrope having $M/R = 0.145$. Three frequencies are given for each of the fundamental modes having spherical harmonic index l in the range $2 \leq l \leq 5$: ω_E the exact relativistic frequency of this mode, $\omega_{\rm MVS}$ the frequency calculated with the relativistic-Cowling approximation of McDermott et al. (1983), and $\omega_{\rm F}$ the frequency calculated with a modified Cowling approximation proposed by Finn (1988). These results show that the Cowling approximation can be used to estimate the frequencies—with accuracies in the 5%–10% range—of the fundamental $l = m$ f-modes which contribute most significantly to the gravitational-radiation instability (Ipser & Lindblom 1991a). The Cowling approximation may also be used to describe the pulsations of fluid configurations—such as accretion disks—in which self-gravitational effects are negligible. In this case the Cowling approximation—which only neglects the self-gravitation of the pulsations—becomes an exact description of the pulsations. Thus we expect that the Cowling approximation should have a number of interesting applications in the study of relativistic astrophysical systems such as neutron stars and the accretion disks associated with compact X-ray sources and active galactic nuclei.

In this paper we develop an elegant reformulation of the relativistic Cowling approximation which describes arbitrary adiabatic pulsations of differentially rotating fluid configurations such as those used to model accretion disks and rotating stars. In § 2 we analyze the consequences of imposing the conservation laws on the perturbations (including the gravitational perturbations) of rotating relativistic fluid configurations. We show that all of the hydrodynamic degrees of freedom in these perturbations may be expressed in terms of a single scalar potential δU. Thus, the problem of evaluating the general pulsations is reduced to determining δU plus the metric-tensor perturbations. In § 3 we use this decomposition of the pulsation equations to obtain a very simple formulation of the Cowling approximation. We show that δU (which completely determines the pulsation in this approximation) is determined by a linear (typically elliptic) second-order partial differential equation. We also derive a variational principle from which the frequencies of the modes in the Cowling approximation may be evaluated. In § 4 we present explicit coordinate representations of the Cowling equation and its variational principle in two simple cases of astrophysical interest: the general adiabatic pulsations of nonrotating stars, and the barotropic pulsations of rigidly rotating stars.

2. THE ADIABATIC PULSATIONS OF ROTATING FLUIDS

We consider the equilibrium stellar models or accretion disks that are represented in general-relativity theory by stationary and axisymmetric spacetimes having perfect-fluid stress-energy tensors,

$$T^{ab} = \rho u^a u^b + p q^{ab} , \tag{1}$$

TABLE 1

FREQUENCIES OF THE F-MODES IN THE COWLING APPROXIMATION[a]

l	ω_E/Ω_0	ω_{MVS}/Ω_0	ω_F/Ω_0
2	1.387	1.676	1.731
3	1.829	2.014	2.043
4	2.160	2.294	2.312
5	2.436	2.539	2.551

[a] The frequencies are given in units of $\Omega_0^2 = 3M/4R^3$.

where ρ is the energy density, p is the pressure, u^a is the unit ($u^a u_a = -1$) four-velocity of the fluid, and $q^{ab} = g^{ab} + u^a u^b$. Tensor indices are lowered and raised with the spacetime metric g_{ab} and its inverse g^{ab}. We assume that these equilibrium states are convection-free so that the fluid velocity is purely rotational, i.e.,

$$u^a = \gamma(t^a + \Omega\varphi^a) , \tag{2}$$

where t^a and φ^a are the Killing fields that generate the stationary and rotational symmetries of this spacetime. This assumption is based on our expectation that any convective velocities will be extremely small so that they can be treated as perturbations. The energy-momentum conservation law, $\nabla_a T^{ab} = 0$, for these equilibrium fluids reduces to the condition

$$u^b \nabla_b u_a \equiv -\frac{\nabla_a \gamma}{\gamma} + \gamma\varphi^b u_b \nabla_a \Omega = -\frac{\nabla_a p}{\rho + p} , \tag{3}$$

where ∇_a is the metric-compatible covariant derivative on this spacetime.

Two additional geometrical properties of these equilibrium spacetimes play a role in our discussion. We assume (without loss of generality in asymptotically flat spacetimes, Carter 1970) that the Killing fields t^a and φ^a commute,

$$\varphi^b \nabla_b t^a = t^b \nabla_b \varphi^a . \tag{4}$$

This ensures that there exist functions t and φ such that $\partial/\partial t = t^a \partial/\partial x^a$ and $\partial/\partial\varphi = \varphi^a \partial/\partial x^a$. It follows (Kundt & Trumper 1966 and Carter 1969) that such convection-free spacetimes are foliated by two-surfaces that are everywhere orthogonal to the trajectories of the Killing fields. Coordinates r and θ may be chosen on these orthogonal two-surfaces so that the spacetime metric has a particularly simple form when expressed in these coordinates:

$$ds^2 \equiv g_{ab}\, dx^a\, dx^b = -e^{2\nu} dt^2 + e^{2(\zeta-\nu)}(dr^2 + r^2 d\theta^2) + B^2 e^{-2\nu} r^2 \sin^2\theta (d\varphi - \alpha\, dt)^2 , \tag{5}$$

where ν, ζ, α, and B are independent of t and φ. The existence of these orthogonal two-surfaces is also equivalent (by Frobenius's theorem) to the following conditions on the Killing fields:

$$\epsilon^{abcd} t_a \varphi_b \nabla_c t_d = 0 , \tag{6}$$

$$\epsilon^{abcd} t_a \varphi_b \nabla_c \varphi_d = 0 , \tag{7}$$

where ϵ^{abcd} is the volume element (i.e., the totally antisymmetric tensor whose components in an orthonormal basis have the values ± 1 or 0).

Our primary concern in this paper is the pulsations of relativistic stellar models and accretion disks. We confine our attention here to the study of infinitesimal pulsations which may be approximated as linear perturbations of the equilibrium fluid states discussed above. In the discussion that follows any quantity with the prefix δ (e.g., $\delta\rho$, δu^a, and δg_{ab}) is to be interpreted as the Eulerian perturbation of that quantity (i.e., the difference between the value of that quantity and its equilibrium value at a given point in the spacetime). Those quantities without prefix (e.g., ρ, u^a, g_{ab} and the operator ∇_a) are to be interpreted as the equilibrium values of those quantities. The equations for the Eulerian perturbations are obtained by linearizing the full time-dependent equations about an equilibrium solution. The perturbed energy-momentum conservation laws obtained in this way are

$$u^a \nabla_a \delta\rho + \delta\hat{u}^a \nabla_a \rho + (\rho + p)\nabla_a \delta\hat{u}^a = -\tfrac{1}{2}(\rho + p) u^a \nabla_a (q^{bc}\, \delta g_{bc}) , \tag{8}$$

$$(\rho + p)(q^a_b u^c \nabla_c \delta\hat{u}^b + \delta\hat{u}^b \nabla_b u^a) + (\delta\rho + \delta p) u^b \nabla_b u^a + q^{ab} \nabla_b \delta p = (\rho + p)\delta F^a , \tag{9}$$

where $\delta\hat{u}^a = q^a_b \delta u^b$, and the "force" δF^a depends linearly on the metric perturbation:

$$\delta F^a = -q^{ab} u^c \nabla_c (\delta g_{bd} u^d) + \tfrac{1}{2} q^{ab} u^c u^d \nabla_b \delta g_{cd} - u^c u^d\, \delta g_{cd}\, u^b \nabla_b u^a . \tag{10}$$

Equation (9) involves only the projected velocity perturbation $\delta\hat{u}^a$. The remaining component, $u_a \delta u^a$, is determined from the condition that the four-velocity is always a unit vector:

$$u_a \delta u^a = -\tfrac{1}{2} u^a u^b\, \delta g_{ab} . \tag{11}$$

These equations, (8)–(11), would determine the perturbed fluid variables if the metric perturbations were known. In general, of course, the metric perturbations must be determined along with the fluid perturbations by solving the perturbed Einstein equation

$$\delta G^{ab} = 8\pi\, \delta T^{ab} . \tag{12}$$

The remainder of this section will focus on the problem of transforming the conservation laws, equations (8) and (9), into a form from which the dynamics of the fluid variables may be more easily analyzed.

The dynamics of a fluid is determined by the equations that govern the evolution of the fluid variables—the conservation laws—plus a thermodynamic description of the fluid matter—the equation of state. In order to determine the evolution of the fluid perturbations via equations (8) and (9) it is necessary to provide this additional thermodynamic information in the form of a relation between the density and pressure perturbations. In this paper we adopt a generalization of the condition that the perturbation be adiabatic, i.e., that the specific entropy of a fluid element remain unchanged as the fluid evolves. Our generalization of this condition is

$$\Delta p = \frac{p\Gamma}{\rho + p} \Delta\rho , \tag{13}$$

where Γ is an arbitrary stationary and axisymmetric function on the equilibrium spacetime. This equation generalizes the standard adiabatic relationship between Δp and $\Delta\rho$ by allowing the function Γ to be completely arbitrary. For adiabatic perturbations Γ must be given by $p\Gamma = (\rho + p)(\partial p/\partial\rho)_s$, where s is the specific entropy of the fluid.[1] The prefix Δ used in equation (13) refers to the Lagrangian perturbation in that quantity (i.e., the difference between the value of that quantity and its equilibrium value associated with a given element of the fluid). The Lagrangian perturbations of the density and pressure are related to their Eulerian counterparts by the equations: $\Delta\rho = \delta\rho + \xi^a\nabla_a\rho$ and $\Delta p = \delta p + \xi^a\nabla_a p$. The vector field ξ^a is the "Lagrangian displacement," which may be defined as a "potential" for the perturbed velocity:

$$\delta\hat{u}^a = q^a_b(u^c\nabla_c\xi^b - \xi^c\nabla_c u^b) . \tag{14}$$

We note that the perturbed velocity in this equation is unchanged by adding to ξ^a any function multiplied by u^a. This invariance represents a gauge degree of freedom in the Lagrangian displacement (see Schutz & Sorkin 1977).

Assume now that all of the Eulerian perturbation quantities have temporal and azimuthal dependence $e^{i\omega t + im\varphi}$, where ω and m are constants. The constant m must be an integer if the perturbations are to be single-valued. For these perturbations equation (14) becomes an algebraic relationship between the Lagrangian displacement ξ^a and the velocity perturbation $\delta\hat{u}^a$:

$$\xi^a = -\frac{i}{\sigma\gamma}\left(q^a_b - \frac{i}{\sigma}\, q^a_c\varphi^c\nabla_b\Omega\right)\delta\hat{u}^b , \tag{15}$$

where $\sigma = \omega + m\Omega$. (We have fixed the gauge in ξ^a by requiring $\xi^a u_a = 0$.) Equations (13) and (15) can be used in turn to express the density perturbation $\delta\rho$ in terms of a scalar potential δU and $\delta\hat{u}^a$:

$$\delta\rho = (\rho + p)^2\left(\frac{\delta U}{p\Gamma} + \frac{iA_a\,\delta\hat{u}^a}{\sigma\gamma}\right) , \tag{16}$$

where

$$\delta U = \frac{\delta p}{\rho + p} , \tag{17}$$

and

$$A_a = \frac{\nabla_a\rho}{(\rho + p)^2} - \frac{\nabla_a p}{(\rho + p)p\Gamma} . \tag{18}$$

We note that A_a vanishes for fluid systems (such as cold neutron stars) in which the pressure depends only on the density of the fluid. However, for systems—such as accretion disks—in which temperature significantly affects the equation of state, A_a will not vanish. Using equations (16) and (17) the perturbed momentum-conservation equation (9) can now be transformed into the particularly simple form

$$iQ^{-1}_{ab}\,\delta\hat{u}^b = -q^b_a\nabla_b\delta U - (\rho + p)A_a\,\delta U + \delta F_a , \tag{19}$$

where

$$Q^{-1}_{ab} = \sigma\gamma q_{ab} - 2iq^c_b\nabla_c u_a + i\gamma q_{ac}\,\varphi^c\nabla_b\Omega - \frac{1}{\sigma\gamma}\nabla_a pA_b . \tag{20}$$

This equation is algebraic in $\delta\hat{u}^a$! Thus, the perturbed momentum-conservation law can be solved *algebraically* for the velocity perturbation in terms of a scalar potential δU and the gravitational perturbations δg_{ab}. This result extends to the general adiabatic perturbations of differentially rotating fluid states the analogous result derived by Friedman (1988) for the stationary ($\omega = 0$)

[1] It can be shown (see Ipser & Lindblom 1991b) that for every Γ there exists a function X such that $p\Gamma = (\rho + p)(\partial p/\partial\rho)_X$. It also follows that $\Delta X = 0$, and so these perturbations have the formal structure of the traditional adiabatic perturbations even though X may not be the physical specific entropy of the fluid.

perturbations of barotropic ($A_a = 0$) uniformly rotating ($\nabla_a \Omega = 0$) fluids. Equation (20) can also be interpreted as an extension to general-relativity theory of an analogous result in the Newtonian theory of stellar pulsations (Ipser & Managan 1985; Ipser & Lindblom 1991b).

The tensor Q_{ab}^{-1} has no real inverse since $Q_{ab}^{-1}u^a = Q_{ab}^{-1}u^b = 0$. However, there will exist a Q^{ab} such that $Q^{ac}Q_{cb}^{-1} = q_b^a$ as long as the determinant of Q_{ab}^{-1} (defined with respect to the three-volume element $\epsilon^{abc} = \epsilon^{abcd}u_d$) does not vanish. This determinant of $(Q^{-1})^a_b$ is a scalar which may be expressed as

$$\det (Q^{-1})^a_b = \frac{1}{\sigma\gamma}[\sigma^4\gamma^4 - \sigma^2\gamma^2(A^a\nabla_a p + 2\Omega^a\omega_a) + 2\omega^a A_a \Omega^b \nabla_b p] \equiv \frac{\sigma^3\gamma^3}{\Lambda}, \tag{21}$$

where ω^a is the vorticity of the fluid

$$\omega^a = \epsilon^{abcd}u_b \nabla_c u_d, \tag{22}$$

and Ω^a is a relativistic generalization of the angular-velocity vector

$$\Omega^a = \tfrac{1}{2}\gamma\epsilon^{abcd}u_b(\nabla_c t_d + \Omega\nabla_c \varphi_d). \tag{23}$$

We note that in the convection-free spacetimes of interest here ω^a and Ω^a are orthogonal to the two-surfaces spanned by the Killing fields: $\omega^a t_a = \omega^a\varphi_a = \Omega^a t_a = \Omega^a\varphi_a = 0$. This follows directly from equations (6) and (7).

Whenever the determinant in equation (21) does not vanish, Q_{ab}^{-1} can be "inverted" in the sense described above. In that case this inverse may be expressed in the form

$$Q^{ab} = \frac{\Lambda}{\sigma^3\gamma^3}\left[(\sigma^2\gamma^2 - A^c\nabla_c p)q^{ab} - 2\omega^a\Omega^b + \nabla^a p A^b - i\hat\varphi^a\epsilon^{bc}\left(\sigma\gamma\omega_c - \frac{\omega^d A_d}{\sigma\gamma}\nabla_c p\right) + 2i\hat\varphi^b\epsilon^{ac}\left(\sigma\gamma\Omega_c - \frac{\Omega^d\nabla_d P}{\sigma\gamma}A_c\right)\right], \tag{24}$$

where $\hat\varphi^a$ is the unit linear combination of the Killing fields that is orthogonal to u^a,

$$\hat\varphi^a = \frac{q^{ab}\varphi_b}{\varpi\gamma}, \tag{25}$$

with $\bar\omega^2 = -t^a t_a \varphi^b\varphi_b + (t^a\varphi_a)^2$. The antisymmetric tensor ϵ^{ab} is the volume element on the two-surfaces that are orthogonal to the Killing trajectories,

$$\epsilon^{ab} = \epsilon^{abcd}\hat\varphi_c u_d. \tag{26}$$

When the tensor Q^{ab} does exist, all of the fluid perturbation variables can be expressed in terms of the scalar potential δU and the gravitational perturbations δg_{ab}. In particular it follows directly from equations (16), (17), and (19) that

$$\delta u^a = iQ^{ab}\nabla_b\delta U + i(\rho + p)Q^{ab}A_b\delta U - iQ^{ab}\delta F_b + \tfrac{1}{2}u^a u^b u^c\delta g_{bc}, \tag{27}$$

$$\delta\rho = \frac{(\rho + p)^2}{\sigma\gamma}\left\{\left[\frac{\sigma\gamma}{p\Gamma} - (\rho + p)A_a Q^{ab}A_b\right]\delta U - A_a Q^{ab}(\nabla_b\delta U - \delta F_b)\right\}, \tag{28}$$

and

$$\delta p = (\rho + p)\delta U. \tag{29}$$

To this point we have used only the perturbed momentum-conservation law, equation (9), to express the fluid-perturbation variables $\delta\rho$, δp, and δu^a in terms of the scalar potential δU and the gravitational perturbations δg_{ab}. The energy-conservation law, equation (8), may also be used to obtain an equation for δU in terms of δg_{ab}. Using equations (27) and (28) to replace the $\delta\rho$ and $\delta\hat u^a$ that appear in this expression, we obtain the following representation of equation (8),

$$\nabla_a[(\rho + p)Q^{ab}\nabla_b\delta U] - Q^{ab}\nabla_a p\nabla_b\delta U + \Psi\delta U = \delta F, \tag{30}$$

where

$$\Psi = (\rho + p)^2\left[\frac{\sigma\gamma}{p\Gamma} - (\rho + p)A_a Q^{ab}A_b\right] + \nabla_a[(\rho + p)^2 Q^{ab}A_b] - (\rho + p)Q^{ab}\nabla_a p A_b + \frac{i}{\varpi\gamma}(m + \sigma\gamma u^a\varphi_a)(\rho + p)^2(Q^{bc} - Q^{cb})\hat\varphi_b A_c, \tag{31}$$

and the "force" δF depends linearly on the metric perturbations,

$$\delta F = -\tfrac{1}{2}\gamma\sigma(\rho + p)q^{ab}\delta g_{ab} - (\rho + p)^2 A_a Q^{ab}\delta F_b + \nabla_a[(\rho + p)Q^{ab}\delta F_b] - Q^{ba}\delta F_a\nabla_b p. \tag{32}$$

We note that in order to put equation (8) into the form given in equation (30), it is necessary to use the identity

$$2\epsilon^{ab}\Omega_a\omega_b = \epsilon^{ab}A_a\nabla_b p, \tag{33}$$

which is satisfied by the equilibrium fluid states considered here. It is obtained by evaluating the integrability condition for the equilibrium conservation law, equation (3). The right side of equation (33) is equal to $-\epsilon^{ab}\nabla_a[(\rho + p)^{-1}\nabla_b p]$. This term equals

$\epsilon^{ab}\nabla_a(u^c\nabla_c u_b)$ as a consequence of equation (3). The left side of equation (33) is obtained by performing a series of complicated transformations on this expression. These transformations are summarized by the identities

$$\epsilon^{ab}\nabla_a(u^c\nabla_c u_b) = \epsilon^{ab}\nabla_a(\gamma\varphi^c u_c)\nabla_b\Omega = \gamma^2\varpi\epsilon^{ab}\nabla_a\Omega\hat{\varphi}^c(\nabla_c u_b - \nabla_b u_c) = -\gamma^2\varpi\epsilon^{abc}\nabla_a\Omega\nabla_b u_c = \gamma^2\varpi\epsilon_{ab}\omega^a\epsilon^{bc}\nabla_c\Omega = 2\epsilon^{ab}\Omega_a\omega_b\,. \quad (34)$$

3. THE COWLING APPROXIMATION

In the last section we analyzed the general adiabatic perturbations of rotating relativistic fluid states. We showed that as a consequence of the conservation laws, the hydrodynamic perturbations δu^a, $\delta\rho$, and δp are completely determined by the scalar potential δU and the gravitational perturbations δg_{ab}! This fact will probably play an important role in the eventual complete description of the pulsations of rotating relativistic fluid systems. Here, however, we explore the consequences of this fact in a more limited context. The Cowling approximation in general relativity is obtained by setting $\delta g_{ab} = 0$ in the fluid equations. This approximation has been found to be useful for estimating the frequencies of the higher order modes of relativistic stellar models (Lindblom & Splinter 1990a, b). For fluid configurations like accretion disks—where self-gravitational effects are negligible—this approximation becomes an exact description of the pulsations.

The results of the last section show that in the Cowling approximation $\delta g_{ab} = 0$ the general adiabatic pulsations of rotating relativistic fluid systems are completely determined by the single scalar potential δU. In particular, setting $\delta g_{ab} = 0$ in equations (27)–(29) we obtain

$$\delta u^a = iQ^{ab}\nabla_b\delta U + i(\rho + p)Q^{ab}A_b\delta U\,, \quad (35)$$

$$\delta\rho = \frac{(\rho+p)^2}{\sigma\gamma}\left\{\left[\frac{\sigma\gamma}{p\Gamma} - (\rho+p)A_aQ^{ab}A_b\right]\delta U - A_aQ^{ab}\nabla_b\delta U\right\}, \quad (36)$$

and

$$\delta p = (\rho + p)\delta U\,, \quad (37)$$

where A^a and Q^{ab} are given as before by equations (18) and (24). The potential δU is determined in this approximation by the linear second-order equation

$$\nabla_a[(\rho+p)Q^{ab}\nabla_b\delta U] - Q^{ab}\nabla_a p\nabla_b\delta U + \Psi\delta U = 0 \quad (38)$$

as a consequence of equation (30).

While equations (35)–(38) provide a complete description of the relativistic Cowling approximation, it is not—from a geometrical point of view—the most natural way to express the equations. Because of the symmetries of the equilibrium spacetime and the assumed dependence $e^{i\omega t + im\varphi}$ of the perturbation variable δU, equations (35)–(38) are really two-dimensional. In equations (35)–(38) they are written, however, in terms of the four-dimensional covariant derivative ∇_a. It would be more natural to write these equations in terms of geometrical structures that are intrinsic to the two-surfaces on which they are defined. Therefore, we introduce the metric, $h^{ab} = g^{ab} + u^au^b - \hat{\varphi}^a\hat{\varphi}^b$, that describes the geometry of the two-surfaces that are orthogonal to the Killing trajectories (i.e., the r-θ plane in the coordinates of eq. [5]). The covariant derivative associated with this metric is denoted D_a; its action on scalars for example, is given by $D_a f = h_a^b\nabla_b f$. Also, introduce the rescaled perturbation potential $\delta V = \delta U/\sigma\gamma$. In terms of these quantities equation (38) can be rewritten in the following manifestly two-dimensional form

$$\mathscr{L}_\omega(\delta V) \equiv D_a[\varpi(\rho+p)H^{ab}D_b\delta V] + \varpi\sigma\gamma\Phi\,\delta V = 0\,. \quad (39)$$

In this equation, H^{ab} is defined by

$$H^{ab} = \sigma\gamma h_c^a h_d^b Q^{cd} = \frac{\Lambda}{\sigma^2\gamma^2}\,[(\sigma^2\gamma^2 - A^cD_c p)h^{ab} - 2\omega^a\Omega^b + D^a pA^b]\,. \quad (40)$$

This tensor is symmetric as a consequence of equation (33); thus, it is Hermitian for real values of the pulsation frequency. The scalar Φ is defined by

$$\Phi = \Psi + H^aD_a p - \frac{1}{\varpi}D_a[\varpi(\rho+p)H^a] + \frac{1}{\varpi}D_a\left[\frac{\varpi}{\sigma^2\gamma^2}(\rho+p)H^{ab}D_b(\sigma\gamma)\right]$$
$$- \frac{\Lambda(\rho+p)}{\varpi^2\sigma^3\gamma^5}(m + \sigma\gamma u^a\varphi_a)^2(\sigma^2\gamma^2 - A^b\nabla_b p)\,, \quad (41)$$

where

$$H^a = \frac{2\Lambda}{\varpi\sigma^4\gamma^5}(m + \sigma\gamma u^b\varphi_b)\epsilon^{ac}(\sigma^2\gamma^2\Omega_c - \Omega^dD_d pA_c)\,. \quad (42)$$

The scalar Φ is real for real values of the frequency. In order to put the pulsation equation into the form of equation (39) a number of identities satisfied by the equilibrium fluid state are needed. One identity—needed to convert the four-dimensional divergences into two-dimensional divergences—is $u^a\nabla_a u_b - \hat{\varphi}^a\nabla_a \hat{\varphi}_b = \varpi^{-1}\nabla_b \varpi$. Another identity—needed to remove the term proportional to $D_a \delta V$ from equation (39)—is the following:

$$\frac{i\sigma}{\varpi}(\rho + p)(m + \sigma\gamma u^d \varphi_d)(Q^{bc} + Q^{cb})h^a_b \hat{\varphi}_c - H^{ab}D_b p + \frac{\rho + p}{\sigma\gamma} H^{ab}D_b(\sigma\gamma) = 0 . \tag{43}$$

This second identity can be derived using the following consequence of equation (33),

$$\epsilon^{ab}(\omega^c A_c D_b p - 2\Omega^c D_c p A_b) + \varpi\gamma^2(A^c D_c p D^a \Omega + 2\Omega^c D_c \Omega \omega^a - A^c D_c \Omega D^a p) = 0 . \tag{44}$$

A variational principle has been found to be a useful tool for estimating the frequencies of the modes of stellar pulsation (e.g., Managan 1986). Therefore, we have constructed a variational principle from which the frequencies of the adiabatic pulsations of relativistic accretion disks and rotating stars may be estimated in the Cowling approximation. We assume that in the equilibrium state the fluid occupies only a finite spatial region of the spacetime. We assume that the density ρ and the pressure p vanish on the smooth boundary of the fluid region. And finally, we assume that the spacetime has a regular rotation axis on which the rotational Killing field φ^a vanishes. Then we define the inner product,

$$\mathscr{H}_\omega(\delta\bar{V}, \delta V) = \int \delta\bar{V}^* \mathscr{L}_\omega(\delta V) d^2x = \int [-\varpi(\rho + p)H^{ab}D_a \delta\bar{V}^* D_b \delta V + \varpi\sigma\gamma\Phi\delta\bar{V}^*\delta V]d^2x , \tag{45}$$

where δV and $\delta\bar{V}$ are independent potentials and * denotes complex conjugation. The integral in equation (45) is performed over the two-dimensional surface that is orthogonal to the Killing trajectories using the proper two-dimensional volume element d^2x on this surface. The second equality in equation (45) is obtained as the result of an integration by parts. The boundary integral that is introduced from the integration by parts vanishes as a consequence of our assumptions: the boundary of the fluid region on this two-dimensional surface may include only the rotation axis (where $\varpi = 0$) and the surface of the fluid (where $\rho + p = 0$). This inner product is Hermitian, $\mathscr{H}_\omega(\delta\bar{V}, \delta V) = \mathscr{H}^*_\omega(\delta V, \delta\bar{V})$, for real values of the frequency ω as a consequence of the facts that H^{ab} is Hermitian and Φ is real. Now consider the expression

$$\mathscr{H}_\omega(\delta V, \delta V) = 0 \tag{46}$$

as an equation for the frequency ω as a function of the trial potential δV. Since $\mathscr{H}_\omega(\delta\bar{V}, \delta V)$ is Hermitian, the frequencies computed in this way are stationary with respect to infinetesimal variations in δV whenever δV is the normal-mode eigenfunction of equation (39) with ω the corresponding eigenvalue. Thus equation (46) is a variational principle for the pulsation frequencies of rotating relativistic stellar models and accretion disks in the Cowling approximation.

4. TWO SPECIAL CASES

Some explicit coordinate representation of the relativistic Cowling equations, (39) and (46), will be needed by anyone who wishes to use this approximation to evaluate the pulsation frequencies of rotating stars, either by analytical or numerical means. While it is straightforward to write out these equations for the completely general case, the resulting expressions are rather complicated. Instead, we have chosen here to give explicit coordinate representations of these equations in two simple cases of astrophysical interest: first, the general adiabatic pulsations of nonrotating stars, and second, the barotropic pulsations of rigidly rotating stars.

The geometry of a static-spherical star is usually described in the coordinate system in which the spacetime metric takes the form

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 , \tag{47}$$

where ν and λ are functions of r alone. In this background geometry the potential δV may be separated into spherical harmonics: $\delta V = \delta V(r)Y^l_m$. Using equations (35) and (36) the density perturbation $\delta\rho$ and the components of the perturbed velocity δu^a may be expressed in terms of the potential δV:

$$\delta\rho = -\frac{(\rho + p)^2 A\omega e^{\nu - 2\lambda}}{\omega^2 - n^2}\frac{d\delta V(r)}{dr} Y^l_m + (\rho + p)\omega e^{-\nu}\left[\frac{\rho + p}{p\Gamma} - \frac{(\rho + p)^2 A^2 e^{2(\nu-\lambda)} + n^2}{\omega^2 - n^2}\right]\delta V(r)Y^l_m , \tag{48}$$

$$\delta u^r = \frac{i\omega^2 e^{-2\lambda}}{\omega^2 - n^2}\left[\frac{d\delta V(r)}{dr} + \frac{\delta V(r)}{\rho + p}\frac{dp}{dr} + A\,\delta V(r)\right]Y^l_m , \tag{49}$$

$$\delta u^\theta = i\frac{\delta V(r)}{r^2}\frac{\partial Y^l_m}{\partial\theta} , \tag{50}$$

$$\delta u^\varphi = -\frac{m\,\delta V(r)}{r^2\sin^2\theta} Y^l_m , \tag{51}$$

and $\delta u^t = 0$, and where A and n^2 are given by

$$A = \frac{1}{(\rho + p)^2}\left(\frac{d\rho}{dr} - \frac{\rho + p}{p\Gamma}\frac{dp}{dr}\right) , \tag{52}$$

and

$$n^2 = e^{2(\nu-\lambda)} A \frac{dp}{dr} . \tag{53}$$

(The quantity n is a relativistic generalization of the Brunt-Vaisala frequency.) The potential δV is determined by equation (39), which reduces in this case to the second-order ordinary differential equation

$$\frac{d}{dr}\left[\frac{(\rho + p)r^2\omega^2 e^{\nu-\lambda}}{\omega^2 - n^2} \frac{d\,\delta V(r)}{dr}\right] + \Upsilon(\omega)\delta V(r) = 0 , \tag{54}$$

where $\Upsilon(\omega)$ is defined by

$$\Upsilon(\omega) = -l(l+1)(\rho + p)e^{\nu+\lambda} + (\rho + p)e^{-\nu} \frac{d}{dr}\left[\frac{(\rho + p)Ar^2\omega^2 e^{2\nu-\lambda}}{\omega^2 - n^2}\right] + \frac{(\rho + p)^2}{p\Gamma} \frac{r^2\omega^4 e^{\lambda-\nu}}{\omega^2 - n^2} + e^{-\nu} \frac{d}{dr}\left[\frac{r^2\omega^2 e^{2\nu-\lambda}}{\omega^2 - n^2} \frac{dp}{dr}\right] . \tag{55}$$

This version of the relativistic Cowling approximation—given by a single second-order equation—is somewhat simpler than the original McDermott et al. (1983) formulation—defined by a coupled pair of first-order equations. The variational principle, equation (46), reduces in this special case to

$$0 = \int_0^\infty \left[-\frac{(\rho + p)r^2\omega^2 e^{\nu-\lambda}}{\omega^2 - n^2}\left|\frac{d\,\delta V(r)}{dr}\right|^2 + \Upsilon(\omega)|\,\delta V(r)\,|^2\right] dr , \tag{56}$$

which is also simpler than the expression given by Lindblom & Splinter (1990a) for the standard formulation of the Cowling approximation.

The second special case considered here is the simplest example of the pulsations of rotating stars: the barotropic ($A_a = 0$) perturbations of rigidly rotating ($\nabla_a \Omega = 0$) stars. The coordinates used to express the spacetime metric in equation (5) are often used to describe rotating relativistic stars. Since numerical methods have been developed to evaluate the equilibrium stellar models in these coordinates (Butterworth & Ipser 1976; Friedman, Ipser, & Parker 1986), we have chosen to use them to express the Cowling equations as well. The scalar γ (from eq. [2]) and the nonvanishing components of the vorticity vector ω^a (from eq. [22]) are given in terms of the metric functions $\nu(r, \theta)$, $\zeta(r, \theta)$, $\alpha(r, \theta)$, and $B(r, \theta)$ of equation (5), by

$$\gamma^2 = [e^{2\nu} - (\Omega - \alpha)^2 B^2 e^{-2\nu} r^2 \sin^2 \theta]^{-1} , \tag{57}$$

$$\omega^r = \frac{e^{2(\nu-\zeta)}}{B\gamma^2 r^2 \sin\theta} \frac{\partial S}{\partial \theta} , \tag{58}$$

$$\omega^\theta = -\frac{e^{2(\nu-\zeta)}}{B\gamma^2 r^2 \sin\theta} \frac{\partial S}{\partial r} , \tag{59}$$

where

$$S \equiv \gamma \varphi^a u_a = \gamma^2(\Omega - \alpha)B^2 e^{-2\nu} r^2 \sin^2 \theta . \tag{60}$$

The perturbations in the density $\delta\rho$ and the four-velocity δu^a are determined, via equations (35) and (36), by the potential δV:

$$\delta\rho = \sigma\gamma \frac{(\rho + p)^2}{p\Gamma} \delta V , \tag{61}$$

$$\delta u^r = i \frac{\Lambda}{\sigma^2\gamma^3}[\sigma^2\gamma^2 e^{2(\nu-\zeta)} - (\omega^r)^2] \frac{\partial(\gamma\,\delta V)}{\partial r} - i \frac{\Lambda\omega^r\omega^\theta}{\sigma^2\gamma^3} \frac{\partial(\gamma\,\delta V)}{\partial \theta} - i \frac{\Lambda\omega^\theta(m + \sigma S)}{\sigma\gamma^2 B \sin\theta} \delta V , \tag{62}$$

$$\delta u^\theta = i \frac{\Lambda}{r^2\sigma^2\gamma^3}[\sigma^2\gamma^2 e^{2(\nu-\zeta)} - r^2(\omega^\theta)^2] \frac{\partial(\gamma\,\delta V)}{\partial \theta} - i \frac{\Lambda\omega^r\omega^\theta}{\sigma^2\gamma^3} \frac{\partial(\gamma\,\delta V)}{\partial r} + i \frac{\Lambda\omega^r(m + \sigma S)}{\sigma\gamma^2 B r^2 \sin\theta} \delta V , \tag{63}$$

$$\delta u^\varphi = \frac{\Lambda(1 + \Omega S)}{\sigma\gamma^2 B r \sin\theta}\left[\frac{r\omega^\theta}{\gamma} \frac{\partial(\gamma\,\delta V)}{\partial r} - \frac{\omega^r}{r\gamma} \frac{\partial(\gamma\,\delta V)}{\partial \theta} - \frac{\sigma(m + \sigma S)}{Br \sin\theta} \delta V\right] , \tag{64}$$

$$\delta u^t = \frac{S\,\delta u^\varphi}{1 + \Omega S} , \tag{65}$$

where $\sigma = \omega + m\Omega$ is the frequency of the mode as measured in the rotating frame of the star, and Λ is the dimensionless scalar (from eq. [21]) that is proportional to the determinant of Q^{ab}. For this case Λ has the value

$$\Lambda = \frac{\sigma^2\gamma^2}{\sigma^2\gamma^2 - [(\omega^r)^2 + r^2(\omega^\theta)^2]e^{2(\zeta-\nu)}} . \tag{66}$$

The potential δV is determined in turn by equation (39), which is given in this case by the two-dimensional partial differential equation

$$0 = \frac{\partial}{\partial r}\left[\Lambda(\rho+p)\frac{Br^2 \sin\theta}{e^{2(\nu-\zeta)}}\left\{\left[e^{2(\nu-\zeta)} - \frac{(\omega^r)^2}{\sigma^2\gamma^2}\right]\frac{\partial \delta V}{\partial r} - \frac{\omega^r\omega^\theta}{\sigma^2\gamma^2}\frac{\partial \delta V}{\partial \theta}\right\}\right] + \frac{\partial}{\partial \theta}\left[\Lambda(\rho+p)\frac{Br^2\sin\theta}{e^{2(\nu-\zeta)}}\left\{\left[\frac{e^{2(\nu-\zeta)}}{r^2} - \frac{(\omega^\theta)^2}{\sigma^2\gamma^2}\right]\frac{\partial \delta V}{\partial \theta} - \frac{\omega^r\omega^\theta}{\sigma^2\gamma^2}\frac{\partial \delta V}{\partial r}\right\}\right] + \Upsilon(\sigma)\delta V\,, \tag{67}$$

where $\Upsilon(\sigma)$ is defined by

$$\Upsilon(\sigma) = \sigma^2\gamma^2\frac{(\rho+p)^2}{p\Gamma}\frac{Br^2\sin\theta}{e^{2(\nu-\zeta)}} - \Lambda(\rho+p)\frac{(m+\sigma S)^2}{\gamma^2 B e^{2(\nu-\zeta)}\sin\theta} - \gamma^2\frac{\partial}{\partial r}\left[\Lambda(\rho+p)\frac{(m+\sigma S)r^2\omega^\theta}{\sigma\gamma^4 e^{2(\nu-\zeta)}}\right] + \gamma^2\frac{\partial}{\partial\theta}\left[\Lambda(\rho+p)\frac{(m+\sigma S)\omega^r}{\sigma\gamma^4 e^{2(\nu-\zeta)}}\right] + \gamma\frac{\partial}{\partial r}\left[\Lambda(\rho+p)\frac{Br^2\sin\theta}{\gamma^2 e^{2(\nu-\zeta)}}\left\{\left[e^{2(\nu-\zeta)} - \frac{(\omega^r)^2}{\sigma^2\gamma^2}\right]\frac{\partial\gamma}{\partial r} - \frac{\omega^r\omega^\theta}{\sigma^2\gamma^2}\frac{\partial\gamma}{\partial\theta}\right\}\right] + \gamma\frac{\partial}{\partial\theta}\left[\Lambda(\rho+p)\frac{Br^2\sin\theta}{\gamma^2 e^{2(\nu-\zeta)}}\left\{\left[\frac{e^{2(\nu-\zeta)}}{r^2} - \frac{(\omega^\theta)^2}{\sigma^2\gamma^2}\right]\frac{\partial\gamma}{\partial\theta} - \frac{\omega^r\omega^\theta}{\sigma^2\gamma^2}\frac{\partial\gamma}{\partial r}\right\}\right]. \tag{68}$$

Equation (67) is elliptic as long as $\Lambda > 0$. The variational principle for the frequency $\sigma = \omega + m\Omega$, equation (46), can be expressed in this case as the two-dimensional integral

$$0 = \int_0^\infty dr\int_0^\pi d\theta\left[\Upsilon(\sigma)|\delta V|^2 - \Lambda(\rho+p)\frac{Br^2\sin\theta}{e^{2(\nu-\zeta)}}\left\{\left[e^{2(\nu-\zeta)} - \frac{(\omega^r)^2}{\sigma^2\gamma^2}\right]\left|\frac{\partial\delta V}{\partial r}\right|^2 - 2\frac{\omega^r\omega^\theta}{\sigma^2\gamma^2}\mathrm{Re}\left(\frac{\partial \delta V}{\partial\theta}\frac{\partial\delta V^*}{\partial r}\right) + \left[\frac{e^{2(\nu-\zeta)}}{r^2} - \frac{(\omega^\theta)^2}{\sigma^2\gamma^2}\right]\left|\frac{\partial\delta V}{\partial\theta}\right|^2\right\}\right]. \tag{69}$$

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