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Causal Theories of Dissipative Relativistic Fluids

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A very wide class of theories for dissipative relativistic fluids is analyzed. General techniques for constructing explicit theories are discussed. The conditions under which these theories have causal evolution equations are determined. The general properties (including stability) of the equilibrium solutions of these theories are evaluated. The requirement that the theory have the appropriate number and kind of equilibrium solutions is a strong constraint on the structure of the fluid theory. The properties of the shock-wave solutions of these theories are briefly considered. Most causal fluid theories have no solutions capable of describing strong shock waves. © 1991 Academic Press, Inc.

I. INTRODUCTION

In nature we observe fluids that manifest dissipative effects. What is the correct relativistic description of such materials? This is more than merely a question of principle. In a neutron star, for example, we expect the matter to be a fluid having a sound speed close to that of light. Dissipation (viscosity and thermal conductivity) in this fluid plays an important role in at least two situations: in determining the structure of the supernova shock wave that accompanies the formation of the neutron star and in determining the stability of the neutron star. Thus, a relativistic theory of a dissipative fluid is needed to describe the macroscopic properties of such a system.

In the non-relativistic limit, there exists a theory for dissipative fluids that is simple, natural, and remarkably successful: the Navier–Stokes–Fourier theory [1]. In this theory, the fluid is described by six tensor fields: a particle-number density, an energy density, a fluid velocity, a heat-flow vector, and the trace-free and trace parts of a stress tensor. These fields are subject to a system of six first-order differential equations that includes the conservation of particle number, conservation of energy,

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and conservation of momentum. This theory also manifests a version of the second law of thermodynamics, in that there are expressions for an entropy density and an entropy current, algebraic functions of the fluid variables, that satisfy a local entropy law as a consequence of the fluid equations.

Since the non-relativistic theory of dissipative fluids is so successful, it is natural to seek a relativistic theory that is a suitable generalization of it. Various attempts have been made to do this. The most straightforward generalizations of the Navier–Stokes–Fourier theory are those of Eckart [2] and Landau and Lifshitz [1]. Unfortunately, neither of these theories has causal evolution equations and neither admits stable equilibrium solutions [3]. More complicated (and more successful) generalizations are those of Israel and Stewart [4, 5] and Liu, Müller, and Ruggeri [6]. However, these theories impose on the detailed structure of the fluid severe restrictions that are not well motiviated physically.

The purpose of this paper is to explore the basic features of a much wider class of relativistic generalizations of the Navier–Stokes–Fourier theory. We consider all theories of a relativistic dissipative fluid that share with the Navier–Stokes–Fourier theory the following three properties: (a) The state of the fluid at each point of space-time is characterized by a finite collection of tensor fields at that point. This property represents a drastic departure from the microscopic description of a fluid, involving, e.g., a distribution function on phase space. It is a significant—and possibly unachievable—assumption that a physically realistic theory have this property. (b) Local laws of conservation of particle number and stress energy follow from the fluid equations. (c) A local entropy law follows from the fluid equations. While it is perhaps natural to assume some entropy law on the macroscopic level, the existence of a local entropy law seems less fundamental than the existence of the local conservation laws.

In Section II, we introduce the class of fluid theories having these three basic properties. We determine when the fluid equations of such a theory are causal. In Section III, we introduce two specific examples of such theories. One of these is a generalization of the fluid theories of divergence type [6, 7], while the other includes causal generalizations of the Eckart and Israel–Stewart theories. In Section IV, we consider the equilibrium states of these general fluid theories. We find that many of the features of such states (e.g., the form of the particle-number current and stress energy) are common to all of these fluid theories. The requirement that the equilibrium states be appropriate—in terms of number and character—is a severe constraint on the structure of the theory. In Section V we investigate the stability of these equilibrium states. We find that causality of the fluid equations implies stability of the homogeneous non-rotating equilibrium solutions in all these theories and of all equilibrium states in certain theories. In Section VI, we seek solutions that describe shock waves in these theories. Generally speaking, these fluid theories, with the possible exception those of divergence type [6, 7], appear to admit no solutions whatever for strong shocks. Finally in Section VII, we discuss a number of open questions and areas for further work.

II. General Fluid Theories

We consider fluid theories in which the state of the fluid is characterized by a finite collection of space-time tensor fields. Let φ^A denote these fields, where upper case indices stand for the entire set of tensor indices represented in this collection of fields. We refer to a point, φ^A, in this space of fluid fields as a fluid *state*. We restrict consideration to fluid theories in which the field equations take the form:

$$M^m{}_{AB}\nabla_m\varphi^B = I_A. \tag{1}$$

Here $M^m{}_{AB}$ and I_A are algebraic functions of the fluid fields, φ^A, and the space-time metric, g_{ab}. Lower case letters are space-time indices; and repeated indices indicate contraction. The requirement in Eq. (1) that the system be first order is not a serious restriction, for a higher-order system can always be reduced to first order simply by introducing additional tensor fields. The requirement that the equations be quasi-linear, however, is more severe. Since the indices A and B that appear in Eq. (1) refer to the same set of tensor indices, there are in Eq. (1) the same number of equations as fields. We use the term *solution* of the fluid equations to mean the fields φ^A on space-time satisfying Eq. (1).

The system of Eqs. (1) is called *symmetric* if $M^m{}_{AB}$ is symmetric in A and B. The system is called *hyperbolic* [8] if it is symmetric, and if the vector

$$E^m = \tfrac{1}{2}M^m{}_{AB}Z^AZ^B \tag{2}$$

lies in the future of some space-like three-dimensional sub-space of the tangent space, for all non-vanishing Z^A. The system is called *causal* if it is symmetric, and if E^m lies within the future light cone (i.e., if E^m is a future-directed timelike vector), for all non-vanishing Z^A. Hyperbolicity guarantees that the system of equations has a well-posed initial-value formulation. Causality guarantees, in addition, that no fluid-signal speed exceeds the speed of light. For physical reasons, we are primarily interested in fluid theories that have causal evolution equations. However, it will be convenient, initially, to consider all theories having equations in the form of Eq. (1), without assuming causality, hyperbolicity, or even symmetry.

In order that Eq. (1) represent a fluid theory (as opposed to an arbitrary classical field theory), we require that the conservation laws of particle number and stress energy, and an entropy law, be consequences of it. First, we require that there be specified fields N^m (the particle-number current) and N^A, algebraic functions of φ^A and g_{ab}, that satisfy

$$N^AM^m{}_{AB} = \frac{\partial N^m}{\partial\varphi^B} \tag{3}$$

and

$$N^AI_A = 0. \tag{4}$$

These conditions are precisely those needed to ensure that particle-number conservation, $\nabla_m N^m = 0$, follow from the field equations. Similarly, we require that there be specified fields T^{ma} (the stress-energy tensor) and T^{aA} that satisfy

$$T^{aA} M^m{}_{AB} = \frac{\partial T^{ma}}{\partial \varphi^B} \tag{5}$$

and

$$T^{aA} I_A = 0. \tag{6}$$

These conditions are those needed to ensure stress-energy conservation, $\nabla_m T^{ma} = 0$. Finally, we require that there be specified fields S^m (the entropy current) and S^A that satisfy

$$S^A M^m{}_{AB} = \frac{\partial S^m}{\partial \varphi^B} \tag{7}$$

and

$$S^A I_A \geqslant 0. \tag{8}$$

These conditions are those needed to ensure an entropy law, $\nabla_m S^m = S^A I_A \geqslant 0$. This represents the second law of thermodynamics in the theory.

To summarize, by a *fluid theory* we mean a collection of tensors $M^m{}_{AB}$, I_A, N^m, T^{ma}, S^m, N^A, T^{aA}, and S^A (all algebraic functions of the fluid fields, φ^A, and the space-time metric, g_{ab}) that satisfy Eqs. (3)–(8). Numerous examples fit into this general framework: ordinary relativistic perfect fluids, as well as the Eckart [2], Landau–Lifshitz [1], Israel–Stewart [4, 5], and the divergence-type [6, 7] theories of relativistic dissipative fluids. Some of the perfect-fluid and divergence-type theories are known to be causal, while the Eckart theory is known to be symmetric but not hyperbolic. It is not known whether the Israel–Stewart or the Landau–Lifshitz theories are even symmetric.

In the remainder of this section, we analyze how causal fluid theories may be constructed. For theories having symmetric field equations, the fields N^m, T^{ma}, S^m, N^A, T^{aA}, and S^A are not independent. Rather, they must satisfy the following equations:

$$N^A \frac{\partial S^m}{\partial \varphi^A} - S^A \frac{\partial N^m}{\partial \varphi^A} = 0, \tag{9}$$

$$T^{aA} \frac{\partial S^m}{\partial \varphi^A} - S^A \frac{\partial T^{ma}}{\partial \varphi^A} = 0, \tag{10}$$

$$T^{aA} \frac{\partial N^m}{\partial \varphi^A} - N^A \frac{\partial T^{ma}}{\partial \varphi^A} = 0, \tag{11}$$

$$T^{aA} \frac{\partial T^{mb}}{\partial \varphi^A} - T^{bA} \frac{\partial T^{ma}}{\partial \varphi^A} = 0. \tag{12}$$

These conditions follow from Eqs. (3), (5), and (7) and the symmetry of $M^m{}_{AB}$. Equation (9), for example, is the result of contracting Eq. (3) with S^B, Eq. (7) with N^B, and subtracting. Note that these equations do not involve $M^m{}_{AB}$ or I_A. There are at least two approaches to finding solutions of these equations. In the first approach, arbitrarily specify the fields N^m, T^{ma}, and S^m as functions of the fluid fields. Then take Eqs. (9)–(12) as a system of linear equations for N^A, T^{aA}, and S^A. This system consists of 56 ($=4+16+16+20$) equations on $6K$ ($=K+4K+K$) unknowns, where K is the dimension of the field-space. Thus, we are guaranteed a nonzero solution provided $K \geqslant 10$. In the second approach, specify the fields N^A, T^{aA}, and S^A as functions of the fluid fields, and then take Eqs. (9)–(12) as a system of first-order differential equations for N^m, T^{ma}, and S^m. A nonzero solution of this system will exist only if appropriate integrability conditions, involving the Lie brackets of N^A, T^{aA}, and S^A regarded as vector fields on field space, are satisfied.

We now turn to the problem of finding $M^m{}_{AB}$ and I_A that satisfy Eqs. (3)–(8), given fields N^m, T^{ma}, S^m, N^A, T^{aA}, and S^A that satisfy Eqs. (9)–(12).

Consider first the problem of finding an I_A that satisfies Eqs. (4), (6), and (8). Clearly, $I_A=0$ is always one solution. More interesting solutions are those given by $I_A = P^B_A G_{BC} P^C_D S^D$, where G_{BC} is symmetric and positive semi-definite, and where P^A_B satisfies $P^A_B N^B=0$ and $P^A_B T^{aB}=0$. Such G_{BC} and P^A_B can, in fact, be constructed explicitly from φ^A and g_{ab} provided that there exists some timelike vector function u^m of φ^A and g_{ab}. The G_{BC} can be constructed in various ways. For example, one possible G_{BC} is that whose action in lowering the index of a fluid-field vector is the result of lowering (or raising) the space-time indices of each space-time tensor that make up that vector using a positive-definite space-time metric such as $-g_{ab}u^c u_c + 2u_a u_b$. This G_{BC} is positive definite. For causal theories another possibility, also positive-definite, is $G_{BC} = -u_m M^m{}_{BC}$. Finally, a third possibility is $G_{BC} = Q_B Q_C$, where Q_B is any co-vector constructed from φ_A and g_{ab}. Any positive linear combination of these G's is also positive semi-definite. The P^A_B can also be constructed in a number of ways. One possibility, when G_{BC} is positive-definite, is the tensor that projects G-orthogonal to N^A and T^{aA}.

Consider next the problem of finding an $M^m{}_{AB}$ that satisfies Eqs. (3), (5), and (7). It is convenient to consider separately two cases: (a) when N^A, T^{aA}, and S^A are linearly independent, and (b) when they are linearly dependent. We refer to the fluid states, φ^A, defined by these cases as *imperfect-fluid* states and *perfect-fluid* states, respectively. As we shall see (in Section IV) the perfect-fluid states share many of the physical features of ordinary perfect fluids, while in the imperfect-fluid states dissipation plays a role.

Consider first the imperfect-fluid states. For these, Eqs. (9)–(12) are the necessary and sufficient conditions for the existence of a symmetric $M^m{}_{AB}$ satisfying Eqs. (3), (5), and (7). This $M^m{}_{AB}$ is unique up to the addition of any $\bar{M}^m{}_{AB}$, symmetric in A and B, that annihilates N^A, T^{aA}, and S^A:

$$N^A \bar{M}^m{}_{AB} = 0, \qquad T^{aA} \bar{M}^m{}_{AB} = 0, \qquad S^A \bar{M}^m{}_{AB} = 0. \tag{13}$$

As we have seen, a fluid theory is causal provided the vector E^m of Eq. (2) is future-directed time-like for every non-vanishing Z^A. Consider, in particular, the Z^A having the form

$$Z^A = \alpha N^A + \beta_a T^{aA} + \gamma S^A, \tag{14}$$

for some α, β_a, and γ. Substituting this Z^A into Eq. (2) and using Eqs. (3), (5), and (7), we obtain

$$\bar{E}^m = \frac{1}{2}\left(\alpha \frac{\partial N^m}{\partial \varphi^A} + \beta_a \frac{\partial T^{ma}}{\partial \varphi^A} + \gamma \frac{\partial S^m}{\partial \varphi^A}\right)(\alpha N^A + \beta_b T^{bA} + \gamma S^A). \tag{15}$$

Thus, a necessary condition for causality is that this $\bar{E}^m$ be future-directed timelike for all non-vanishing α, β_a, and γ. This necessary condition involves only the fields N^m, T^{ma}, S^m, N^A, T^{aA}, and S^A, and not $M^m{}_{AB}$. We note that, if the $\bar{E}^m$ of Eq. (15) are all future-directed timelike, then there always exists a choice of $\bar{M}^m{}_{AB}$ that makes the final theory causal. In fact, it is straightforward to construct such an $\bar{M}^m{}_{AB}$ whenever there exists a timelike vector function u^m of φ^A and g_{ab}. In this case we may construct, as discussed above, a positive-definite G_{AB} and projection tensor $\bar{P}^A_B$, that satisfies $\bar{P}^A_B N^B = \bar{P}^A_B S^B = 0$ and $\bar{P}^A_B T^{aB} = 0$. Then $\bar{M}^m{}_{AB} = -\Phi^2 u^m G_{CD} \bar{P}^C_A \bar{P}^D_B$, for any non-zero Φ, is symmetric, satisfies Eq. (13) and makes a timelike contribution to E^m. The resulting theory is necessarily causal for Φ^2 sufficiently large (in fact, for $\Phi^2 > |u_m M^m{}_{AB} Z^A Z^B / G_{CD} \bar{P}^C_E Z^E \bar{P}^D_F Z^F|$, where Z^A is the G-unit vector that minimizes $-u_m M^m{}_{AB} Z^A Z^B$).

Consider now the perfect-fluid states. Let us assume that the linear dependency between N^A, T^{aA}, and S^A can be expressed as

$$S^A = -\zeta N^A - \zeta_a T^{aA}, \tag{16}$$

where ζ and ζ_a are algebraic functions of φ^A and g_{ab} [9]. That is, we assume that the entropy law is a consequence of particle-number and stress-energy conservation, and that these conservation laws are themselves independent of each other. Taking the appropriate linear combination of Eqs. (3), (5), and (7), we find the following additional condition on the fields N^m, T^{ma}, and S^m:

$$\frac{\partial S^m}{\partial \varphi^A} = -\zeta \frac{\partial N^m}{\partial \varphi^A} - \zeta_a \frac{\partial T^{ma}}{\partial \varphi^A}. \tag{17}$$

This equation and Eqs. (11) and (12) are the necessary and sufficient conditions that there exists an $M^m{}_{AB}$ satisfying Eqs. (3), (5), and (7). Taking the appropriate linear combination of Eqs. (4), (6), and (8), it follows that in a perfect-fluid state the entropy-production density vanishes:

$$S^A I_A = 0. \tag{18}$$

The necessary condition for the theory to be causal is analogous to the condition for the inperfect-fluid states: that $\bar{E}^m$ of Eq. (15) be timelike for all α, β_a, but with

$\gamma = 0$. Again, when this holds there always exists an $\bar{M}^m{}_{AB}$ satisfying Eq. (13) that makes the final theory causal.

A complete fluid theory will include both perfect- and imperfect-fluid states. Thus, the fields $M^m{}_{AB}$ and I_A for such a theory must be chosen as described above for each set of fluid states separately, and must also be chosen to have smooth dependence on φ^A as φ^A passes from one set of states to the other. This choice is made difficult by the fact that N^A, T^{aA}, and S^A span a six-dimensional space for imperfect-fluid states, but, because of Eq. (16), only a five-dimensional space for perfect-fluid states. Thus the constraints on $M^m{}_{AB}$ and I_A in Eqs. (3)–(8) are of different dimension in the two cases. One way of guaranteeing the existence of smooth $M^m{}_{AB}$ and I_A for all fluid states is the following: Assume that the functions ζ and ζ_a (defined originally only for perfect-fluid states) can be extended smoothly to all fluid states, and that a function ψ defined on all states and vanishing on perfect-fluid states, can be introduced, such that κ^A and κ^m_A defined by

$$\kappa^A = \frac{1}{\psi}(S^A + \zeta N^A + \zeta_a T^{aA}), \tag{19}$$

$$\kappa^m_A = \frac{1}{\psi}\left(\frac{\partial S^m}{\partial \varphi^A} + \zeta \frac{\partial N^m}{\partial \varphi^A} + \zeta_a \frac{\partial T^{am}}{\partial \varphi^A}\right), \tag{20}$$

have smooth limits at the perfect-fluid states. It is not difficult to show that, under these circumstances, smooth $M^m{}_{AB}$ and I_A exist.

III. Examples

We consider first a rather general example that illustrates the formalism developed in Section II. We assume that it is possible to choose the fluid fields φ^A to consist of one scalar field, φ, one co-vector field, φ_a, and some additional tensor fields $\varphi^{\bar{A}}$ (possibly including additional scalar and co-vector fields)—i.e., to choose $\varphi^A = (\varphi, \varphi_a, \varphi^{\bar{A}})$—such that in these variables,

$$S^A = -\varphi^A, \tag{21}$$

$$N^A = (1, 0, 0), \tag{22}$$

$$T^{aA} = (0, \delta^a_b, 0). \tag{23}$$

This is a strong assumption. Essentially, it amounts to the requirement that the components of S^A can be taken as the fluid fields, and that among these fields are a scalar and a co-vector whose corresponding fluid equations are the particle-number and stress-energy conservation laws, respectively. In geometrical terms, Eqs. (21)–(23) are equivalent to the following commutation relations among N^A, T^{aA}, and S^A, regarded as vector fields on the space of fluid fields:

$$N^A \frac{\partial T^{aB}}{\partial \varphi^A} - T^{aA} \frac{\partial N^B}{\partial \varphi^A} = 0, \tag{24}$$

$$T^{aA} \frac{\partial T^{bB}}{\partial \varphi^A} - T^{bA} \frac{\partial T^{aB}}{\partial \varphi^A} = 0, \tag{25}$$

$$N^A \frac{\partial S^B}{\partial \varphi^A} - S^A \frac{\partial N^B}{\partial \varphi^A} = N^B, \tag{26}$$

$$T^{aA} \frac{\partial S^B}{\partial \varphi^A} - S^A \frac{\partial T^{aB}}{\partial \varphi^A} = T^{aB}. \tag{27}$$

To complete this fluid theory we must, first, satisfy Eqs. (9)–(12). Here we have specified the fields N^A, T^{aA}, and S^A and must solve for the physical fields N^m, T^{am}, and S^m in terms of the φ^A. The following expressions satisfy Eqs. (9)–(12),

$$N^m = \frac{\partial^2 X}{\partial \varphi \, \partial \varphi_m}, \tag{28}$$

$$T^{am} = \frac{\partial^2 X}{\partial \varphi_a \, \partial \varphi_m}, \tag{29}$$

$$S^m = \frac{\partial X}{\partial \varphi_m} - \varphi^A \frac{\partial^2 X}{\partial \varphi^A \, \partial \varphi_m}, \tag{30}$$

for any function, X, of φ^A and g_{ab}. It is not difficult to check that, conversely, every solution of Eqs. (9)–(12) is given by Eqs. (28)–(30) for some X.

Having found fields N^m, T^{ma}, S^m, N^A, T^{aA}, and S^A that satisfy Eqs. (9)–(12), the next step in the construction this fluid theory is to find $M^m{}_{AB}$ and I_A that satisfy Eqs. (3)–(8). The existence of an $M^m{}_{AB}$ that satisfies these equations is guaranteed by Eqs. (9)–(12); and it is easy to verify that one such solution is

$$M^m{}_{AB} = \frac{\partial^3 X}{\partial \varphi_m \, \partial \varphi^A \, \partial \varphi^B}. \tag{31}$$

The resulting theory (with any appropriately chosen I_A) is a generalization (to an arbitrary number of fluid fields) of the fluid theories of divergence type [6, 7]. Of course the solution of Eqs. (3), (5), and (7) is not unique. More general theories are obtained by adding to $M^m{}_{AB}$ any symmetric $\bar{M}^m{}_{AB}$ that satisfies Eq. (13).

The causality of a fluid theory is determined by the quadratic form E^m of Eq. (2). As we showed in Section II, from any given fluid theory a causal theory can be obtained by adding an appropriate $\bar{M}^m{}_{AB}$, provided the restricted quadratic form, $\bar{E}^m$, of Eq. (15) is future-directed timelike for all α, β_a, and γ. For this example, $\bar{E}^m$ is given by

$$\bar{E}^m = (\alpha + \gamma\varphi)^2 \frac{\partial^3 X}{\partial\varphi_m\, \partial\varphi^2} + 2(\alpha + \gamma\varphi)(\beta_a + \gamma\varphi_a) \frac{\partial^3 X}{\partial\varphi_m\, \partial\varphi\, \partial\varphi_a}$$
$$+ (\beta_a + \gamma\varphi_a)(\beta_b + \gamma\varphi_b) \frac{\partial^3 X}{\partial\varphi_m\, \partial\varphi_a\, \partial\varphi_b} + 2(\alpha + \gamma\varphi)\, \gamma\varphi^{\bar{A}} \frac{\partial^3 X}{\partial\varphi_m\, \partial\varphi\, \partial\varphi^{\bar{A}}}$$
$$+ 2(\beta_a + \gamma\varphi_a)\, \gamma\varphi^{\bar{A}} \frac{\partial^3 X}{\partial\varphi_m\, \partial\varphi_a\, \partial\varphi^{\bar{A}}} + \gamma^2 \varphi^{\bar{A}}\varphi^{\bar{B}} \frac{\partial^3 X}{\partial\varphi_m\, \partial\varphi^{\bar{A}}\, \partial\varphi^{\bar{B}}}. \tag{32}$$

The Eckart [2] theory is a spacial case of the fluid theories considered in this first example. In the Eckart theory, the additional dynamical fields, $\varphi^{\bar{A}}$, consist of a single symmetric trace-free tensor φ_{ab}. The potential X in this case is given by $X = \alpha(\varphi, \mu) - \mu^{-1}\varphi_a\varphi_b\varphi^{ab}$, where $\mu = \varphi_a\varphi^a$, and $\alpha(\varphi, \mu)$ is an arbitrary smooth function [7]. Since X is linear in $\varphi^{\bar{A}}$, the last term in Eq. (32) vanishes identically, and so $\bar{E}^m$ is zero, rather than timelike, for $\alpha = -\gamma\varphi$, $\beta_a = -\gamma\varphi_a$, and $\gamma \neq 0$. We conclude that it is not possible to modify this representation of the Eckart theory, by the addition of any $\bar{M}^m{}_{AB}$, to yield a causal theory.

As a second example, consider the case in which the fluid fields can be taken to be N^a and T^{ab} themselves:

$$\varphi^A = (N^a, T^{ab}). \tag{33}$$

These fields could be re-expressed by decomposing them into a pair of thermodynamic potentials, a unit timelike fluid-velocity vector, a spatial heat-flow vector, and the trace-free and trace parts of a spatial stress-tensor. But these are precisely the fields that occur in the Navier–Stokes–Fourier, the Eckart, the Landau–Lifshitz, the Israel–Stewart, and the Liu–Müller–Ruggeri theories. Thus, this second example is the case of a "normal" dissipative fluid.

To determine the theory, we must specify the fields N^A, T^{mA}, S^A, S^m, $M^m{}_{AB}$, and I_A as functions of N^a, T^{ab}, and g_{ab}. We write the components of N^A, T^{mA}, and S^A as follows:

$$N^A = (n^a, n^{ab}), \tag{34}$$

$$T^{mA} = (t^{ma}, t^{mab}), \tag{35}$$

$$S^A = (s^a, s^{ab}). \tag{36}$$

The fields n^{ab}, t^{mab}, and s^{ab} are symmetric in a and b. Similarly, we write the components of $M^m{}_{AB}$ and I_A such that the fluid equations, (1), become

$$M^m{}_{ac} \nabla_m N^c + M^m{}_{acd} \nabla_m T^{cd} = I_a, \tag{37}$$

$$M^m{}_{cab} \nabla_m N^c + M^m{}_{abcd} \nabla_m T^{cd} = I_{ab}. \tag{38}$$

The fields $M^m{}_{ab}$, $M^m{}_{cab}$, $M^m{}_{abcd}$, and I_{ab} are symmetric in a and b, and $M^m{}_{abcd} = M^m{}_{cdab}$.

In order that the theory be symmetric, these fields must satisfy Eqs. (9)–(12). Evaluating the derivatives $\partial T^{ma}/\partial\varphi^A$ and $\partial N^m/\partial\varphi^A$ therein using Eq. (33), these equations become:

$$n^a \frac{\partial S^m}{\partial N^a} + n^{ab}\frac{\partial S^m}{\partial T^{ab}} = s^m, \tag{39}$$

$$t^{na}\frac{\partial S^m}{\partial N^a} + t^{nab}\frac{\partial S^m}{\partial T^{ab}} = s^{nm}, \tag{40}$$

$$t^{ab} = n^{ab}, \tag{41}$$

$$t^{mab} = t^{amb}. \tag{42}$$

The last two equations imply, respectively, that t^{ab} and t^{mab} are totally symmetric. One method for solving these equations, for example, is first to choose n_a and S^m arbitrarily as functions of N^a, T^{ab}, and g_{ab}, then to choose symmetric t^{ab} and t^{mab} such that the left side of Eq. (40) is symmetric, and finally to choose s^m, s^{nm}, and n^{ab} to have the values given by Eqs. (39)–(41).

Finally, we must specify $M^m{}_{AB}$ and I_A satisfying Eqs. (3)–(8). For simplicity, we limit consideration to examples that are generic in the sense that t^{ab} is invertible. In terms of the fields defined above, Eqs. (3), (5), and (7) then become

$$(n^{ab} - n^r t^{-1}_{rs} t^{sab})\, M^m{}_{abcd} = -n^r t^{-1}_{r(c}\, \delta^m_{d)}, \tag{43}$$

$$(s^{ab} - s^r t^{-1}_{rs} t^{sab})\, M^m{}_{abcd} = \frac{\partial S^m}{\partial T^{ab}} - s^r t^{-1}_{r(c}\, \delta^m_{d)}, \tag{44}$$

$$M^m{}_{ab} = -t^{-1}_{ac} t^{-1}_{bd} t^{mcd} + t^{-1}_{ar} t^{rcd} t^{-1}_{bs} t^{spq} M^m{}_{cdpq}, \tag{45}$$

$$M^m{}_{cab} = t^{-1}_{c(a}\, \delta^m_{b)} - t^{-1}_{cn} t^{nrs} M^m{}_{rsab}. \tag{46}$$

Parenthesis surrounding tensor indices denote symmetrization, e.g., $t^{(ab)} = \frac{1}{2}(t^{ab} + t^{ba})$. One method for solving these equations, for example, is first to choose $M^m{}_{abcd}$ to satisfy Eqs. (43) and (44), and then to choose $M^m{}_{ab}$ and $M^m{}_{cab}$ to have the values given by Eqs. (45) and (46). Similarly, Eqs. (4), (6), and (8) become

$$(n^{ab} - n^r t^{-1}_{rs} t^{sab})\, I_{ab} = 0, \tag{47}$$

$$(s^{ab} - s^r t^{-1}_{rs} t^{sab})\, I_{ab} \geqslant 0, \tag{48}$$

$$I_a = -t^{-1}_{ab} t^{bcd} I_{cd}. \tag{49}$$

One method of solving these equations, for example, is first to choose I_{ab} to satisfy Eqs. (47) and (48), and then to choose I_a to have the value given by Eq. (49). The final fluid equations, (37) and (38), can now be simplified using Eqs. (43)–(49):

$$\nabla_m T^{ma} = 0, \tag{50}$$

$$M^m{}_{abcd}(\nabla_m T^{cd} - t^{-1}_{rs} t^{scd} \nabla_m N^r) + t^{-1}_{c(a} \nabla_{b)} N^c = I_{ab}. \tag{51}$$

For this example, the restricted quadratic form $\bar{E}_m$ of Eq. (15) is given by

$$\bar{E}^m = \frac{1}{2}\left(\alpha^2 n^m + \beta_a \beta_b t^{mab} + \gamma^2\left[s^a \frac{\partial S^m}{\partial N^a} + s^{ab}\frac{\partial S^m}{\partial T^{ab}}\right]\right) + \alpha\beta_a t^{am} + \alpha\gamma s^m + \gamma\beta_a s^{am}. \tag{52}$$

This quadratic form must be future-directed timelike for all α, β_a, and γ if the fluid theory is to be causal. Thus, causality requires in particular that n^m, $s^a\,\partial S^m/\partial N^a + s^{ab}\,\partial S^m/\partial T^{ab}$, and $t^{mab}\beta_a\beta_b$ be future-directed timelike for all β_a.

If the Israel–Stewart theory has a representation as a symmetric theory, then it must be a particular case of this example. It is apparently not known whether or not the Israel–Stewart theory has such a representation.

The Eckart theory is known to be a particular case of this example. It is convenient to decompose the fields N^a and T^{ab} for this case as

$$N^a = nu^a, \tag{53}$$

$$T^{ab} = \rho u^a u^b + (p+\tau)(g^{ab} + u^a u^b) + u^a q^b + u^b q^a + \tau^{ab}, \tag{54}$$

where u^a is a unit timelike vector, and q^a and the symmetric-trace-free τ^{ab} are orthogonal to it. In the Eckart theory, the entropy current is given by

$$S^a = snu^a + \frac{1}{T} q^a. \tag{55}$$

There are a variety of ways to represent the Eckart theory as a symmetric system in the form of Eq. (1) (in contrast to a generic fluid theory, which has a unique representation—up to an over-all scale—in this form). For the case of Eckart, the most general such representation, up to over-all scale, is given by

$$M^m{}_{abcd} = u_a u_b \delta^m_{(c} u_{d)} + u_{(a}\delta^m_{b)} u_c u_d + \left[u^m + \frac{2}{T}\left(\frac{\partial T}{\partial \rho}\right)_n q^m\right] u_a u_b u_c u_d, \tag{56}$$

$$t^{ab} = -\frac{n}{T}\left(\frac{\partial T}{\partial \rho}\right)_n [g^{ab} + (1+\omega)\, u^a u^b], \tag{57}$$

$$t^{abc} = \left[\omega\,\frac{n}{T}\left(\frac{\partial T}{\partial n}\right)_\rho - 1\right] u^a u^b u^c - \frac{6}{T}\left(\frac{\partial T}{\partial \rho}\right)_n q^{(a} u^b u^{c)} + \omega^{abc}, \tag{58}$$

$$n^a = -\omega\,\frac{n}{T}\left(\frac{\partial T}{\partial \rho}\right)_n \left(\frac{\partial n}{\partial \rho}\right)_T u^a, \tag{59}$$

$$I_{ab} = -T\left(\frac{\partial \rho}{\partial T}\right)_n \left[\frac{1}{\kappa T} u_{(a} q_{b)} - \frac{1}{2\eta_1}\tau_{ab} - \frac{1}{3\eta_2}\tau(g_{ab} + u_a u_b)\right] + u_a u_b\left[\frac{n}{\eta_2}\left(\frac{\partial \rho}{\partial n}\right)_T \tau - \frac{2}{\kappa T} q^m q_m - T\left(\frac{\partial \rho}{\partial T}\right)_n u_m \omega^{mcd}\left(\frac{1}{2\eta_1}\tau_{cd} + \frac{1}{3\eta_2}\tau g_{cd}\right)\right]. \tag{60}$$

In these formulae the scalar ω and the totally-symmetric tensor ω_{abc} satisfying $\omega^{abc}u_b u_c = 0$ are otherwise arbitrary. These merely specify the representation of the theory. The scalars κ, η_1, and η_2 are the thermal conductivity and viscosity coefficients, respectively. The remaining fields required to specify the theory are obtained directly from the equations given previously: s^a from Eq. (39), s^{ab} from Eq. (40), n^{ab} from Eq. (41), $M^m{}_{ab}$ from Eq. (45), $M^m{}_{cab}$ from Eq. (46), and I_a from Eq. (49).

These representations of the Eckart theory are considerably more complicated than its representation as a divergence-type theory in the first example.

IV. Equilibrium Solutions

We think of a dissipative physical system as being in equilibrium whenever its dynamics is time reversible. That is, we regard a solution of the fluid equations as an *equilibrium* solution if its time reverse is also a solution. In terms of the dynamical fields of a fluid theory, we define the action of *time reversal* in terms of the tensor character of each field: tensor fields of odd rank, such as N^m, change sign under time reversal, while tensor fields of even rank, such as T^{ab}, are unchanged.

For an equilibrium solution, we must have $I_A = 0$. To see this note that the rank of each tensor expresson on the left of Eq. (1) is always one larger than the rank of the corresponding component of I_A. Thus under time reversal the two sides of Eq. (1) acquire a relative minus sign, and so its right side must vanish for any solution whose time reverse is also a solution.

Consider now a particular equilibrium state of the fluid. Since the entropy production density, $\sigma \equiv S^A I_A$, is assumed non-negative for *all* states of the fluid, its value in this equilibrium state (zero) is its minimum. It follows that the first variation of σ under arbitrary variations in φ^A must also vanish at this equilibrium state: $\delta\sigma = S^A\,\delta I_A = 0$. We now require that the fluid theory be generic in the sense that under such variations all values of δI_A may be achieved that are compatible with the conservation laws, i.e., all values satisfying $N^A\,\delta I_A = 0$ and $T^{aA}\,\delta I_A = 0$. (For example, the Eckart [2] theory, with finite viscosity coefficients and thermal conductivity, satisfies this requirement.) The vanishing of the first variation of the entropy production density, $\delta\sigma$, now implies that S^A has nonzero components only in the conservation-law directions:

$$S^A = -\zeta N^A - \zeta_a T^{aA}. \tag{61}$$

These tensors, ζ and ζ_a, are functions of φ^A and g_{ab}. We conclude that, in a generic dissipative-fluid theory, the equilibrium states are (in the terminology of Section II) perfect-fluid states.

We next investigate the forms taken by the physical fields N^m, T^{ma}, and S^m in such a perfect-fluid state. To this end, we introduce the vector

$$X^m = S^m + \zeta N^m + \zeta_a T^{ma}. \tag{62}$$

Taking the derivative of X^m with respect to φ^A and using Eq. (17), we obtain

$$\frac{\partial X^m}{\partial \varphi^A} = N^m \frac{\partial \zeta}{\partial \varphi^A} + T^{ma} \frac{\partial \zeta_a}{\partial \varphi^A}. \tag{63}$$

This equation implies that X^m is a function of ζ and ζ_a alone, its dependence on φ^A in turn arising from the dependence of ζ and ζ_a on φ^A. Equation (63) also implies that N^m and T^{ma} are related to X^m by

$$N^m = \frac{\partial X^m}{\partial \zeta} \tag{64}$$

and

$$T^{ma} = \frac{\partial X^m}{\partial \zeta_a}. \tag{65}$$

From the symmetry of T^{ma} in Eq. (65), it follows that there exists a scalar generating function X (an algebraic function of ζ and ζ_a) that determines X^m:

$$X^m = \frac{\partial X}{\partial \zeta_m}. \tag{66}$$

But any scalar function of ζ and ζ_a is a function of ζ and $\mu = \zeta_a \zeta^a$ alone, and so we have $X = X(\zeta, \mu)$. Thus, Eqs. (64) and (65) become

$$N^m = 2 \frac{\partial^2 X}{\partial \zeta \, \partial \mu} \zeta^m \tag{67}$$

and

$$T^{ma} = 4 \frac{\partial^2 X}{\partial \mu^2} \zeta^m \zeta^a + 2 \frac{\partial X}{\partial \mu} g^{ma}. \tag{68}$$

These in turn determine S^m through Eq. (62):

$$S^m = -2 \left(\zeta \frac{\partial^2 X}{\partial \zeta \, \partial \mu} + 2\mu \frac{\partial^2 X}{\partial \mu^2} \right) \zeta^m. \tag{69}$$

Note that these expressions, Eqs. (67)–(69), are identical in form to those of an ordinary perfect fluid: the entropy and number currents are parallel to the timelike eigenvector of the stress-energy tensor, and the stress tensor is isotropic in that frame. Defining the standard thermodynamic variables n, ρ, p, s, and u^m in the usual way—$N^m = nu^m$, $T^{ma} = (\rho + p) u^m u^a + pg^{ma}$, and $S^m = snu^m$—we obtain, via Eqs. (67)–(69), expressions [7] for these variables in terms of the derivatives of X. The resulting expressions automatically satisfy the first law of thermodynamics,

$$d\rho = nT \, ds + \frac{\rho + p}{n} dn, \tag{70}$$

provided the temperature T is taken to be $T=(-\mu)^{-1/2}$. Inverting these expressions, we obtain the fields ζ and ζ_a in terms of the standard thermodynamic variables:

$$\zeta=\frac{\rho+p}{nT}-s \tag{71}$$

and

$$\zeta_a=\frac{u_a}{T}. \tag{72}$$

For any fluid theory, the physical fields N^m, T^{am}, and S^m are, as we have just seen, functions only of ζ and ζ_a in any equilibrium state. In addition, for a generic fluid theory, *all* of the fluid fields, φ^A, are functions only of ζ and ζ_a in any equilibrium state. Indeed, Eq. (61), in which N^A, T^{aA}, and S^A are given algebraic functions of φ^A, can in the generic case be inverted to yield φ^A in terms of ζ and ζ_a. We note that the resulting function, $\varphi^A(\zeta, \zeta_a)$, is determined once and for all by the fluid theory. Thus, the values of ζ and ζ_a specify completely the particular equilibrium state.

In addition to these algebraic constraints on the fluid fields in an equilibrium state, there are constraints on the space-time derivatives of those fields as well. The fluid equations, (1), when applied to an equilibrium solution, become a system of homogeneous linear equations on the derivatives $\nabla_m\zeta$ and $\nabla_m\zeta_a$. If there are "too many" fluid equations, the only equilibrium solutions will satisfy $\nabla_m\zeta=0$ and $\nabla_m\zeta_a=0$. These conditions are unphysical. For example, these conditions admit only homogeneous non-rotating solutions in flat space-time, and no solutions whatever for static spherically-symmetric self-gravitating fluid objects. Thus, in order to allow a "sufficient number" of equilibrium solutions to account for the observed properties of laboratory fluids, the number of field equations, and so the number of fluid fields must be limited. We expect, from the observation that laboratory fluids admit rotating equilibrium configurations, that the fluid equations will fix only the symmetric part of $\nabla_m\zeta_a$. This suggests that the appropriate fluid fields, φ^A, consist of one vector and one symmetric second-rank-tensor field (e.g., N^m and T^{ma}), and consequently that the fluid equations consist of one vector and one symmetric second-rank-tensor equation. We assume, for the remainder of this section, that this is the case.

In this case, the fluid equations, when evaluated at an equilibrium state, will include three scalar equations: (a) the equation formed by contracting the vector equation with ζ_a; (b) the equation formed by contracting the tensor equation with g_{ab}; and (c) the equation formed by contracting the tensor equation with $\zeta_a\zeta_b$. These three will be a system of homogeneous linear equations on the three scalar derivatives involving ζ and ζ_a: $\zeta^m\nabla_m\zeta$, $\zeta^m\nabla_m\mu$, and $\nabla_m\zeta^m$ (where $\mu=\zeta^a\zeta_a$). In the generic situation the only solution will be

$$\zeta^m\nabla_m\zeta=\zeta^m\nabla_m\mu=\nabla_m\zeta^m=0. \tag{73}$$

There will, in addition, be two vector equations: (a) the vector equation itself; and (b) the equation formed by contracting the tensor equation with ζ_a. These two will be a system of homogeneous linear equations on the three vector derivatives involving ζ and ζ_a: $\nabla_m \zeta$, $\nabla_m \mu$, and $\zeta^m \nabla_m \zeta_a$. In the generic situation, this system will require that some scalar function, Θ, of ζ and μ be constant, and that $\zeta^m \nabla_m \zeta_a$ be proportional to the gradient of some scalar function of ζ and μ. The latter condition may be expressed in terms of a scalar function Υ:

$$\zeta^m \nabla_m \zeta_a + \mu \nabla_a \log[(-\mu)^{1/2} \Upsilon] = 0. \tag{74}$$

Finally, there will be one symmetric tensor equation on the one symmetric tensor derivative involving ζ and ζ_a: $\nabla_{(a}\zeta_{b)}$. In the generic situation, the only solution will, in light of Eq. (74), be $\nabla_{(a}\zeta_{b)} = -\zeta_{(a}\nabla_{b)} \log \Upsilon$. But this means that $\Theta_a = \Upsilon \zeta_a$ satisfies Killing's equation. Thus the equilibrium equations become:

$$\nabla_a \Theta = 0, \tag{75}$$

$$\nabla_a \Theta_b + \nabla_b \Theta_a = 0. \tag{76}$$

Note that $\Theta(\zeta, \zeta_a)$ and $\Theta_a(\zeta, \zeta_b)$ are determined once and for all by the fluid theory, i.e., they are the same functions for all equilibrium solutions. In the generic case, these functions can be inverted to obtain ζ and ζ_a (and hence φ^A) as functions of Θ and Θ_a. Inserting $\varphi^A(\Theta, \Theta_a)$ into Eq. (1), it follows from the above that the coefficients of the anti-symmetric derivatives of Θ_a must vanish:

$$M^m{}_{AB} \frac{\partial \varphi^B}{\partial \Theta_a} - M^a{}_{AB} \frac{\partial \varphi^B}{\partial \Theta_m} = 0. \tag{77}$$

This equation in effect determines Θ and Θ_a for a given fluid theory.

To illustrate these remarks, consider the first example of Section III. From Eqs. (21)–(23), we have

$$\frac{\partial \varphi^A}{\partial \zeta_a} = T^{aA}. \tag{78}$$

Substituting and using Eqs. (5) and (29), we find that Eq. (77)—but with ζ, ζ_a replacing Θ, Θ_a—holds for any $M^m{}_{AB}$ in this example. Hence, we have

$$\Theta = \zeta, \tag{79}$$

$$\Theta_a = \zeta_a, \tag{80}$$

for these theories. The physical interpretation of Eqs. (79) and (80), using Eqs. (71) and (72), is that the fluid motion is stationary and rigid, and that the temperature, suitably red-shifted, is constant throughout the equilibrium fluid. In fact, Eqs. (79) and (80) also hold for the Eckart, Landau–Lifshitz, Israel–Stewart, and the divergence-type theries.

Do Eqs. (79) and (80) hold for all of the fluid theories introduced in Section II? The answer is, apparently, no. To see this, fix a fluid theory that does satisfy Eqs. (79) and (80). Now modify this theory by replacing $M^m{}_{AB}$ by $M^m{}_{AB} + \bar{M}^m{}_{AB}$, where $\bar{M}^m{}_{AB}$ satisfies Eq. (13). This modification will not affect any of the fields N^m, T^{ma}, S^m, N^A, T^{aA}, S^A, or I_A, and so it will not change the dependence of these fields on ζ and ζ_a in equilibrium states. Thus the function $\varphi^A(\zeta, \zeta_a)$ will be unchanged. However, this modification will change Θ and Θ_a defined by Eq. (77), provided only that

$$\bar{M}^m{}_{AB} \frac{\partial \varphi^B}{\partial \zeta_a} - \bar{M}^a{}_{AB} \frac{\partial \varphi^B}{\partial \zeta_m} \neq 0. \tag{81}$$

It appears that such an $\bar{M}^m{}_{AB}$ can be found. Consider, for instance, the second example of Section III. For the perfect-fluid states, let $\bar{M}^m{}_{AB}$ be given by

$$\bar{M}^m{}_{abcd} = u^m u_{(a} q_{b)(c} u_{d)}, \tag{82}$$

$$\bar{M}^m{}_{cab} = -t^{-1}_{cn} t^{nrs} \bar{M}^m{}_{rsab}, \tag{83}$$

$$\bar{M}^m{}_{ab} = t^{-1}_{ar} t^{rcd} t^{-1}_{bs} t^{spq} \bar{M}^m{}_{cdpq}, \tag{84}$$

where u^m is the unit vector proportional to N^m and $q_{ab} = g_{ab} + u_a u_b$. This $\bar{M}^m{}_{AB}$ satisfies Eq. (13). Indeed, it annihilates T^{aA} by Eqs. (83) and (84), and it annihilates N^A and S^A by these equations and the fact that $\bar{M}^m{}_{abcd}$ of Eq. (82) annihilates every symmetric tensor constructed from g_{ab} and u_a—in particular, $n^{ab} - n^r t^{-1}_{rs} t^{sab}$ and $s^{ab} - s^r t^{-1}_{rs} t^{sab}$. For the imperfect-fluid states, let $\bar{M}^m{}_{AB}$ again be given by Eqs. (82)–(84), but now projected, as described in Section II, orthogonally to N^A, T^{aA}, and S^A. Thus, we obtain an $\bar{M}^m{}_{AB}$, defined for all states, satisfying Eq. (13). Furthermore, substituting directly using Eqs. (71) and (72), we see that this $\bar{M}^m{}_{AB}$ also satisfies Eq. (81). We conclude that Eqs. (79) and (80) need not hold for a general fluid theory. Note that, were the original theory hyperbolic, we could, by choosing $\bar{M}^m{}_{AB}$ to be a small multiple of that given above, retain hyperbolicity.

Tolman [10] has given a physical argument that suggests that Eqs. (79) and (80) should hold quite generally for a reasonable fluid theory. This argument suggests that it might be appropriate to rule out fluid theories not satisfying these equations.

V. Stability of the Equilibrium Solutions

In certain theories of dissipative relativistic fluids, causality of the fluid equations implies stability of the equilibrium solutions [7, 11]. In this section we explore the conditions under which this relationship between causality and stability holds for more general theories.

Consider a smooth one-parameter family, $\varphi^A(\lambda)$, of solutions of the fluid equations (1), which for $\lambda = 0$ is an equilibrium solution. (We assume for simplicity that the space-time metric g_{ab} is independent of λ.) Denote by $\delta\varphi^A$ the derivative of this family with respect to λ, evaluated at $\lambda = 0$. To determine the evolution of this perturbation, $\delta\varphi^A$, we differentiate the fluid equations (1) with respect to λ and evaluate at $\lambda = 0$:

$$M^m{}_{AB} \nabla_m \delta\varphi^B + \frac{\partial M^m{}_{AB}}{\partial \varphi^C} \delta\varphi^C \nabla_m \varphi^B = \delta I_A. \tag{85}$$

Here, and for the remainder of this section, a field not preceded by δ is to be evaluated at $\lambda = 0$, i.e., in the equilibrium solution.

To investigate the stability of this equilibrium solution, we introduce the quadratic form

$$E^m = \tfrac{1}{2} M^m{}_{AB} \delta\varphi^A \delta\varphi^B. \tag{86}$$

We now assume that our theory has causal evolution equations. Then this E^m must be future-directed timelike for all non-vanishing fluid perturbations $\delta\varphi^A$, and so $E(\Sigma)$, defined by

$$E(\Sigma) = -\int_\Sigma E^m \, dS_m \tag{87}$$

(where the integral is over a Cauchy surface Σ) is a positive-definite norm on $\delta\varphi^A$. We use this norm to monitor the evolution of $\delta\varphi^A$ as follows: The difference between the $E(\Sigma)$'s evaluated on two Cauchy surfaces is given by

$$E(\Sigma_2) - E(\Sigma_1) = \int_\Omega \nabla_m E^m \, d\Omega, \tag{88}$$

where the integral is over the region, Ω, of space-time between the two Cauchy surfaces. Were the integrand, $\nabla_m E^m$, non-positive for all fluid perturbations, then the norm, $E(\Sigma)$, would be a non-increasing function of time bounded below by zero; and so this equilibrium solution would be stable.

Evaluating the divergence of E^m, using Eq. (85), we obtain

$$\nabla_m E^m = \delta\varphi^A \delta I_A + J^m_{ABC} \delta\varphi^A \delta\varphi^B \nabla_m \varphi^C, \tag{89}$$

where we have set

$$J^m_{ABC} = \frac{\partial M^m{}_{AB}}{\partial \varphi^C} - \frac{1}{2} \frac{\partial M^m{}_{AC}}{\partial \varphi^B} - \frac{1}{2} \frac{\partial M^m{}_{BC}}{\partial \varphi^A}. \tag{90}$$

We now choose (essentially without loss of generality) the fluid variables to be

$$\varphi^A = -S^A. \tag{91}$$

The first term on the right of Eq. (89) now becomes $-\delta S^A \delta I_A = -\delta^2 \sigma$, i.e., minus the second variation of the entropy-proudction density σ. Since σ achieves its minimum value (zero) in equilibrium states, this term is non-positive. The second term on the right in Eq. (89) may be written as

$$J^m_{ABC}\,\delta\varphi^A\,\delta\varphi^B\,\nabla_m\varphi^C=\frac{1}{2}\left(J^m_{ABC}\frac{\partial\varphi^C}{\partial\Theta_a}-J^a_{ABC}\frac{\partial\varphi^C}{\partial\Theta_m}\right)\delta\varphi^A\,\delta\varphi^B\,\nabla_m\Theta_a, \tag{92}$$

using the fact (from Section IV) that an equilibrium solution φ^A depends only on a constant Θ and a Killing field Θ_a. This term vanishes identically for those equilibrium solutions in which $\nabla_a\Theta_b=0$, e.g., those for which Θ_a is a translation Killing field in flat space-time. Thus, these particular equilibrium solutions are always stable in every causal fluid theory. For which causal fluid theories must *every* equilibrium solution be stable? Since any timelike Killing field Θ_a is possible on the right in Eq. (92), this term will be non-positive for every equilibrium solution only if it is zero, i.e., only if

$$J^m_{ABC}\frac{\partial\varphi^C}{\partial\Theta_a}-J^a_{ABC}\frac{\partial\varphi^C}{\partial\Theta_m}=0. \tag{93}$$

Thus, Eq. (93) is a sufficient condition that every equilibrium solution of a causal fluid theory be stable.

To illustrate this condition, we consider the first example of Section III. As we saw in Section IV, all of these theories satisfy Eqs. (79) and (80). Thus, Eq. (93) is evaluated using Eqs. (77), (78), (23), and (5) and becomes

$$\frac{\partial M^m{}_{AB}}{\partial\zeta_a}-\frac{\partial M^a{}_{AB}}{\partial\zeta_m}=0, \tag{94}$$

i.e., $M^m{}_{AB}=\partial X_{AB}/\partial\zeta_m$ for some X_{AB}. Thus, causality implies stability for all of the equilibrium solutions for certain of these theories, including via Eq. (31) all theories of divergence type [6, 7]. But Eq. (93) is not satisfied in general, even for the theories of this example.

VI. Shock Waves

Physical fluids are known to manifest shock waves—rapid transitions of the fluid state that occur as fluid elements pass from one space-time region into another. A stringent test of any macroscopic fluid theory is its ability to describe the behavior of a fluid during such a severe non-equilibrium process. In the theory of a perfect-fluid, for example, a shock wave is described by a solution of the fluid equations that is discontinuous across a timelike three-surface—the shock front. Such a solution can, by virtue of the divergence form of the perfect-fluid equations, be given mathematical meaning. In the Navier–Stokes–Fourier theory, in contrast, a shock wave is described by a smooth solution of the fluid equations. What, if any, is the appropriate description of shock waves within the general fluid theories of

Section II? We argue in this section that, for most of these fluid theories, there is no available description at all for sufficiently strong shock waves.

Consider a smooth, stationary [12], plane-symmetric solution of the fluid equations (1) in flat space-time. That is, fix a space-like unit translation, $x^a = \nabla^a x$, in flat space-time, and demand that the derivative of the fluid fields be proportional to it: $\nabla_a \varphi^A = (d\varphi^A/dx)\nabla_a x$. Then Eq. (1) simply becomes

$$x_m M^m{}_{AB}(x^a \nabla_a \varphi^B) = I_A. \tag{95}$$

The conservation laws for this solution can be completely integrated: the fields $N^m x_m$ and $T^{am} x_m$ must be constant throughout space-time. The entropy law becomes $x^a \nabla_a (S^m x_m) \geqslant 0$.

We now impose, as a further condition on this solution, that, in each of the limits $x \to \infty$ and $x \to -\infty$, the fluid goes to some perfect-fluid state: $N^a = nu^a$, $T^{ab} = (\rho + p)\, u^a u^b + pg^{ab}$, and $S^a = snu^a$. The perfect-fluid states achieved in the two limits will in general be different. But, by conservation, the combinations

$$N^a x_a = n(u^a x_a), \tag{96}$$

$$T^{ab} x_b = (\rho + p)(u^b x_b) u^a + px^a, \tag{97}$$

must have the same values in the limit $x \to \infty$ as in the limit $x \to -\infty$. Similarly, the combination

$$S^a x_a = snu^a x_a, \tag{98}$$

must be no smaller in the limit $x \to \infty$ than in the limit $x \to -\infty$. These, the Taub conditions (i.e., the relativistic Rankine–Hugoniot conditions) [13], constrain the asymptotic perfect-fluid states between which transitions are allowed. For example, fix the perfect-fluid equation of state, $s(\rho, n)$, and let there be given values of the perfect-fluid fields, ρ, n, and u^a, as $x \to -\infty$. Then there will normally be just one other set of values for these fields as $x \to \infty$ that preserves the combinations in Eqs. (96)–(97) while not decreasing the combination in Eq. (98).

Equation (95) is a system of ordinary differential equations to determine the spatial variation of the fluid fields through the shock solution. We are guaranteed a unique smooth solution of this system except where $x_m M^m{}_{AB}$ becomes non-invertible, i.e., except where

$$\det[x_m M^m{}_{AB}] = 0. \tag{99}$$

This equation is precisely the condition that x_m be normal to a characteristic surface of the system [8]. We expect that there will be no continuous solution of Eq. (95) across any x-value at which Eq. (99) holds. We shall now argue that $x_m M^m{}_{AB}$ must be non-invertible somewhere for all sufficiently strong shock solutions in essentially all of these causal fluid theories. We shall thus conclude that there exist no continuous solutions of the fluid equations for strong shocks in these theories [14].

There are two types of speeds associated with a fluid state in this problem. The first is the characteristic speeds, λ, of the fluid equations, i.e., the roots of

$$\det[(-\lambda u_m + w_m) M^m{}_{AB}] = 0, \tag{100}$$

where w^a is any unit vector orthogonal to u^a. For a causal fluid theory each λ is real (since these tensors are symmetric in A and B, and $-u_m M^m{}_{AB}$ is positive definite) and satisfies $|\lambda| \leqslant 1$. The other type of speed is the physical speed v of a fluid element, relative to the shock front, given by

$$\frac{v}{(1-v^2)^{1/2}} = u^a x_a. \tag{101}$$

It is a general feature of these theories that Eq. (99) is satisfied precisely when the physical fluid speed is equal to one of the characteristic speeds. To see this, compare Eqs. (99) and (100), using that

$$x^a = \frac{-vu^a + w^a}{(1-v^2)^{1/2}}, \tag{102}$$

for some unit vector w^a orthogonal to u^a.

We now consider a causal fluid theory in which: (a) all the characteristic speeds are strictly less than the speed of light, i.e., $|\lambda| < 1$; and (b) the equation of state for the perfect-fluid states is such that the solutions to the Taub conditions satisfy $v_+ > c_s^+$ and $c_s^- > v_- \geqslant 0$, where v_+ and v_- are the physical fluid speeds in the asymptotic perfect-fluid states before and after the shock, and c_s^+ and c_s^- are the corresponding adiabatic sound-speeds, each given by $c_s^2 = (\partial p/\partial \rho)_s$. The first condition holds in a generic fluid theory, while the second is known to hold for numerous specific equations of state [15] (and may well hold in general).

Consider a solution of Eq. (95) whose incoming fluid speed v_+ is larger than the largest characteristic speed of the system. For this solution the matrix $x_m M^m{}_{AB}$ must be positive definite when evaluated in the asymptotic perfect-fluid state preceding the shock. This follows because $x_m M^m{}_{AB}$ is positive definite (by assumption) for x_m on the past light cone, and in this case x_m will by Eq. (102) be "closer to the past light cone" than the normal to any characteristic surface. Consider next the sub-matrix that is formed from the inner products of N^A and T^{aA} with $x_m M^m{}_{AB}$. In the asymptotic perfect-fluid states, this sub-matrix is just the matrix that appears in the usual perfect-fluid equations. It is staightforward to verify that this perfect-fluid sub-matrix is positive definite whenever $v > c_s$, while its signature contains at least one minus when $v < c_s$. Whenever the signature of this sub-matrix contains a minus, then so must the full matrix $x_m M^m{}_{AB}$ [16]. But we have assumed that the fluid speed v_- is smaller than c_s in the asymptotic region following the shock. Thus the perfect-fluid sub-matrix, and also the full matrix $x_m M^m{}_{AB}$, cannot be positive definite in this asymptotic region.

Thus, we have shown that under these conditions the matrix $x_m M^m{}_{AB}$ must be positive definite in the incoming asymptotic region, and not positive definite in the outgoing. It follows that this matrix must be non-invertible somewhere in between. We therefore expect that there will be no continuous solution of Eq. (95) in this case.

This breakdown of the shock equations for strong shocks (i.e., those with sufficiently large fluid speeds) was first noted by Grad [17], in the context of a non-relativistic fluid theory, and by a number of authors [14, 18, 19, 20], in the context of the Israel–Stewart theory. For the Israel–Stewart theory's version of a classical Maxwell gas [21] the largest characteristic speed is only about $1.76c_s$, so this breakdown is a severe limitation on the theory's ability to describe shocks.

Finally, we remark that it is not clear what it means for a discontinuous φ^A to be a "solution" of Eq. (1). For example, since Eq. (1) is non-linear, distributional solutions of that equation do not, as the equation is written, make sense. Thus, it appears that in general, the causal fluid theories of Section II have no solutions at all capable of describing strong shocks. But note that, for those theories whose equations can be written as divergences [6, 7], distributional solutions—and also shock solutions—presumably can (in analogy with those for perfect fluids) be defined.

VII. Conclusion

In this section, we discuss a number of open questions and areas for further work. For the fluid theories constructed in Section II, it is generally appropriate on physical grounds to restrict the fluid fields φ^A by various algebraic inequalities. Examples of such restrictions are that the particle-number current N^m be future-directed timelike, that the stress-energy tensor T^{ma} satisfy a suitable energy condition, or that the thermodynamic variables satisfy various inequalities. Some such restriction is always necessary in order that the theory be causal, since, e.g., no theory of Section II is causal for the fluid state $\varphi^A = 0$. What is the complete list of physically reasonable restrictions on the φ^A? Is there a fluid theory that is causal for all φ^A allowed by these conditions?

A major complication is writing down examples of fluid theories, via the construction of Section II, is the fact that the perfect-fluid and the imperfect-fluid states were treated differently there. Is there some way of reformulating this subject, e.g., by a specific choice of variables, such that the two types of states are treated on a more equal footing? Does there exist a simple, systematic procedure for explicitly writing out all the fluid theories introduced in Section II? Does there exist a simple systematic procedure for determining whether a given fluid theory (e.g., the Israel–Stewart theory) is symmetric in the sense of Section II?

In Section IV, we briefly discussed the issue of whether in equilibrium the Θ-variables (those that satisfy simple space-time equations, (75) and (76)) must be identical to the ζ-variables (those directly linked to the thermodynamics, Eqs. (71) and (72)). For many theories of interest—in particular, all of the theories of divergence type—these two are identical. But, as we saw in Section II, there are examples of theories for which these variables differ. Is there a solid physical argument that rejects such theories? Is there a simple characterization of those theories for which the two types of variables are identical?

In Section V, we gave a sufficient condition, Eq. (93), that the equilibrium states of a causal theory be stable. Is this condition also necessary? As it stands, Eq. (93) is quite complicated. Is there some simpler version or physical interpretation of this equation?

In Section VI, we argued that causal theories admit no continuous solutions to represent sufficiently strong shocks. Thus, if shocks are to be described at all within these theories, it will have to be by discontinuous solutions of Eq. (1). What is this to mean mathematically? Can, for example, a "solution, not necessarily continuous" of Eq. (1) be defined in such a way that the system has a well-posed initial-value formulation? Such a definition appears more feasible for theories of divergence type than for a general theory. Even after shock solutions have been given mathematical meaning, there remains a related question. Do there exist solutions, suitable in terms of number and character, to describe shocks in all situations of physical interest?

All of the theories of Section II generalize, in some sense, the Navier–Stokes–Fourier theory. Is it true, in some precise sense, that these theories give rise to the Navier–Stokes–Fourier theory as an appropriate (e.g., low velocity and large length-scale) limit? Is there a systematic procedure for taking this limit? Are there observations that will distinguish these theories from their limits—and from each other? Are there, for example, observation-based conditions on the $M^{m}{}_{AB}$ other than those already incorporated in this paper?

Finally, it would be of great interest to find a rigorous derivation, starting from a microphysical description of the matter, of any relativistic theory of a dissipative fluid.

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