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POST-NEWTONIAN FREQUENCIES FOR THE PULSATIONS OF RAPIDLY ROTATING NEUTRON STARS

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ABSTRACT

The formalism for computing the oscillation frequencies of rapidly rotating stars in the post-Newtonian approximation is reviewed and extended. Numerical results are presented for the frequencies of the $l = m$ *f*-modes of rapidly rotating neutron stars. The ratios of the critical angular velocities (where the mode frequencies pass through zero) to $\sqrt{\pi G \bar{\rho}_0}$ (with $\bar{\rho}_0$ the average density) are lower than their Newtonian counterparts by up to 10%. Thus post-Newtonian effects tend to enhance the gravitational-radiation-induced instability in rotating stars.

Subject headings: radiation mechanisms: gravitational — stars: neutron — stars: oscillations — stars: rotation

1. INTRODUCTION

It is well known that gravitational radiation tends to make all rotating stars unstable (Chandrasekhar 1970; Friedman & Schutz 1978). Viscosity, however, tends to counteract this instability so that only sufficiently rapidly rotating stars are in fact unstable (Lindblom & Detweiler 1977; Lindblom & Hiscock 1983). In order to determine which stars are stable, therefore, a detailed calculation of the pulsations of rapidly rotating stars must be carried out which includes both the effects of viscosity and gravitational radiation. Such calculations are very difficult. The problem of finding solutions to the pulsation equations for rapidly rotating relativistic stellar models has never been seriously attempted, let alone solved. Various approximate calculations have been completed however. For example, the equations that describe the pulsations of rapidly rotating Newtonian stars have been solved, including the effects of viscosity and gravitational radiation (Ipser & Lindblom 1989, 1990, 1991). These calculations are unrealistic due to their neglect of relativistic effects in the equations for the structure and pulsations of the stars and due to their use of very idealized equations of state for the stellar matter. More realistic calculations have also been carried out using the full relativistic equations and using more realistic equations of state (Cutler & Lindblom 1987; Cutler, Lindblom, & Splinter 1990), but these calculations are limited to nonrotating stars.

Although idealized, these calculations do give some approximate understanding of the gravitational-radiation instability in rotating neutron stars. The shear viscosity of neutron-star matter scales with temperature like T^{-2}. Therefore, in sufficiently cold neutron stars, the viscosity is so large that it completely suppresses the gravitational-radiation instability in all rotating stars. The approximate calculations described above indicate that this complete suppression occurs when $T < 10^7$ K. In hotter stars the instability may occur, but only in stars rotating faster than about 90% of the maximum equilibrium angular velocity. In the very hottest stars, $T \geq 5 \times 10^{10}$ K, the bulk viscosity (which scales with temperature like T^6) becomes very large and completely suppresses the instability in all rotating stars. While the exact values of the neutron-star viscosities are not known with much precision, the temperature scalings are probably correct. An error of an order of magnitude in the shear viscosity, therefore, would result in an error of only a factor of 3 in the temperature below which the gravitational-radiation instability was suppressed. Similarly an error of an order of magnitude in the bulk viscosity would result in only a 50% error in the suppression temperature.

In an attempt to improve our understanding of these instabilities, we present here the results of another approximate calculation of the pulsation frequencies of rapidly rotating stars. In typical realistic neutron-star models the gravitational field is fairly weak in the sense that GM/c^2R is considerably less than one. Therefore we expect that the structure of the star and its gravitational field can be reasonably well approximated in a post-Newtonian expansion of general relativity. A formalism for calculating the post-Newtonian corrections to the modes of rotating stars has been developed by Cutler (1991) (Paper I). In particular, Cutler showed how the post-Newtonian corrections to the mode frequencies can be determined without solving the full post-Newtonian mode equations. In this paper we use this method to calculate numerically the oscillation frequencies of the $l = m$ *f*-modes of rapidly rotating polytropic neutron-star models. The *f*-modes are the lowest frequency *p*-modes (i.e., the modes that have significant density and pressure perturbations) for given values of l and m. These are the modes that play the most significant role in the gravitational-radiation instability.

The plan of this paper is as follows. In § 2 we review and somewhat extend the post-Newtonian formalism developed in Paper I. We also provide a more thorough discussion than in Paper I of the appropriate boundary conditions to be imposed on the post-Newtonian mode solutions, and their role in determining the post-Newtonian correction to the mode eigenfrequency. In § 3 we present our numerical results for the post-Newtonian oscillation frequencies. We estimate the accuracy of our method by comparing the post-Newtonian frequencies for nonrotating stars with the full, general-relativistic mode frequencies. The post-Newtonian results are found to agree with the exact ones to within a few percent for typical neutron stars. We then compute the frequency corrections for the $l = m$ f-modes of rotating stars, and we determine, in the post-Newtonian approximation, the critical angular velocities where these frequencies vanish. These are the angular velocities where these modes would become unstable to the gravitational-radiation-induced instability in the absence of viscosity. We find that post-Newtonian effects lower, by up to 10%, the ratios of these critical angular velocities to $\sqrt{\pi G \bar{\rho}_0}$ where $\bar{\rho}_0$ is the average density of the star. Thus, post-Newtonian effects tend to make the gravitational-radiation instability more important.

The numerical methods we use for solving the post-Newtonian equations are generally the same as those developed by Ipser & Lindblom (1990) for solving the corresponding Newtonian problem. However some additional techniques are required for solving the equation for the post-Newtonian gravito-magnetic vector potential, and these are described in the Appendix.

2. THE POST-NEWTONIAN APPROXIMATION

This section has a number of related purposes. The first is to review the formalism developed in Paper I for determining the post-Newtonian corrections to the equilibrium structures and the pulsation modes of rotating stellar models. Our treatment is somewhat more general than in Paper I, in the following way. In approximating a given general relativistic star in a post-Newtonian expansion, there exists the freedom to choose which "corresponding" Newtonian star to expand around. In Paper I, the Newtonian stellar model was chosen to have the same angular velocity Ω as the general relativistic star; that is, the post-Newtonian correction to the angular velocity, $\Delta\Omega/c^2$, was set to zero. However, numerical experimentation has shown us that $\Delta\Omega/c^2 = 0$ is not the most useful choice when approximating rapidly rotating stars. Hence our second purpose is to write explicitly, without rederiving them, a number of equations that are more general than the corresponding equations in Paper I, in that they include terms proportional to $\Delta\Omega/c^2$. (Throughout this paper we use the convention that $\Delta Q/c^2$ represents the post-Newtonian correction to some Newtonian quantity Q.) Third, we use this section as an opportunity to discuss more thoroughly the appropriate boundary conditions to be imposed on the post-Newtonian mode solutions. And finally, we derive some additional post-Newtonian formulae—expressions for the kinetic energy, potential energy, and Keplerian angular velocity of rotating, equilibrium stars—that are useful for characterizing these stars.

We begin by reviewing our basic assumptions. We assume the stellar matter is a perfect fluid; that is, its stress-energy tensor has the form

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + p g^{\alpha\beta} , \tag{1}$$

where ϵ is the energy density, p the pressure, and u^α the four-velocity of the stellar fluid, and where $g_{\alpha\beta}$ is the space-time metric. We also assume the fluid has a one-parameter equation of state, $p = p(\epsilon/c^2)$. (We remark that while the bulk motions of matter are required to be nearly Newtonian for the post-Newtonian approximation to be useful, no such requirement is placed on the microscopic properties of matter. Even in Newtonian fluid theory we are free to use an equation of state that is derived using relativistic quantum field theory. Hence the equation of state itself is not expanded in post-Newtonian fashion, but rather we proceed as if we had in our possession the "correct" equation of state, fixed once and for all.) Under these conditions the dynamics of the stellar matter is determined completely by Einstein's equation,

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta} . \tag{2}$$

Realistic neutron-star models typically have surface red-shifts of $z \approx 0.3$, and so we expect in this context the "post-Newtonian solutions " to equation (2) to be useful approximations. Following Chandrasekhar (1965) (and for the notation used here, Paper I) we expand the spacetime metric and the fluid variables as formal series in inverse powers of the speed of light $1/c^n$:

$$ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta = -\left[1 + \frac{2}{c^2}\Phi + \frac{2}{c^4}(\Phi^2 + \Psi) + \mathcal{O}(c^{-6})\right]c^2 dt^2 + \left[\frac{2}{c^3} A_a + \mathcal{O}(c^{-5})\right] c\, dt\, dx^a$$

$$+ \left[e_{ab}\left(1 - \frac{2}{c^2}\Phi\right) + \mathcal{O}(c^{-4})\right] dx^a dx^b , \tag{3}$$

$$\epsilon = \rho c^2 + (\sigma + 2\rho\Phi - \rho v^2) + \mathcal{O}(c^{-2}) , \tag{4}$$

$$p = p(\rho) + \frac{1}{c^2}\frac{dp}{d\rho}(\sigma + 2\rho\Phi - \rho v^2) + \mathcal{O}(c^{-4}) , \tag{5}$$

$$cu^t = 1 + \frac{1}{2c^2}(v^2 - 2\Phi) + \frac{1}{2c^4}\left[\Phi^2 - 3\Phi v^2 - 2\Psi - \frac{1}{4}v^4 + 2v^a(A_a + w_a)\right] + \mathcal{O}(c^{-6}) , \tag{6}$$

$$u^a = \frac{1}{c} v^a + \frac{1}{c^3} w^a + \mathcal{O}(c^{-5}) . \tag{7}$$

Equations (3)–(7) serve as definitions of the Newtonian fields ρ, v^a, and Φ, and of the post-Newtonian fields σ, w^a, Ψ, and A_a, which represent the next-order "corrections" to the Newtonian quantities. Our convention is that Latin letters represent spatial indices which are raised and lowered with the flat Euclidean metric e_{ab} and its inverse e^{ab}. We denote the derivative operator compatible with e_{ab} by D_a, and we use the shorthand $v^2 = v^a v_a$.

Having expanded the metric and stress tensor as above, the next step is to expand Einstein's equation (2) and the associated conservation law, $\nabla_\alpha T^{\alpha\beta} = 0$, as formal power series in $1/c$, and then (following Gunnarsen 1991) to set the coefficient of each power of $1/c$ in these expansions equal to zero. The coefficients of the lowest-order terms in this expansion give the standard Newtonian equations for ρ, v^a, and Φ. The next-order terms give the post-Newtonian equations for the fields σ, w^a, Ψ, and A^a (eqs. [14]–[17] in Paper I). Solutions of the Newtonian and post-Newtonian equations are to be combined as prescribed in equations (3)–(7) to produce an approximate solution of Einstein's equation (2).

We next summarize how to obtain the post-Newtonian corrections to the structure of any given stationary Newtonian "background" stellar model. We restrict our attention to uniformly rotating stars. In full general relativity, the condition of uniform rotation is just that the fluid four-velocity have the form

$$u^\alpha \propto t^\alpha + \varphi^\alpha \Omega_{\rm GR} , \tag{8}$$

where $t^\alpha = (\partial/\partial t)^\alpha$ and $\varphi^\alpha = (\partial/\partial\varphi)^\alpha$ are the timelike and rotational Killing fields, and, the angular velocity $\Omega_{\rm GR}$ is constant throughout the star. Expanding $\Omega_{\rm GR}$ as

$$\Omega_{\rm GR} = \Omega + \Delta\Omega/c^2 + \mathcal{O}(c^{-4}) \tag{9}$$

and using equations (8)–(9) in equations (6)–(7), we find that the post-Newtonian correction w^a to the Newtonian velocity $v^a = \Omega\varphi^a$ is given by

$$w^a = \frac{1}{2}\left(v^2 - 2\Phi + 2\frac{\Delta\Omega}{\Omega}\right)v^a . \tag{10}$$

There are two free parameters to choose in selecting which (rigidly rotating) Newtonian stellar model to associate with a given general-relativistic star. We like to use the post-Newtonian corrections to the mass and angular velocity of the star, $\Delta M/c^2$ and $\Delta\Omega/c^2$, as these two parameters. Given a choice of $\Delta\Omega$ (which determines w^a by eq. [10]), the other fields characterizing the post-Newtonian corrections to this stationary model, A^a, σ, and Ψ, satisfy the equations:

$$D^bD_b A^a = 16\pi G\rho v^a , \tag{11}$$

$$D^aD_a\Psi = 4\pi G(\sigma + \rho v^2 + 3p) , \tag{12}$$

$$\Delta C = \frac{1}{4}v^4 - \Psi - 2\Phi v^2 + v^aA_a + v^2\frac{\Delta\Omega}{\Omega} - \frac{1}{\rho}\frac{dp}{d\rho}(\sigma - \rho v^2 + 2\rho\Phi) + \int_0^p \frac{\bar{p}\,d\bar{p}}{[\rho(\bar{p})]^2} , \tag{13}$$

where ΔC is a constant. Equations (11)–(13) may be solved in two steps. First, solve the Poisson-type equation (11) for A^a. (Note that A^a is completely determined by the background Newtonian solution and the boundary condition $A^a \to 0$ as $r \to \infty$.) Second, use equation (13) to eliminate σ from equation (12). The resulting equation is

$$D^aD_a(\Psi + \Delta C) + 4\pi G\rho\frac{d\rho}{dp}(\Psi + \Delta C) = 4\pi G(2\rho v^2 + 3p - 2\rho\Phi) + 4\pi G\rho\frac{d\rho}{dp}\left\{\frac{1}{4}v^4 - 2\Phi v^2 + v^aA_a + v^2\frac{\Delta\Omega}{\Omega} + \int_0^p\frac{\bar{p}\,d\bar{p}}{[\rho(\bar{p})]^2}\right\} . \tag{14}$$

This is an elliptic equation for $\Psi + \Delta C$ whose right side depends only on the background Newtonian solution and the previously determined post-Newtonian field A^a. A boundary condition for this potential must be specified in order that this equation have a unique solution. This boundary condition is equivalent to specifying the post-Newtonian correction to the mass, $\Delta M/c^2$, for

$$\Delta M = \frac{1}{4\pi G}\int n^aD_a(\Psi + \Delta C)d^2x , \tag{15}$$

where the integral is to be performed over any closed two-surface which contains the entire star. Once $\Psi + \Delta C$ is known, the post-Newtonian field σ is determined by equation (13).

Having found the corrections to the stationary stellar model, we want to calculate the post-Newtonian corrections to the pulsation frequencies. Fortunately, the formalism developed by Ipser & Lindblom (1989, 1990) for solving the Newtonian pulsation equations is readily adapted to the post-Newtonian equations. We begin by briefly summarizing that Newtonian formalism. Restrict attention to a single mode, having time and angular dependence $e^{-i\omega t + im\varphi}$. Define the scalar potential δU by

$$\delta U \equiv \delta\Phi + \frac{1}{\rho}\frac{dp}{d\rho}\delta\rho , \tag{16}$$

and the constant Hermitian tensor field Q^{ab} by

$$Q^{-1}_{ab} = -(\omega - m\Omega)e_{ab} - 2iD_b v_a . \tag{17}$$

Then, the Euler equation for the fluid acceleration can be "solved" explicitly:

$$\delta v^a = iQ^{ab}D_b\delta U . \tag{18}$$

Eliminating δv^a from the remaining equations in favor of δU, the Newtonian mode equations are reduced to the following system of second-order (typically elliptic) equations in δU and $\delta\Phi$:

$$P\begin{pmatrix}\delta U\\ \delta\Phi\end{pmatrix} \equiv \begin{pmatrix} D_a(\rho Q^{ab}D_b\delta U) - (\omega - m\Omega)\rho\dfrac{d\rho}{dp}(\delta U - \delta\Phi) \\ (\omega - m\Omega)\rho\dfrac{d\rho}{dp}(\delta U - \delta\Phi) - \dfrac{\omega - m\Omega}{4\pi G}D^aD_a\delta\Phi \end{pmatrix} = 0 . \tag{19}$$

The frequency ω plays the role of an eigenvalue in equation (19) since only for certain discrete values of ω do the solutions satisfy the appropriate boundary conditions. These boundary conditions are that $r\delta\Phi \to 0$ as $r \to \infty$ and that δU be smooth on the surface of the star. Since δU is not defined outside the star, by δU being smooth at the surface we mean that there exists *some* smooth extension of δU into the exterior of the star. This second condition implies the usual boundary condition: that the perturbed fluid density vanishes at the surface of the perturbed star. This implication can be seen by evaluating equation (19) at the star's surface. While smoothness of δU is almost certainly a stronger condition than is actually needed here (e.g., C^2 is probably sufficient), it makes our discussion considerably simpler.

The post-Newtonian corrections to the pulsations of a star are described by the fields $\delta\sigma$, δw^a, $\delta\Psi$, and δA^a, which satisfy the linearized (with respect to the amplitude of the pulsation) post-Newtonian equations. We seek a solution of these linearized post-Newtonian equations that, when added to the Newtonian mode solution as prescribed in equations (3)–(7), yields a mode solution of the general-relativistic pulsation equations—up to higher-order terms in powers of $1/c$. The frequency of the mode in full general relativity, $\omega_{\rm GR}$, will not in general equal the Newtonian frequency, ω, but rather

$$\omega_{\rm GR} = \omega + \Delta\omega/c^2 + \mathscr{O}(c^{-4}) , \tag{20}$$

where $\Delta\omega/c^2$ is the post-Newtonian correction to the frequency. This expression (20) determines, then, the time dependence of the linearized post-Newtonian fields. For instance, the expansion of the general-relativistic equation, $\partial_t \delta\epsilon = -i\omega_{\rm GR}\delta\epsilon$ to post-Newtonian order,

$$\partial_t \delta(\rho c^2 + \sigma + 2\rho\Phi - \rho v^2) = -i(\omega + \Delta\omega/c^2)\delta(\rho c^2 + \sigma + 2\rho\Phi - \rho v^2) + \mathscr{O}(c^{-2}) , \tag{21}$$

implies that $\delta\sigma$ has time dependence:

$$\partial_t \delta\sigma = -i\omega\,\delta\sigma - i\Delta\omega\,\delta\rho . \tag{22}$$

Thus, in order that the combined Newtonian and post-Newtonian terms have sinusoidal time dependence—up to higher order terms in the expansion—the post-Newtonian terms themselves must have the nonsinusoidal time dependence given in equation (22). Analogous results apply as well to the time dependencies of δw^a and $\delta\Psi$.

The first step in obtaining the post-Newtonian mode solution is to solve the equation,

$$D^b D_b \delta A_a = 16\pi G(\delta\rho\, v_a + \rho\,\delta v_a) - i\omega D_a \delta\Phi , \tag{23}$$

for δA^a. Thus, δA^a is completely determined by the Newtonian mode solution and appropriate boundary conditions. (In the Appendix we describe two "tricks" that facilitate the numerical solution of eq. [23].) To solve for the remaining fields $\delta\sigma$, δw^a, and $\delta\Psi$, we begin by defining the scalar potential δW:

$$\delta W \equiv \delta\Psi + \frac{1}{\rho}\frac{dp}{d\rho}\,\delta\sigma . \tag{24}$$

Then, in complete analogy with equation (18), the field δw^a can be eliminated in favor of δW:

$$\delta w^a = iQ^{ab}[D_b \delta W - i\Delta\omega\,\delta v_b - i(\omega - m\Omega)\delta A_b - D_b(v^c \delta A_c) - \delta B_b] , \tag{25}$$

where the quantity δB_a depends only on the previously determined Newtonian and stationary post-Newtonian fields,

$$\begin{aligned}\delta B_a = {}& \frac{1}{\rho} D_a\left\{\left[(v^2 - 2\Phi)\frac{d}{d\rho}\left(\rho\frac{dp}{d\rho}\right) - \sigma\frac{d^2p}{d\rho^2}\right]\delta\rho - 2\rho\frac{dp}{d\rho}(\delta\Phi - v^b\delta v_b)\right\} - \frac{\delta\rho}{\rho} D_a\Psi - \left(2\Phi + v^2 + \frac{p+\sigma}{\rho}\right)D_a\delta\Phi \\ & - \left(\Phi + \frac{1}{2}v^2 - \frac{p+\sigma}{\rho} - \frac{\Delta\Omega}{\Omega}\right)D_a\delta U + iv_a\left\{\frac{\omega}{\rho}\frac{dp}{d\rho}\,\delta\rho - (2\omega - 3m\Omega)\delta\Phi - [(\omega - m\Omega)v_b + 4iD_b\Phi]\delta v^b\right\} \\ & - D_a\Phi\left[2v^b\delta v_b + 2\,\delta\Phi + \left(v^2 + 2\Phi + \frac{dp}{d\rho}\right)\frac{\delta\rho}{\rho}\right] + D_a(v^bA_b)\frac{\delta\rho}{\rho} - i\omega\frac{\Delta\Omega}{\Omega}\delta v_a - v^bD_a v_b\left[2v_c\,\delta v^c + \left(2\Phi - \frac{dp}{d\rho} - 2\frac{\Delta\Omega}{\Omega}\right)\frac{\delta\rho}{\rho}\right] \\ & + (D_a A_b - D_b A_a)\delta v^b .\end{aligned} \tag{26}$$

The post-Newtonian mode equations then reduce to two coupled (typically elliptic) equations in δW and $\delta\Psi$,

$$P\begin{pmatrix}\delta W\\ \delta\Psi\end{pmatrix} = \begin{pmatrix}\Delta\omega[\delta\rho + iD_a(\rho Q^{ab}\delta v_b)] + dX_W\\ -(\omega - m\Omega)\delta X_\Psi\end{pmatrix}, \tag{27}$$

where P is the operator defined in equation (19). Thus, the post-Newtonian pulsation equation (27) is just an inhomogeneous generalization of the Newtonian equation (19). The right side of equation (27) depends only on the post-Newtonian correction to the frequency $\Delta\omega/c^2$, and the previously determined fields. In particular the quantities δX_W and δX_Ψ depend only on previously determined fields:

$$\begin{aligned}\delta X_W = {}& -\rho(\omega + m\Omega)\,\delta\Phi + D_a\{\rho Q^{ab}[\delta B_b + i(\omega - m\Omega)\delta A_b + D_b(v^c\,\delta A_c)]\} + m\Omega\left[\rho v^a\,\delta v_a - \left(\frac{dp}{d\rho} + \frac{\Delta\Omega}{\Omega}\right)\delta\rho\right] \\ & + iD_a\left[\left(\sigma + p + \rho\Phi - \frac{1}{2}\rho v^2\right)\delta v^a\right],\end{aligned} \tag{28}$$

$$\delta X_\Psi = \left(v^2 + 3\frac{dp}{d\rho}\right)\delta\rho + 2\rho v_a\,\delta v^a . \tag{29}$$

We now turn to the question of the appropriate boundary conditions to impose on the solutions of equation (27). We demand that $r\delta\Psi \to 0$ as $r \to \infty$ in order to ensure that the mass of the perturbed star is the same as the mass of the equilibrium model. We also demand as a second boundary condition that δW have some smooth extension to the exterior of the star. This second condition implies (for the class of equations of state of interest to us) the usual boundary condition: that the perturbed density vanishes on the perturbed surface of the star to post-Newtonian order. We digress, briefly, to establish this implication.

We have assumed that the Newtonian potential δU is smooth at the star's surface, and so it follows that δv^a and δA^a must also be smooth as consequences of equations (18) and (23). The smoothness of δW would imply the smoothness of the post-Newtonian velocity perturbation δw^a via equation (25), if δB^a were smooth. The expression for δB^a, equation (26), contains a number of terms proportional to ρ^{-1} that are not obviously bounded, let alone smooth, at the surface of the star. However, for the equations of state of interest to us, these singular terms cancel. A sufficient assumption on the equation of state is that $\rho^{-1}(dp/d\rho)$ is a strictly positive, smooth function of ρ for $\rho \geq 0$. While this assumption is probably stronger than is needed, it includes the $\Gamma = 2$ polytropes that are the subject of our numerical analysis in § 3. Under this assumption, it follows (after a fair amount of algebraic manipulation) that δB^a is smooth at the surface of the star. Thus, the smoothness of δW implies the smoothness of δw^a for the equations of state considered here. We remark that for these equations of state the density ρ goes continuously to zero at the star's surface while σ, δp, and $D_a \rho$ are finite but not zero at the surface.

We next introduce the vector field $\Delta x^a/c^2$ that represents the post-Newtonian correction to the location of the star's surface. It is defined by the requirement that the density ϵ vanishes on the surface of the general-relativistic star; thus, to post-Newtonian order

$$\sigma + \Delta x^a D_a \rho = 0 , \tag{30}$$

evaluated on the surface of the Newtonian star. Using equation (30) and the Newtonian mass-conservation law, it follows that

$$\Delta x^b D_b[\delta v^a D_a \rho - i(\omega - m\Omega)\delta\rho] = \sigma D_a \delta v^a , \tag{31}$$

also evaluated on the star's surface. Now, the post-Newtonian energy-conservation law (eq. [46] of Paper I), evaluated at the surface of the star where $\rho = 0$, reduces to the condition,

$$\sigma D_a \delta v^a + \delta w^a D_a \rho + \delta v^a D_a \sigma - i(\omega - m\Omega)\delta\sigma - i(\Delta\omega - m\Delta\Omega)\delta\rho + (\Phi - \tfrac{1}{2}v^2)\delta v^a D_a \rho = 0 . \tag{32}$$

Thus, combining equation (31) with (32) we obtain

$$\Delta x^b D_b[\delta v^a D_a \rho - i(\omega - m\Omega)\delta\rho] + \delta v^a D_a(\sigma + 2\rho\Phi - \rho v^2) - i(\omega - m\Omega)(\tfrac{1}{2}v^2 - \Phi)\delta\rho - i(\omega - m\Omega)\delta(\sigma + 2\rho\Phi - \rho v^2) + \delta w^a D_a \rho - i(\Delta\omega - m\Delta\Omega)\delta\rho = 0 . \tag{33}$$

The motion of a fluid element on the star's surface will follow the perturbed motion of this surface iff $(u^\alpha \nabla_\alpha \epsilon) = 0$ on the surface. This condition is equivalent to the usual boundary condition that the density remain zero on the perturbed surface of the star. Using equations (4), (6), (7), and (10), the quantity $\delta(u^\alpha \nabla_\alpha \epsilon)$ can be expanded to post-Newtonian order:

$$\begin{aligned}\delta(u^\alpha \nabla_\alpha \epsilon) = c[\delta v^a D_a \rho - i(\omega - m\Omega)\delta\rho] + c^{-1}\Delta x^b D_b[\delta v^a D_a \rho - i(\omega - m\Omega)\delta\rho] + c^{-1}[\delta v^a D_a(\sigma + 2\rho\Phi - \rho v^2)\\ - i(\omega - m\Omega)\delta(\sigma + 2\rho\Phi - \rho v^2) - i(\omega - m\Omega)(\tfrac{1}{2}v^2 - \Phi)\delta\rho + \delta w^a D_a \rho - i(\Delta\omega - m\Delta\Omega)\delta\rho] + \mathcal{O}(c^{-3}) .\end{aligned} \tag{34}$$

The quantity $\delta(u^\alpha \nabla_\alpha \epsilon)$ is to be evaluated on the surface of the general-relativistic star, while the expressions on the right side of equation (34) are to be evaluated on the surface of the Newtonian star. We see that equation (34) implies that $\delta(u^\alpha \nabla_\alpha \epsilon) = 0$ as a consequence of equation (33) and the Newtonian mass-conservation law. Thus, we conclude that the smoothness of δW at the star's surface implies the usual form of the boundary condition: that the perturbed density vanishes on the perturbed surface of the star to post-Newtonian order.

We can now derive an expression for the post-Newtonian change in the frequency, $\Delta\omega/c^2$, of a pulsation mode. Note that the operator P in the post-Newtonian mode equation (27) has a nontrivial kernel, namely the Newtonian solution $(\delta U, \delta\Phi)$. Hence equation (27) will not have any solution at all unless the source on its right side has a certain inner product with $(\delta U^*, \delta\Phi^*)$. Take the inner product of each side of equation (27) with the row vector $(\delta U^*, \delta\Psi^*)$, and integrate over the interior of the star (where $\rho > 0$). If the operator P acting on $(\delta W, \delta\Psi)$ were Hermitian, then the resulting left side would vanish. Consequently, the vanishing of the integrals on the right side of equation (27) would fix the value of $\Delta\omega$. However, the boundary conditions on $(\delta W, \delta\Psi)$ are such that P is only "almost" Hermitian.

The failure of P to be Hermitian when it operates on $(\delta W, \delta\Psi)$ can be traced to the failure of $D_a \delta\Psi$ to be continuous at the boundary of the star. Since σ jumps discontinuously to zero at the star's boundary, the post-Newtonian mass-conservation law (e.g., eq. [32]) implies that $\delta\sigma$ will behave there like a Dirac delta function. Equation (32) implies that

$$\delta\sigma \sim -i \frac{D_a(\sigma\delta v^a)}{\omega - m\Omega} , \tag{35}$$

where " $\sim$ " means that the difference between the two functions is everywhere finite. Then, from equation (27) it follows that

$$D^a D_a \delta\Psi \sim -i4\pi G \frac{D_a(\sigma\delta v^a)}{\omega - m\Omega} . \tag{36}$$

Integrating this expression over a thin shell enclosing the star's surface, we find by Stokes's theorem that

$$n^a[D_a \delta\Psi_{\text{in}} - D_a \delta\Psi_{\text{out}}] = -i4\pi G \frac{\sigma\delta v_a n^a}{\omega - m\Omega} , \tag{37}$$

where $D_a\,\delta\Psi_{\rm out}$ is evaluated just outside the star's surface, while $D_a\,\delta\Psi_{\rm in}$ and the right side of equation (37) are evaluated just inside; and n^a is the outward-pointing normal to the surface. This condition implies, in turn, that $\delta\Psi^*$ and $\delta\Psi$ satisfy the identity,

$$\int(\delta\Phi^* D^a D_a\,\delta\Psi - \delta\Psi D^a D_a\,\delta\Phi^*)\,d^3x = \int \delta\Phi^* n^a[(D_a\,\delta\Psi)_{\rm in} - (D_a\,\delta\Psi)_{\rm out}]d^2x\,, \tag{38}$$

where the integral on the left side is taken over the star's interior, where $\rho > 0$. This identity is readily derived by using the fact that $\delta\Phi$ and $\delta\Psi$ are solutions of the (source-free) Laplace equation in the exterior which fall to zero faster than $1/r$. It is the nonvanishing of the right side of equation (38) that is responsible for the non-Hermiticity of the operator P. Therefore, when we multiply equation (27) by $(\delta U^*, \delta\Phi^*)$ and integrate over the stellar interior, the left side becomes

$$\int(\delta U^*, \delta\Phi^*)\cdot P\begin{pmatrix}\delta W\\ \delta\Psi\end{pmatrix}d^3x = i\int \sigma\delta\Phi^*\delta v_a n^a\,d^2x\,. \tag{39}$$

Combining this with equation (27) gives the following expression for $\Delta\omega$,

$$\Delta\omega\int(\rho\delta v_a^*\,\delta v^a + \delta\rho\delta U^*)d^3x = \int[(\omega - m\Omega)\delta X_\Psi\,\delta\Phi^* - \delta X_W\,\delta U^*]d^3x + i\int\sigma\delta\Phi^*\delta v_a n^a\,d^2x\,. \tag{40}$$

These integrals completely determine $\Delta\omega$ in terms of the previously determined functions which describe the equilibrium structures of the Newtonian and post-Newtonian stars, and the Newtonian pulsation mode. This expression for $\Delta\omega$ does *not* depend on the functions δW and $\delta\Psi$ which determine the structure of the post-Newtonian mode itself. It can be shown from this expression that $\Delta\omega$ is real whenever ω is real. Once $\Delta\omega$ has been determined, equation (27) can be solved for the potentials δW and $\delta\Psi$. However, it is not necessary to determine these potentials if only the frequency of the mode is desired.

We now derive some formulae that are useful for characterizing the time-independent, rotating stellar models (see also Cutler & Lindblom 1991). The rotation rates of stars are often parameterized by the dimensionless quantity $\tau = -K/W$, the ratio of the rotational kinetic energy of the star to its gravitational potential energy. We use the general-relativistic definitions of these quantities given by Friedman et al. (1986):

$$K_{\rm GR} = \frac{\Omega}{2c}\int(\epsilon + p)u^\alpha\varphi_\alpha u^\beta\,dS_\beta\,, \tag{41}$$

$$W_{\rm GR} = \int\{[2(\epsilon + p)u^\alpha u^\beta + (\epsilon - p)g^{\alpha\beta}]t_\alpha - \epsilon u^\beta\}dS_\beta - K_{\rm GR}\,, \tag{42}$$

where t_α is the globally timelike Killing field, φ_α is the rotational Killing field, and the integrals are performed over a t = constant hypersurface with volume element dS_β. Writing $K_{\rm GR} = K + \Delta K/c^2 + \mathcal{O}(c^{-4})$ and $W_{\rm GR} = W + \Delta W/c^2 + \mathcal{O}(c^{-4})$, and using the expansions in equations (3)–(7) for the various quantities that appear in these integrals, expressions may be obtained for K, W, ΔK, and ΔW. The first-order terms in these expansions yield the standard Newtonian expressions for K and W:

$$K = \tfrac{1}{2}\int\rho v^2\,d^3x\,, \tag{43}$$

$$W = \int(\rho\Phi + \rho v^2 + 3p)d^3x = \tfrac{1}{2}\int\rho\Phi\,d^3x\,. \tag{44}$$

(The second equality in eq. [44] is a consequence of the equilibrium equations.) The post-Newtonian corrections to these quantities are given by the second-order terms in the expansions of equations (41) and (42):

$$\Delta K = \frac{1}{2}\int\left[\left(\sigma + p - 4\rho\Phi + 2\rho\,\frac{\Delta\Omega}{\Omega}\right)v^2 + \rho v^a A_a\right]d^3x\,, \tag{45}$$

$$\Delta W = -\Delta K + \frac{1}{2}\int\left[\rho\left(-13v^2\Phi - \Phi^2 + \frac{1}{4}\,v^4 + 2\Psi + 6v^2\,\frac{\Delta\Omega}{\Omega} + 2v^a A_a\right)\right.$$
$$\left. + 4p(v^2 - 3\Phi) + \sigma(3v^2 + 2\Phi) + 6\,\frac{dp}{d\rho}\,(\sigma + 2\rho\Phi - \rho v^2)\right]d^3x\,. \tag{46}$$

We next consider the post-Newtonian correction to the terminal angular velocity of a sequence of rotating models. A sequence of stellar models (Newtonian or general relativistic) terminates when the star's angular velocity Ω is equal to $\Omega_{\rm K}$ (the angular velocity of a test particle in orbit at the star's surface in the equatorial plane), referred to as the "Keplerian" angular velocity. For Newtonian stellar models the Keplerian angular velocity is given by the expression

$$\Omega_{\rm K}^2 = \frac{1}{r}\,\frac{d\Phi}{dr}\,, \tag{47}$$

where the quantity on the right side is to be evaluated in the equatorial plane at the surface of the star. Expanding $\Omega_{\rm K,GR} = \Omega_{\rm K} + \Delta\Omega_{\rm K}/c^2 + \mathcal{O}(c^{-4})$ and solving the geodesic equation for the metric in equation (3), we obtain the following expression for the post-Newtonian correction to the Keplerian angular velocity of such a test particle:

$$\Delta\Omega_{\rm K} = \frac{1}{2r\Omega_{\rm K}}\left[2\Phi\,\frac{d\Phi}{dr} + \frac{d\Psi}{dr} - \frac{\Omega_{\rm K}}{\Omega}\frac{d(v^a A_a)}{dr} + \Omega_{\rm K}^2\,\frac{d(r^2\Phi)}{dr} + \left(\frac{d^2\Phi}{dr^2} - \Omega_{\rm K}^2\right)\Delta r\right], \tag{48}$$

where the quantities on the right side are evaluated in the equatorial plane at the surface of the Newtonian star, and $\Delta r/c^2$ represents the post-Newtonian change on the surface of the star at the equator (as defined in eq. [30]).

3. NUMERICAL RESULTS

We have used the formalism described in the previous section to calculate numerically the post-Newtonian corrections to the pulsation frequencies of rapidly rotating neutron stars. The numerical methods used to solve the equations in this paper are essentially the same as those described in Ipser & Lindblom (1989, 1990). (The extension needed to solve the vector Poisson equation for the gravito-magnetic potential is described in the Appendix.) This numerical method takes advantage of the axisymmetry and approximate spherical symmetry of the equilibrium stellar models, and the sinusoidal angular dependence and definite parity of the pulsation modes. The equations were solved on a two-dimensional radial grid consisting of 1600 evenly spaced points on each of 20 radial spikes having angles in the range $0 < \theta \leq \pi/2$.

For this study we have selected a simple polytropic equation of state,

$$p = \kappa\epsilon^2/c^4\,, \tag{49}$$

where κ is a constant. While there are various definitions of "relativistic polytrope" in the literature, in this paper we refer to equations of state having the form $p = \kappa\epsilon^\Gamma/c^{2\Gamma}$ as polytropic. The particular equation of state used here, with $2 = \Gamma \equiv 1 + 1/n$, is commonly referred to as an $n = 1$ polytrope. This value of Γ produces stellar models having central condensations (i.e., the ratios between the central and the average densities) that are similar to more realistic neutron-star models. The choice $\kappa = 10^5$ (in cgs units) results in stellar models that are similar in size to more realistic neutron-star models. However, the specific value of κ does not change our results in any significant way. Solutions to the equations having different values of κ are related by a simple scaling. Given a solution $g_{\alpha\beta}$ to the Einstein equation with a polytropic fluid source, $p = \kappa\epsilon^\Gamma/c^{2\Gamma}$, the rescaled metric $(\kappa'/\kappa)^{1/(\Gamma-1)}g_{\alpha\beta}$ is a new solution with fluid source satisfying $p = \kappa'\epsilon^\Gamma/c^{2\Gamma}$. Thus, all physical quantities scale in a simple way under the transformation $\kappa \to \kappa'$, and many dimensionless quantities of interest are completely independent of κ. This scaling applies to all solutions of Einstein's equation with polytropic fluid source, including those which may be approximated in a post-Newtonian expansion. Thus, equations (1)–(7) determine the appropriate scalings of each of the Newtonian and post-Newtonian fields under this transformation.

Consider a sequence of rotating equilibrium general-relativistic stellar models, all with the same mass $M_{\rm GR}$ and with angular velocities ranging from zero to the terminal, Keplerian angular velocity. Our aim is to approximate these stellar models as some appropriate sequence of rotating Newtonian stars plus post-Newtonian corrections. We begin by selecting the nonrotating Newtonian model to associate with the nonrotating general-relativistic star in this sequence. After some numerical experimentation (see also Balbinski et al. 1985), we found it convenient to associate nonrotating models having the same GM/c^2R ratios, where M is the gravitational mass, and R is the radius of the star. Next, we fix the remaining Newtonian models in the sequence by requiring them to have the same mass as this nonrotating model. Post-Newtonian corrections are now added to these Newtonian models to approximate the original general-relativistic sequence. Thus, we constrain the post-Newtonian change in GM/c^2R to be zero for the nonrotating model. This constraint is imposed by adjusting the boundary condition to equation (14) so that $\Delta M/M = \Delta R/R$. And finally, the remainder of the constant-mass sequence is fixed by requiring that the post-Newtonian change in the mass $\Delta M/c^2$ be independent of angular velocity.

So far, we have selected particular Newtonian and post-Newtonian constant-mass sequences to approximate our original sequence of rotating general-relativistic stellar models. Except for the zero-angular-velocity members of the sequence, however, we have yet to specify how individual members of the two sequences are to be identified. It might seem natural to identify the Newtonian and the general-relativistic models having the same angular velocity: $\Delta\Omega = 0$. However, numerical experimentation shows that this is in fact a poor choice. The reason is, roughly speaking, that general-relativistic gravity is stronger than Newtonian gravity. Thus, a general-relativistic star will be less distorted in shape by its rotation than its Newtonian counterpart rotating at the same angular velocity. It is more appropriate, therefore, to associate models whose angular velocities are related in some more dynamically meaningful way. Various studies (e.g., Friedman et al. 1986, 1989) have shown that sequences of rotating stellar models all terminate when the ratio $\Omega/\Omega_0 \approx 0.6$, where $\Omega_0^2 = \pi G\bar{\rho}_0 \equiv 3GM_0/4R_0^3$ (with M_0 the mass and R_0 the radius of the nonrotating star in the sequence). This result applies to both Newtonian and general-relativistic stellar models and is essentially independent of the equation of state of the stellar fluid. This ratio is, therefore, a dynamically meaningful measure of the star's angular velocity. Thus, we choose to associate the Newtonian stellar model with its general-relativistic counterpart having the same ratio Ω/Ω_0. Thus, to post-Newtonian order we set $\Delta(\Omega/\Omega_0) = 0$. (This condition is equivalent to $\Delta\Omega/\Omega = \Delta\Omega_0/\Omega_0$.) In summary, we approximate a constant-mass sequence of relativistic stellar models by first constructing a constant-mass sequence of Newtonian models, and then adding post-Newtonian corrections. The post-Newtonian corrections are chosen so that ΔM is independent of angular velocity and satisfies $\Delta M/M = \Delta R/R$ for the nonrotating model, and so that $\Delta\Omega$ satisfies $\Delta\Omega/\Omega = \Delta\Omega_0/\Omega_0$.

In this paper we are primarily interested in determining the frequencies ω_m of the $l = m$ f-modes (with $2 \leq m \leq 6$) since these are the modes that are most likely to participate in the gravitational-radiation induced secular instability. It is convenient to express these frequencies as ratios with Ω_0. For Newtonian polytropic stellar models ω_m/Ω_0 depends only on the angular velocity of the star, Ω/Ω_0, and the polytropic exponent Γ. The post-Newtonian corrections to these ratios may also be written in an analogous invariant form for polytropic stellar models. For the sequences of rotating stellar models described above, the post-Newtonian correction to the frequency of the mode ω_m/Ω_0 (for a given value of Ω/Ω_0) is directly proportional to GM_0/c^2R_0, that is, the dimensionless quantity $(R_0/GM_0)\Delta(\omega_m/\Omega_0)$ depends only on Ω/Ω_0 and the polytropic exponent Γ (where M_0 is the mass and R_0 is the radius of the nonrotating star in the sequence). This result follows directly from the following scaling relations for post-Newtonian polytropes. Let the equation of state be $p = \kappa\epsilon^\Gamma/c^{2\Gamma}$, and let $(\rho, \Phi, v^a; \sigma, \Psi, w^a, A^a)$ represent a solution of the Newtonian and post-Newtonian equations. Then $(\bar\rho, \bar\Phi, \bar v^a; \bar\sigma, \bar\Psi, \bar w^a, \bar A^a)$ defined by

$$\begin{aligned}
\bar\rho(\vec{x}, t) &= \lambda^2\rho(\lambda^{2-\Gamma}\vec{x}, \lambda t)\,, & \bar\sigma(x, t) &= \lambda^{2\Gamma}\sigma(\lambda^{2-\Gamma}\vec{x}, \lambda t)\,,\\
\bar\Phi(\vec{x}, t) &= \lambda^{2\Gamma-2}\Phi(\lambda^{2-\Gamma}\vec{x}, \lambda t)\,, & \bar\Psi(\vec{x}, t) &= \lambda^{4\Gamma-4}\Psi(\lambda^{2-\Gamma}\vec{x}, \lambda t)\,,\\
\bar v^a(\vec{x}, t) &= \lambda^{\Gamma-1}v^a(\lambda^{2-\Gamma}\vec{x}, \lambda t)\,, & \bar w^a(\vec{x}, t) &= \lambda^{3\Gamma-3}w^a(\lambda^{2-\Gamma}\vec{x}, \lambda t)\,,\\
& & \bar A^a(\vec{x}, t) &= \lambda^{3\Gamma-3}A^a(\lambda^{2-\Gamma}\vec{x}, \lambda t)\,,
\end{aligned} \tag{50}$$

is also a solution for any value of the constant λ with the same (including κ) equation of state. Now consider the affect of this scaling on the constant-mass sequences of Newtonian and post-Newtonian models described above. Under this scaling $\bar M = \lambda^{3\Gamma-4}M$ and $\bar\Omega/\bar\Omega_0 = \Omega/\Omega_0$, so this scaling maps a constant-mass Newtonian sequence into another constant-mass sequence. Moreover, $\Delta\bar M = \lambda^{5\Gamma-6}\Delta M$, $\Delta(\bar\Omega/\bar\Omega_0) = \lambda^{2\Gamma-2}\Delta(\Omega/\Omega_0)$, and $\Delta(\bar M/\bar R) = \lambda^{4\Gamma-4}\Delta(M/R)$. These last three relations insure that this scaling preserves the conditions that defined the post-Newtonian sequence described above: that $\Delta\bar M$ is constant along the sequence, that $\Delta(\bar\Omega/\bar\Omega_0) = 0$ along the sequence, and that $\Delta(\bar M/\bar R) = 0$ for the zero-angular-velocity model. For rigidly rotating stars there is only a one-parameter family of constant-mass sequences, parameterized by the value of GM_0/c^2R_0 for the nonrotating model in the sequence. Since $\bar M/\bar R = \lambda^{2\Gamma-2}M/R$, this simple scaling suffices to map any sequence to any other sequence (for $\Gamma \neq 1$). Now, examination of the integral expression for $\Delta\omega$, equation (40), reveals that $\Delta(\bar\omega_m/\bar\Omega_0) = \lambda^{2\Gamma-2}\Delta(\omega_m/\Omega_0)$. That is, $\Delta(\omega_m/\Omega_0)$ scales in the same way as M_0/R_0. Hence, $(R_0/GM_0)\Delta(\omega_m/\Omega_0)$ is independent of λ and so may depend only on Ω/Ω_0 and Γ.

For the case of nonrotating stars, we can explore the range of validity of the post-Newtonian approximation for the frequencies of these modes. In Figure 1 we compare the post-Newtonian estimates and the full, general-relativistic values of ω_m/Ω_0 for the $2 \leq l = m \leq 6$ f-modes of $\Gamma = 2$ polytropes. The full, general-relativistic values were obtained using the numerical methods described in Lindblom & Detweiler (1983) and Detweiler & Lindblom (1985). We are primarily interested in the modes that propagate in the direction opposite the star's rotation (since these are the ones responsible for the gravitational-radiation secular instability), so ω_m is taken to be negative. The post-Newtonian estimates $\omega_m/\Omega_0 + c^{-2}\Delta(\omega_m/\Omega_0)$ are linear in M/R and are represented here as the solid lines; the dots represent the general-relativistic values. The post-Newtonian frequencies are tangent to the general-relativistic values at $M/R = 0$ to the level of the numerical accuracy. The post-Newtonian estimates agree with the exact general-relativistic values to within 3%–8% for $GM/Rc^2 = 0.20$. This discrepancy is consistent with the expected magnitude of the second-order post-Newtonian corrections. For further comparison, in Table 1 we list several parameters of the nonrotating Newtonian, post-Newtonian, and general-relativistic $\Gamma = 2$ polytropes (with $\kappa = 10^5$ in cgs units) having $M + c^{-2}\Delta M = 1.400\ M_\odot$. In this case $GM/Rc^2 = 0.167$, and we see that the nonrotating post-Newtonian parameters agree with the general-relativistic ones to within about 4%.

In Figure 2 we illustrate the angular-velocity dependence of the frequencies of the $l = m$ f-modes for the $\Gamma = 2$ polytropes with $GM_0/c^2R_0 = 0.20$ (where M_0 is the mass and R_0 the radius of the nonrotating star in the sequence). For clarity of presentation we plot the frequencies as measured in the rotating frame of the star: $(\omega_m - m\Omega)/\Omega_0$. We note that the post-Newtonian corrections tend to lower the absolute values of the frequencies, with this effect being somewhat more pronounced at the higher angular velocities. We estimate that the accuracy of the Newtonian frequencies presented here is better than 0.1%, while the post-Newtonian

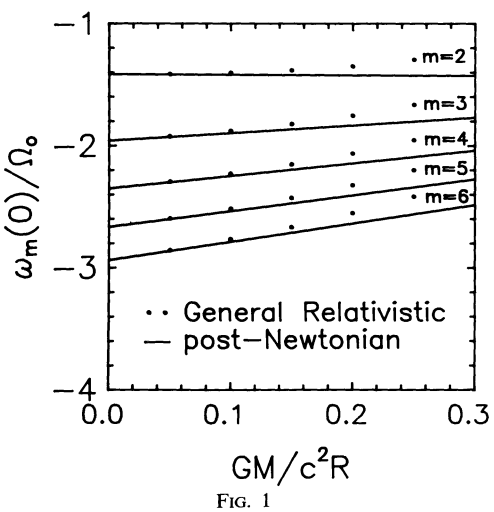

FIG. 1

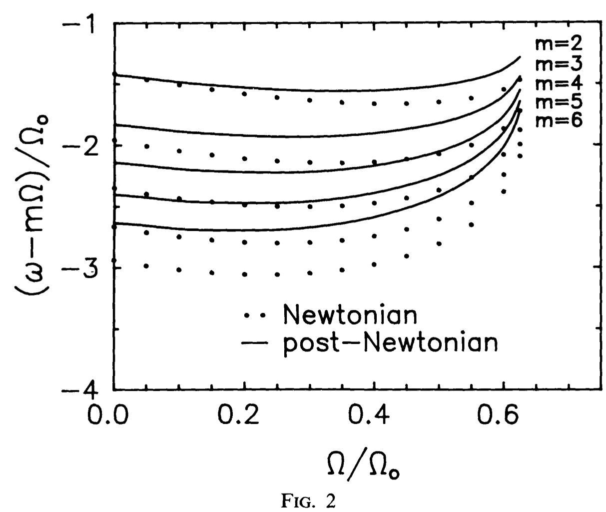

FIG. 2

FIG. 1.—The exact general-relativistic frequencies $(\omega_m/\Omega_0)_{GR}$ and the corresponding post-Newtonian estimates $\omega_m/\Omega_0 + c^{-2}\Delta(\omega_m/\Omega_0)$ are plotted vs. GM/c^2R for the $l = m$ f-modes of nonrotating $\Gamma = 2$ polytropes.

FIG. 2.—Display of the frequencies of the $l = m$ f-modes as viewed in the rotating frame of the star. Both the Newtonian $(\omega_m - m\Omega)/\Omega_0$ and post-Newtonian frequencies $(\omega_m - m\Omega)/\Omega_0 + c^{-2}\Delta[(\omega_m - m\Omega)/\Omega_0]$ are given as functions of the star's angular velocity Ω/Ω_0. Results shown here are for the sequence of $\Gamma = 2$ polytropes with $GM_0/c^2R_0 = 0.20$.

TABLE 1

PARAMETERS FOR A NONROTATING STAR WITH $M + c^{-2}\Delta M = 1.400\ M_\odot$[a]

	Q_N	$Q_N + c^{-2}\Delta Q_{PN}$	Q_{GR}
$M/M_\odot$	1.736	1.400	1.352
R (km)	15.343	12.374	11.959
$\Omega_0(s^{-1})$	6917	8256	8871
$-\omega_2/\Omega_0$	1.416	1.423	1.374
$-\omega_3/\Omega_0$	1.960	1.854	1.801
$-\omega_4/\Omega_0$	2.351	2.177	2.124
$-\omega_5/\Omega_0$	2.667	2.448	2.393
$-\omega_6/\Omega_0$	2.939	2.684	2.629

[a] The Newtonian (Q_N), post-Newtonian ($Q_N + c^{-2}\Delta Q_{PN}$), and general-relativistic (Q_{GR}) value is given for each quantity.

corrections to these frequencies are estimated to be accurate to about 1%. Since the post-Newtonian corrections are about 10% of the Newtonian values for the $GM_0/c^2R_0 = 0.20$ models depicted here, the post-Newtonian frequencies (obtained by adding the post-Newtonian corrections to the Newtonian values) are expected to be accurate to roughly 0.1% as well. These accuracy estimates are based on several computations of the frequencies on grids having different numbers of radial points and spokes. The frequencies presented here were obtained by solving the equations on the largest such grid, which contained 1600 radial points on each of 20 spokes covering the quadrant $0 < \theta \le \pi/2$. These accuracy estimates refer to the precision with which the post-Newtonian equations have been solved, *not* to the accuracy with which the post-Newtonian frequencies approximate the true, general-relativistic frequencies.

For many purposes it is convenient to describe the frequencies of the modes of rotating stars in terms of the dimensionless function α_m (Lindblom 1986; Ipser & Lindblom 1990), defined by

$$\alpha_m(\Omega/\Omega_0) = \frac{\omega_m(\Omega/\Omega_0) - m\Omega}{\omega_m(0)} . \tag{51}$$

The post-Newtonian correction to this function, $c^{-2}\Delta\alpha_m$, is related to the post-Newtonian change in the frequency of the mode by

$$\Delta\alpha_m(\Omega/\Omega_0) = \frac{\Omega_0}{\omega_m(0)}\left[\Delta\,\frac{\omega_m(\Omega/\Omega_0)}{\Omega_0} - \alpha_m(\Omega/\Omega_0)\Delta\,\frac{\omega_m(0)}{\Omega_0}\right] . \tag{52}$$

It follows immediately from equation (52) that $\Delta\alpha_m$, like $\Delta(\omega_m/\Omega_0)$, is proportional to GM_0/c^2R_0 for polytropes. In Figure 3 we display the Newtonian α_m and their post-Newtonian counterparts $\alpha_m + c^{-2}\Delta\alpha_m$ as functions of Ω/Ω_0 for the $l = m$ f-modes of $\Gamma = 2$ polytropes having $GM_0/c^2R_0 = 0.20$. We note that the post-Newtonian functions are smaller than their Newtonian counterparts by as much as 12% for this case. The post-Newtonian function $\alpha_m + c^{-2}\Delta\alpha_m$ for any other value of GM_0/c^2R_0 can be obtained from Figure 3 simply by scaling the difference between the Newtonian and the post-Newtonian functions.

In the absence of viscosity, the $l = m$ f-modes are unstable to the emission of gravitational radiation when the angular velocity of the star exceeds the critical angular velocity where the frequency of the mode passes through zero. Thus it is of great interest to determine the values of these critical angular velocities. The Newtonian equation for the critical angular velocities $\omega_m(\Omega_c/\Omega_0) = 0$

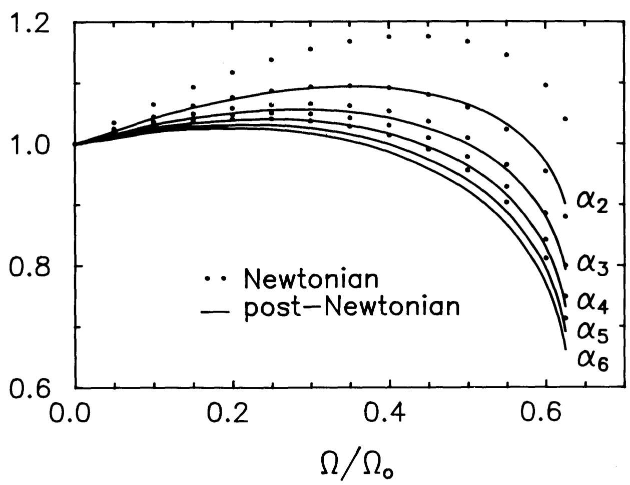

FIG. 3.—Comparison of the Newtonian, α_m, and the corresponding post-Newtonian, $\alpha_m + c^{-2}\Delta\alpha_m$, functions of Ω/Ω_0 for the $l = m$ f-modes of the sequence of $\Gamma = 2$ polytropes with $GM_0/c^2R_0 = 0.20$.

TABLE 2

CRITICAL ANGULAR VELOCITIES

$l = m$	$\frac{\Omega_c}{\Omega_0}$	$\frac{R_0}{GM_0}\Delta\left(\frac{\Omega_c}{\Omega_0}\right)$	τ_c	$\frac{R_0}{GM_0}\Delta\tau_c$
3.........	0.610	−0.169	0.0797	−0.113
4.........	0.560	−0.236	0.0581	−0.088
5.........	0.515	−0.245	0.0453	−0.067
6.........	0.477	−0.244	0.0368	−0.055

can be expressed in the form

$$\frac{\Omega_c}{\Omega_0} = -\frac{\omega_m(0)}{m\Omega_0}\,\alpha_m(\Omega_c/\Omega_0)\,. \tag{53}$$

This equation is easily solved numerically since the function α_m as defined in equation (51) is slowly varying. The post-Newtonian correction to the critical angular velocity $c^{-2}\Delta(\Omega_c/\Omega_0)$ is obtained by finding (to post-Newtonian order) the angular velocities where $\omega_m + c^{-2}\Delta\omega_m = 0$:

$$\Delta\frac{\Omega_c}{\Omega_0} = -\Delta\frac{\omega_m(\Omega_c/\Omega_0)}{\Omega_0}\left[m + \frac{\omega_m(0)}{\Omega_0}\frac{d\alpha_m(\Omega_c/\Omega_0)}{d(\Omega/\Omega_0)}\right]^{-1}\,. \tag{54}$$

We find that the angular velocity reaches its terminal value, Ω_K, before the frequency of the $l = m = 2$ mode, ω_2, goes through zero. The values of the Newtonian critical angular velocities and their post-Newtonian corrections are listed in Table 2 for the $3 \leq l = m \leq 6$ modes. The ratios Ω_c/Ω_0 are lowered by post-Newtonian effects by up to about 10% for the sequence of rotating $\Gamma = 2$ polytropes with $GM_0/c^2R_0 = 0.20$. Thus, post-Newtonian effects tend to reduce this measure of the maximum angular velocity of rotating neutron stars (set by the gravitational-radiation instability) by up to 10%. The lowering of this measure of the critical angular velocities by post-Newtonian effects may be contrasted with the fact that the post-Newtonian Ω_c may nevertheless be *larger* than its Newtonian counterpart since the post-Newtonian Ω_0 is larger. In order to determine the actual upper limit on these angular velocities, however, a more complicated calculation that includes the effects of gravitational-radiation reaction and viscous dissipation on these modes would have to be carried out. The results of such calculations in the Newtonian case (Ipser & Lindblom 1991) suggest, however, that the purely gravitational critical angular velocities associated with the $l = m = 4$ or 5 modes are close to the actual limiting angular velocities of neutron stars whose temperatures are near 10^9–10^{10} K. For those stars whose temperatures lie significantly outside this range, however, the limiting angular velocities are near Ω_K.

Another common measure of the angular velocities of rotating stars is the ratio of the rotational kinetic energy of the star to its gravitational potential energy, $\tau = -K/W$, (i.e., eqs. [41]–[46]). We have evaluated this ratio, and its post-Newtonian correction $\Delta\tau/c^2$, for the stars rotating at the critical angular velocities of the $l = m$ f-modes. For the sequences of rotating post-Newtonian polytropes considered here, $\Delta\tau_c = \Delta\tau(\Omega_c/\Omega_0) + [d\tau(\Omega_c/\Omega_0)/d(\Omega/\Omega_0)]\,\Delta(\Omega_c/\Omega_0)$ is proportional to GM_0/c^2R_0. In Table 2 we present the Newtonian critical value, τ_c, and its post-Newtonian correction, $(R_0/GM_0)\Delta\tau_c$. We note that the values of $\tau_c + c^{-2}\Delta\tau_c$ are reduced from their Newtonian counterparts by almost 25% for $GM_0/c^2R_0 = 0.20$. This large discrepancy is due to the fact that $\tau = \tau(\Omega/\Omega_0)$ is a rather nonlinear function of Ω/Ω_0.

We have also evaluated the terminal, Keplerian angular velocities for the sequences of rotating $\Gamma = 2$ polytropes considered in this paper. We find that the Newtonian value of the Keplerian angular velocity is $\Omega_K/\Omega_0 = 0.635$, while its post-Newtonian analog, $\Omega_K/\Omega_0 + c^{-2}\Delta(\Omega_K/\Omega_0) = 0.634$ for the sequence with $GM_0/c^2R_0 = 0.20$. We have also evaluated the corresponding values of τ_K. We find $\tau_K = 0.1026$ for the Newtonian sequence, and $\tau_K + c^{-2}\Delta\tau_K = 0.1036$ for its post-Newtonian counterpart. That is, to post-Newtonian order, the Newtonian and general relativistic stellar sequences terminate at almost precisely the same values of Ω/Ω_0 and of τ. Since post-Newtonian effects lower the critical values of Ω/Ω_0 for the $l = m$ modes, this implies that the post-Newtonian values of the ratio Ω_c/Ω_K are also smaller than their Newtonian counterparts.

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APPENDIX

The purpose of this Appendix is to describe our method of solving equations (11) and (23) for the stationary and perturbed gravito-magnetic potentials:

$$D^aD_aA^b = 16\pi G\rho v^b\,, \tag{55}$$

$$D^aD_a\delta A^b = 16\pi G(\delta\rho\, v^b + \rho\delta v^b) - i\omega D^b\delta\Phi\,. \tag{56}$$

Since the perturbed quantities on the right sides of these equations are assumed to have angular dependence $e^{im\varphi}$, the solutions will depend in a nontrivial way only on the spherical coordinates r and $\mu = \cos\theta$. However, there are two difficulties encountered in formulating equations (55) and (56) as difference equations on a two-dimensional grid in a compact region of the (r, μ)-plane. First, the source term $-i\omega D^b \delta\Phi$ in equation (56) does not have compact support. Second, while the Cartesian components of A^b and δA^b satisfy simple scalar Poisson equations, these components do not have simple (i.e., sinusoidal) φ-dependence.

The first problem is obviated by the following clever trick, due to Blanchet, Damour, & Schäfer (1990). Let δA_1^b and δA_2^b be the solutions of

$$D^a D_a \delta A_1^b = 16\pi G(\delta\rho\, v^b + \rho\delta v^b)\,, \tag{57}$$

$$D^a D_a \delta A_2^b = i\omega 2\pi G \delta\rho\, x^b\,, \tag{58}$$

where x^b is the radial position vector, that is, the vector whose components in Cartesian coordinates are (x, y, z). It is readily verified that

$$\delta A^b = \delta A_1^b + \delta A_2^b - \tfrac{1}{2} i\omega\, \delta\Phi\, x^b\,, \tag{59}$$

is a solution to equation (56). Thus, the problem of finding a solution to equation (56)—which has a noncompact source—is replaced by the problem of finding solutions to equations (57) and (58)—each of which has a compact source.

The second difficulty arises when we attempt to factor out the "trivial" φ-dependence and express equations (55) and (57)–(58) as two-dimensional scalar equations on the (r, μ) plane. The Cartesian components of the sources on the right sides of equations (57) and (58) do *not* have simple φ-dependence, and hence neither do the Cartesian components of A^b and δA^b. This second problem may be overcome by expressing these equations in terms of a basis which is more intimately related to the rotational symmetry of this problem. The Newtonian mode solutions may be written in the form $\delta\rho = \delta\rho(r, \mu)e^{im\varphi}$ and $\delta\Phi = \delta\Phi(r, \mu)e^{im\varphi}$, where $\delta\rho(r, \mu)$ and $\delta\Phi(r, \mu)$ are real functions. It follows, then, from equation (18) that δv^b has the form

$$\delta v^b = e^{im\varphi}[\delta v^r(r, \mu)r^b + \delta v^\mu(r, \mu)\mu^b + \delta v^\varphi(r, \mu)\varphi^b]\,, \tag{60}$$

where δv^r and δv^μ are imaginary, δv^φ is real, and $r^b = (\partial/\partial r)^b$, $\mu^b = (\partial/\partial\mu)^b$ and $\varphi^b = (\partial/\partial\varphi)^b$. Thus, equations (55) and (57)–(58) each have the general form

$$D^a D_a \xi^b = e^{im\varphi}[i\alpha(r, \mu)r^b + i\beta(r, \mu)\mu^b + \gamma(r, \mu)\varphi^b]\,, \tag{61}$$

where α, β, and γ are real functions of (r, μ). We proceed to reexpress this equation in terms of its components with respect to an "irreducible" basis. Define $\xi^+ \equiv \xi^y - i\xi^x$, $\xi^0 \equiv -i\xi^z$, and $\xi^- \equiv \xi^y + i\xi^x$, where ξ^x, ξ^y, and ξ^z are the Cartesian components of ξ^b. Then, we find that ξ^+, ξ^0, and ξ^- obey the equations

$$D^a D_a \xi^+ = e^{i(m+1)\varphi}(1-\mu^2)^{1/2}\left[\alpha + \gamma r - \frac{\beta r\mu}{1-\mu^2}\right], \tag{62}$$

$$D^a D_a \xi^0 = e^{im\varphi}[\alpha\mu + \beta r]\,, \tag{63}$$

$$D^a D_a \xi^- = e^{i(m-1)\varphi}(1-\mu^2)^{1/2}\left[-\alpha + \gamma r + \frac{\beta r\mu}{1-\mu^2}\right]. \tag{64}$$

The sources on the right sides of these equations have simple sinusoidal φ-dependence, and therefore so do ξ^+, ξ^0, and ξ^-. Thus, equations (62)–(64) become two-dimensional Poisson equations for the real functions $\xi^+(r, \mu)$, $\xi^0(r, \mu)$, and $\xi^-(r, \mu)$ defined by $\xi^+ = \xi^+(r, \mu)e^{i(m+1)\varphi}$, $\xi^0 = \xi^0(r, \mu)e^{im\varphi}$, and $\xi^- = \xi^-(r, \mu)e^{i(m-1)\varphi}$. We also note that in the solution of equation (55) there is a further simplification: $\xi^0(r, \mu) = 0$, and $\xi^+(r, \mu) = \xi^-(r, \mu)$ since $\alpha = \beta = m = 0$. Thus, equation (55) is reduced to a single two-dimensional real-scalar Poisson equation.

For the problem of interest here, the functions on the right sides of equations (62)–(64) have definite parity under the transformation $\mu \to -\mu$ (i.e., reflection about the star's equatorial plane). This implies in turn that the functions ξ^+, ξ^0, and ξ^- also have definite parity. Thus, equations (62)–(64) need only be solved in one quadrant of the (r, μ)-plane: $0 \le r \le R_{\max}$, $0 \le \mu \le 1$, where $R_{\max}$ is somewhat larger than the star's equatorial radius. The parity of these functions is inherited from the parity of the Newtonian pulsations:

$$\delta\rho(r, -\mu) = (-1)^{l-m}\delta\rho(r, \mu)\,, \qquad \delta\Phi(r, -\mu) = (-1)^{l-m}\delta\Phi(r, \mu)\,, \tag{65}$$

where the integers l and m are the spherical-harmonic indices of the zero-angular-velocity limit of these perturbation functions. It follows from equations (18) and (65) that δv^r and δv^φ have parity $(-1)^{l-m}$ while δv^μ has parity $(-1)^{l-m+1}$. These transformations imply in turn that the functions α, β, and γ that appear in equations (55) and (57)–(58) also have definite parity:

$$\alpha(r, -\mu) = (-1)^{l-m}\alpha(r, \mu)\,, \qquad \beta(r, -\mu) = (-1)^{l-m+1}\beta(r, \mu)\,, \qquad \gamma(r, -\mu) = (-1)^{l-m}\gamma(r, \mu)\,. \tag{66}$$

And finally, it follows from equations (64)–(66) that ξ^+, and ξ^- have parity $(-1)^{l-m}$ while ξ^0 has parity $(-1)^{l-m+1}$.

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