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The Coupling of Charged Superfluid Mixtures to the Electromagnetic Field

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The system of equations that describes the macroscopic properties of a mixture of superfluids is derived by generalizing the equations of Andreev and Bashkin to include new vorticity-preserving forces. The affects of these forces on the dynamics is investigated by using a macroscopic phenomenological approach developed by Bekarevich and Khalatnikov. A Hamiltonian formulation of the theory is developed and used to couple the charged components of the fluid to the electromagnetic field. The physically relevant values of the additional vorticity-preserving forces are determined by requiring that each component of the superfluid mixture responds to the electromagnetic field via an appropriate Lorentz force.

I. INTRODUCTION

The equations that describe the macroscopic dynamics of a superfluid were first derived by Landau [1] and have subsequently been generalized in a number of ways. The equations were extended to include the description of a mixture of two superfluids by Khalatnikov [2] and by Andreev and Bashkin [3], while Vardanyan and Sedrakyan [4] generalized the equations to include charged superfluids coupled to the electromagnetic field. Holm and Kupershmidt [5] extended the theory to a mixture of N charged superfluids and developed a Hamiltonian formalism for the resulting fluid equations and electromagnetic coupling. In this paper we generalize these equations still further by including a large class of vorticity-preserving interaction terms in the dynamical equations for the superfluid velocities. A Hamiltonian formulation for the equations is presented and used to couple the charged components of the mixture to the electromagnetic field. The new vorticity-preserving forces play a non-trivial dynamical role at the locations of vortices and have a profound effect on the *natural* coupling of these fluids to the electromagnetic field.

We consider a fluid consisting of a mixture of N species of superfluids, a single "normal" fluid consisting of the excited states of all the superfluid species, and an additional "ordinary" fluid of other particles. To avoid confusion we will use the terms "normal" and "ordinary" when referring respectively to the latter two fluids.

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The specific physical system that we have in mind is the superfluid interior of a neutron star. There the neutrons and protons each form superfluid condensates while the electrons (and muons) form an independent ordinary fluid. Other systems to which the present study may apply include mixtures of He^3 in He^4 and mixtures of protons and neutrons in a heavy-metal crystal (for references see Holm and Kupershmidt [5]).

The macroscopic superfluid equations, of the type introduced by Landau [1], are intended to describe the behavior of the large scale properties of the fluid (e.g., density, temperature, velocity) which have been suitably averaged over distances that are large compared to typical inter-particle separations. The theory of superfluid mixtures being considered here is described by macroscopic variables associated with each component of the fluid: e.g., the total mass density and superfluid velocity, ρ_α and $\mathbf{v}_\alpha$, associated with the αth species of superfluid, the velocity, $\mathbf{v}_{(n)}$, of the normal fluid, and the density and velocity, ρ_e and $\mathbf{v}_e$, of the ordinary fluid. The ordinary fluid is composed primarily of electrons in a neutron star, so we let the subscript e refer to this fluid whatever its composition might be. Note that we allow the normal and ordinary fluids to have independent dynamics at this point in order to allow for the possibility of macroscopic electromagnetic interactions between these fluids. In contrast, we combine all of the excited superfluid states into a single normal fluid. In a neutron-star superfluid this equilibrium should be maintained by the strong interactions of the various species on very short time scales.

For each superfluid species the macroscopic velocity, $\mathbf{v}_\alpha$, is chosen to be proportional to the kinematic portion of the canonical momentum of the particles (or Cooper pairs) which have condensed into the superfluid state. The canonical momentum for each species is given, therefore, by $m_\alpha(\mathbf{v}_\alpha + a_\alpha \mathbf{A})$, where m_α is the mass, $a_\alpha = q_\alpha / m_\alpha c$ is the charge-to-mass ratio of each particle (or Cooper pair) divided by c, the speed of light, and $\mathbf{A}$ is the electromagnetic vector potential. (We use gaussian units for all electromagnetic quantities.) For the simple case considered here, in which the order parameter for the condensed stated is a complex scalar field [6], the canonical momentum is related to the gradient of the phase, S_α, of the order parameter via the London equation:

$$\mathbf{v}_\alpha + a_\alpha \mathbf{A} = \frac{\hbar}{m_\alpha} \nabla S_\alpha. \tag{1}$$

While the macroscopic superfluid equations, of the type described above, have proven to be extremely successful in describing a variety of superfluid phenomena, they do not provide a convenient (or perhaps even acceptable) description of superfluids in which rotation and magnetic fields are present. As a consequence of Eq. (1), the vorticity, $\boldsymbol{\omega}_\alpha$, is linked to the value of the magnetic induction, $\mathbf{B}$, by $\boldsymbol{\omega}_\alpha \equiv \nabla \times \mathbf{v}_\alpha = -a_\alpha \mathbf{B}$, except where the right side of Eq. (1) becomes singular. On the basis of microscopic theory and laboratory experiments, such singular regions are found to assume the form of arrays of one-dimensional quantized vortices [7]. Thus the curl of Eq. (1) takes on the form [8, 9],

$$\boldsymbol{\omega}_\alpha + a_\alpha \mathbf{B} = \frac{2\pi\hbar}{m_\alpha} \sum_i \int \mathbf{v}_{\alpha i}(l)\, \delta^3[\mathbf{r} - \mathbf{r}_{\alpha i}(l)]\, dl, \tag{2}$$

where $\mathbf{r}_{\alpha i}(l)$ is a vector giving the location of the points (parameterized by l) along the ith vortex belonging to species α, and $\mathbf{v}_{\alpha i}(l)$ is the unit vector tangent to the vortex. Due to the extremely small magnitude (in the ground state) of the circulation about each vortex, $2\pi\hbar/m_\alpha$, any fluid undergoing macroscopic rotation (or having a macroscopic magnetic field) must contain a very large spatial density of these vortices. The inter-vortex separation is expected to be much smaller, therefore, than typical macroscopic length scales. For example, in the superfluid interior of a neutron star that rotates at the angular velocity of the Vela pulsar, the inter-vortex spacing is estimated to be about 10^{-3} cm [10]. Similarly a neutron star having a typical magnetic field of 10^{12} G is expected to contain vortex lines with an average spacing of about 10^{-10} cm [10]. It is appropriate, therefore, to perform a second spatial averaging of the superfluid equations to smooth out these singularities. Following Bekarevich and Khalatnikov [11] and Sonin [8] we replace the singular superfluid velocities and magnetic induction which satisfy Eq. (1) with smooth averaged quantities that satisfy

$$\boldsymbol{\omega}_\alpha + a_\alpha \mathbf{B} = \frac{2\pi\hbar}{m_\alpha} n_\alpha \mathbf{v}_\alpha, \tag{3}$$

where n_α is the number of vortices per unit area perpendicular to $\mathbf{v}_\alpha$ (the unit vector parallel to the *average* local direction of the vortices) [12]. To complete this averaging process, the expression for the energy of the fluid must be modified to include the internal energy associated with the vortices (e.g., the kinetic and/or magnetic energy associated with the circulation of fluid about each vortex). In the derivation of the superfluid equations in the following sections we include a simple model for this vortex energy which is a natural generalization of the model proposed by Bekarevich and Khalatnikov [11].

The goal of this paper is to derive a system of equations that describes the dynamics of mixtures of charged superfluids on length scales larger than typical inter-particle and inter-vortex separations. In Section II we present a standard macroscopic derivation of the uncharged ($a_\alpha = 0$) versions of these equations based on the conservation laws. In our derivation, however, we include a large class of new vorticity-preserving forces in the superfluid velocity equations. Special cases of these forces have been considered previously by Bekarevich and Khalatnikov [11]. These additional forces play an important role in the presence of vortices, as illustrated by the vortex-averaged version of the theory considered here. We also present in Section II a Hamiltonian formulation of the uncharged equations which generalizes the work of Holm and Kupershmidt [5]. Using this Hamiltonian formulation, we turn in Section III to the coupling of electromagnetism to the charged components of superfluid mixtures. We find that the new vorticity preserving force terms introduced in Section II have a profound affect on this electromagnetic coupling. By properly choosing these forces, it is possible to allow each component of the charged superfluid mixture to respond to the electromagnetic field via an appropriate Lorentz force law. We think that the electromagnetic coupling proposed here is far more natural, therefore, than those proposed by Vardanyan and Sedrakyan [4] or Holm and Kupershmidt [5].

II. The Uncharged Superfluid Equations

The macroscopic dynamical equations for mixtures of superfluids are derived here using the method developed by Landau [1] and Khalatnikov [2]. One begins by imposing the appropriate conservation laws for this system: mass conservation for each species of particle, the conservation of total momentum, and conservation of entropy for the ordinary and the normal fluid. These equations together with an assumed form for the variation in the energy density (the first law of thermodynamics) imply an equation for the time evolution of the energy density of the fluid. The requirement that this energy-evolution equation be a conservation law is used to fix the remaining undetermined quantities in the theory: the form of the stress energy tensor and the forces that appear in the equations for the fluid velocities.

The equations that describe the evolution of this fluid mixture include the conservation of mass of each species of particle, the conservation of entropy and the total momentum conservation equations:

$$\partial_t \rho_\alpha + \nabla_b(\rho_\alpha v^b_{(n)} + P^b_\alpha) = 0, \tag{4}$$

$$\partial_t \rho_e + \nabla_b(\rho_e v^b_e) = 0, \tag{5}$$

$$\partial_t s_{(n)} + \nabla_b(s_{(n)} v^b_{(n)}) = 0, \tag{6}$$

$$\partial_t s_e + \nabla_b(s_e v^b_e) = 0, \tag{7}$$

$$\partial_t P^a + \nabla_b \pi^{ab} = 0. \tag{8}$$

In these equations ∂_t is the partial derivative with respect to the time coordinate t while ∇_b is the three-dimensional Euclidean covariant derivative (i.e., in Cartesian coordinates ∇_b is just the partial derivative $\partial/\partial x^b$). Latin indices (a, b, c, etc., except e which we reserve for the ordinary fluid) refer to the spatial components of vectors and tensors, and summation over these repeated indices is assumed. The mass current of each species of superfluid particle (measured in the frame co-moving with $v^a_{(n)}$ and including both superfluid and normal phases) is denoted P^a_α. The entropy densities of the ordinary and normal fluids are denoted as s_e and $s_{(n)}$, respectively.

The stress tensor is denoted π^{ab} and the total momentum density of the fluid P^a is taken to be equal to the sum of the mass currents of each species of particle, i.e.,

$$P^a = \rho_e v_e^a + \sum_\alpha (\rho_\alpha v_{(n)}^a + P_\alpha^a). \tag{9}$$

In addition to the conservation laws, Eqs. (4)–(8), dynamical equations for the evolution of the superfluid velocities v_α^a and the ordinary fluid velocity v_e^a must be specified. Without loss of generality we take these equations to have the forms:

$$\partial_t v_\alpha^a + v_\alpha^b \nabla_b v_\alpha^a + \nabla^a(\mu_\alpha - \tfrac{1}{2}|\mathbf{v}_\alpha - \mathbf{v}_{(n)}|^2) = F_\alpha^a, \tag{10}$$

$$\partial_t v_e^a + v_e^b \nabla_b v_e^a + \nabla^a(\mu_e - \tfrac{1}{2}|\mathbf{v}_e - \mathbf{v}_{(n)}|^2) + \frac{s_e}{\rho_e} \nabla^a T_e = F_e^a. \tag{11}$$

(Note that summation over Greek indices is not intended unless explicitly noted.) The left sides of Eqs. (10)–(11) are respectively the Landau equation for the evolution of the superfluid velocity and the Euler equation for the evolution of an ordinary fluid. The forces F_α^a and F_e^a which appear on the right sides of Eqs. (10)–(11) are yet to be determined. The F_α^a must either be curl-free or they must vanish when the superfluid velocities v_α^a are curl-free, however, if the superfluid equations are to ensure that the v_α^a evolve in a curl-free manner consistent with the Landau equation in the absence of vortices. In these equations T_e and $T_{(n)}$ are the temperatures of the ordinary and normal fluids, while μ_α and μ_e are the chemical potentials measured in a frame moving with velocity $v_{(n)}^a$. More precisely, if U_0 is the energy density of the fluid in the frame moving with velocity $v_{(n)}^a$ then these quantities are defined as the indicated coefficients in the first law of thermodynamics

$$\begin{aligned} \mathbf{d}U_0 = {} & T_e\, \mathbf{d}s_e + \mu_e\, \mathbf{d}\rho_e + \tfrac{1}{2}\rho_e \mathbf{d}\, |\mathbf{v}_e - \mathbf{v}_{(n)}|^2 + T_{(n)}\, \mathbf{d}s_{(n)} \\ & + \sum_\alpha \{\mu_\alpha\, \mathbf{d}\rho_\alpha + P_{\alpha b} \mathbf{d}(v_\alpha^b - v_{(n)}^b) + \lambda_\alpha^b\, \mathbf{d}\omega_{\alpha b}\}. \end{aligned} \tag{12}$$

The terms, $\lambda_\alpha^b\, \mathbf{d}\omega_{\alpha b}$, which appear in Eq. (12), generalize the terms proposed by Bekarevich and Khalatnikov [11] to describe the energy associated with the vortices in the fluid. Since $\omega_{\alpha b}$ is proportional to the number density of vortices (see Eq. (3)) the magnitude of λ_α^b is a measure of the energy per vortex. (See Sonin [8] for a recent review of more complicated vortex-energy expressions.) When λ_α^a is set to zero the equations return to their original Landau form in which the superfluid velocities satisfy Eq. (1) and have singular vorticities. When λ_α^a is not zero, the superfluid velocities are to be interpreted as averaged quantities that satisfy Eq. (3). Khalatnikov [13] argues that λ_α^a should be taken to be parallel to ω_α^a in this case and have a magnitude such that $\boldsymbol{\lambda}_\alpha \cdot \boldsymbol{\omega}_\alpha$ is the energy density associated with the fluid circulating about the vortices. We allow λ_α^a to be an arbitrary Galilean invariant vector field.

The superfluid mass currents P^a_α are also defined by the first law of thermodynamics, Eq. (12). Nepomnyashchiĭ [14] shows (on the basis of a particular microscopic theory) that these currents must be the same as those that appear in the mass conservation law, Eq. (4). We assume that the P^a_α as defined by Eq. (12) are equal to those that appear in the mass conservation law, Eq. (4), in general. The precise form of these currents in terms of the fundamental variables of the problem will not play a significant role in our analysis. However, these currents are Galilean invariant vectors and therefore could be written as some linear combinations of the Galilean invariant vector fields in the problem: $v^a_\alpha - v^a_{(n)}$, $v^a_e - v^a_{(n)}$ and ω^a_α. While no detailed physical motivation for including terms in P^a_α proportional to $v^a_e - v^a_{(n)}$ or ω^a_α has ever been proposed, Khalatnikov [13] has speculated that terms proportional to ω^a_α might be needed when the fluid velocities are large. It is probably most appropriate, nevertheless, to think of P^a_α in terms of Andreev and Bashkin's [3] expression

$$P^a_\alpha = \sum_\beta \rho_{\alpha\beta}(v^a_\beta - v^a_{(n)}). \tag{13}$$

The mass density matrix $\rho_{\alpha\beta}$ must be evaluated for any physical system of interest on the basis of some micro-physical model for that system. Explicit expressions for $\rho_{\alpha\beta}$ have been derived by several authors [3, 4, 10, 15]. They find that $\rho_{\alpha\beta}$ must, quite generally, be symmetric in $\alpha\beta$, but that it is not diagonal for many cases of interest (including the neutron–proton superfluids of neutron-star interiors).

The dynamical evolution of an uncharged superfluid mixture would be completely determined by Eqs. (4)–(12) if expressions for the stress tensor π^{ab} and the forces F^a_α and F^a_e were known. The method of Landau [1] and Khalatnikov [2] for determining these quantities is to demand that the evolution of the energy predicted by these equations is in fact a conservation law. The energy density of the fluid U is related to U_0, the energy density measured in the frame of reference moving at velocity $v^a_{(n)}$, by the expression

$$U = U_0 + P_b v^b_{(n)} - \tfrac{1}{2}\rho v^b_{(n)} v_{(n)b}, \tag{14}$$

where ρ denotes the total mass density of the fluid, $\rho = \rho_e + \sum_\alpha \rho_\alpha$. The time evolution of this quantity can be computed with the aid of Eqs. (4)–(12). The resulting expression is

$$\begin{aligned}\partial_t U + \nabla_b U^b = {} & \rho_e(v_{eb} - v_{(n)b})\, F^b_e + \nabla_b v_{(n)a} \Big\{ \pi^{ab} - p g^{ab} - \rho_e v^a_e v^b_e \\ & - \sum_\alpha [v^a_\alpha P^b_\alpha + (\rho_\alpha v^a_{(n)} + P^a_\alpha)\, v^b_{(n)} + g^{ab}\lambda^c_\alpha \omega_{\alpha c} - \lambda^a_\alpha \omega^b_\alpha] \Big\} \\ & + \sum_\alpha [P_{\alpha b} + (\nabla \times \boldsymbol{\lambda}_\alpha)_b][F^b_\alpha + 2(v_{(n)a} - v_{\alpha a})\, \nabla^{[a} v^{b]}_\alpha],\end{aligned} \tag{15}$$

where the energy current U^a and the pressure p are defined by the expressions

$$U^a = T_e s_e v_e^a + T_{(n)} s_{(n)} v_{(n)}^a + (\mu_e - \tfrac{1}{2} |\mathbf{v}_{(n)}|^2)\, \rho_e v_e^a + (\pi^{ba} - p g^{ab})\, v_{(n)b}$$
$$+ \sum_\alpha (\mu_\alpha - \tfrac{1}{2} |\mathbf{v}_{(n)}|^2)(\rho_\alpha v_{(n)}^a + P_\alpha^a)$$
$$+ \sum_\alpha \{\boldsymbol{\lambda}_\alpha \times [\mathbf{F}_\alpha + (\mathbf{v}_\alpha - \mathbf{v}_{(n)}) \times \boldsymbol{\omega}_\alpha]\}^a \tag{16}$$

and

$$p = -U_0 + T_e s_e + T_{(n)} s_{(n)} + \mu_e \rho_e + \sum_\alpha \rho_\alpha \mu_\alpha. \tag{17}$$

The Euclidean metric g_{ab} (i.e., just the identity matrix in Cartesian coordinates) and its inverse g^{ab} (which appears explicitly in Eqs. (15) and (16)) are used to raise and lower tensor indices. Square brackets surrounding a pair of indices indicates anti-symmetrization, e.g., $\nabla^{[a} v_\alpha^{b]} = \frac{1}{2}(\nabla^a v_\alpha^b - \nabla^b v_\alpha^a)$.

Equation (15) will guarantee the local conservation of energy of the fluid if the right side vanishes [16]. It is natural to require that the term proportional to $\nabla_b v_{(n)a}$ on the right vanishes separately by defining the stress tensor as

$$\pi^{ab} = p g^{ab} + \rho_e v_e^a v_e^b + \sum_\alpha [v_\alpha^a P_\alpha^b + (\rho_\alpha v_{(n)}^a + P_\alpha^a)\, v_{(n)}^b + g^{ab} \lambda_\alpha^c \omega_{\alpha c} - \lambda_\alpha^a \omega_\alpha^b]. \tag{18}$$

This expression for π^{ab} is a symmetric tensor if the mass currents P_α^a are given by Eq. (13), with $\rho_{\alpha\beta}$ symmetric in $\alpha\beta$ and $\lambda_\alpha^a = \lambda_\alpha \omega_\alpha^a$ (where λ_α is any scalar function). It is also natural to set the force F_e^a equal to zero. While it is possible that there could exist some non-electromagnetic force acting on the ordinary fluid, we are unaware of any significant force of this type (other than viscous dissipation) for the case of the electron fluid in neutron star matter. Since we are ignoring dissipation in this paper and the electromagnetic forces in this section, we set $F_e^a = 0$. This leaves only the term containing the forces F_α^a. This term will vanish if F_α^a is given by

$$F_\alpha^a = \sum_\beta K_{\alpha\beta}^{ab} [P_{\beta b} + (\nabla \times \boldsymbol{\lambda}_\beta)_b] + 2(v_{(n)b} - v_{\alpha b})\, \nabla^{[a} v_\alpha^{b]}, \tag{19}$$

where $K_{\alpha\beta}^{ab}$ is any tensor that is anti-symmetric in the sense that $K_{\alpha\beta}^{ab} = -K_{\beta\alpha}^{ba}$. The forces F_α^a must vanish, however, when the vorticity of the superfluid velocities vanishes if Eq. (10) is to return to the Landau form in the absence of vortices. The tensor $K_{\alpha\beta}^{ab}$ must vanish, therefore, whenever the vorticity of v_α^a vanishes. A simple example of a tensor that meets these criteria is

$$K_{\alpha\beta}^{ab} = \sum_\gamma 2K_{\alpha\beta\gamma} \nabla^{[a} v_\gamma^{b]}, \tag{20}$$

where $K_{\alpha\beta\gamma}$ is symmetric in $\alpha\beta$. For the remainder of this paper we restrict our attention to this case with $K_{\alpha\beta\gamma}$ an arbitrary function of $s_{(n)}$ and ρ_μ. The forces given by Eqs. (19)–(20) are generalizations of those included by Bekarevich and Khalatnikov [11] in the non-dissipative limit of their equations. It is worth noting that these forces do not vanish when the coefficients λ^a_α are set to zero. In this case the forces become singular at the locations of vortices. By introducing λ^a_α we may interpret F^a_α as the average force the vortices exert on the average flow of the superfluids. These forces may also be considered, therefore, to be generalizations of the vortex elasticity forces introduced by Hall [17] and the "mutual friction" forces introduced by Hall and Vinen [18] and generalized to mixtures of superfluids by Onuki [19].

The main purpose of this paper is to determine the form of the electromagnetic coupling to these superfluid mixtures. This is most easily accomplished by introducing a Hamiltonian formulation of the equations. Holm and Kupershmidt [5] give such a formulation for the special case of the fluid equations presented above when $K_{\alpha\beta\gamma}$ and λ^a_α are zero. Here we generalize their work to include non-zero values for these quantities. The Hamiltonian density is taken to be the energy density of the fluid; thus, the Hamiltonian of the fluid, H, is given by

$$H=\int d^3x\, U. \tag{21}$$

It will be convenient (primarily for the discussion on charged fluids that follows) to relabel the fluid variables, v^a_e, v^a_α, and P^a when they occur in the Hamiltonian form of the equations:

$$u^a_e = v^a_e, \tag{22}$$

$$u^a_\alpha = v^a_\alpha, \tag{23}$$

$$\Upsilon^a = P^a. \tag{24}$$

Thus, using Eqs. (12) and (14), the variation in the Hamiltonian may be written as

$$\delta H=\int d^3x \left\{ T_e\, \delta s_e + (\mu_e - \tfrac{1}{2}|\mathbf{v}_{(n)}|^2)\, \delta\rho_e + \rho_e(u^b_e - v^b_{(n)})\, \delta u_{eb} + T_{(n)}\, \delta s_{(n)} \right.$$
$$\left. + \sum_\alpha (\mu_\alpha - \tfrac{1}{2}|\mathbf{v}_{(n)}|^2)\, \delta\rho_\alpha + \sum_\alpha [P^b_\alpha + (\nabla\times\boldsymbol{\lambda}_\alpha)^b]\, \delta u_{\alpha b} + v^b_{(n)}\, \delta\Upsilon_b \right\}. \tag{25}$$

In addition to the expressions for the Hamiltonian and its variation, the Hamiltonian formulation of the fluid equations must include the definition of a Poisson bracket. In terms of this bracket the evolution equations could then be expressed in the standard Hamiltonian form

$$\partial_t F = -[F, H], \tag{26}$$

where F is an arbitrary smooth function of the dynamical fluid variables (i.e., s_e, ρ_e, u_e^a, $s_{(n)}$, ρ_α, u_α^a, and Υ^a). It is reasonably straightforward to find a bracket through which Eq. (26) reproduces the fluid equations (4)–(11) and which satisfies the anti-symmetry property $[F, G] = -[G, F]$ for arbitrary smooth F and G. The following bracket satisfies these conditions

$$[F, G] = (F, G) - 2 \int d^3x \sum_{\alpha\beta\gamma} \frac{\delta F}{\delta u_\alpha^a} \frac{\delta G}{\delta u_\beta^b} K_{\alpha\beta\gamma} \nabla^{[a} u_\gamma^{b]}, \tag{27}$$

where (F, G) is defined by

$$\begin{aligned}(F, G) = \int d^3x \Bigg\{ & \frac{\delta F}{\delta s_e} \nabla^b \left[\frac{s_e}{\rho_e} \frac{\delta G}{\delta u_e^b} + s_e \frac{\delta G}{\delta \Upsilon^b} \right] + \frac{\delta F}{\delta \rho_e} \nabla^b \left[\frac{\delta G}{\delta u_e^b} + \rho_e \frac{\delta G}{\delta \Upsilon^b} \right] \\ & + \frac{\delta F}{\delta u_e^a} \left[\frac{s_e}{\rho_e} \nabla^a \frac{\delta G}{\delta s_e} + \nabla^a \frac{\delta G}{\delta \rho_e} - \frac{2}{\rho_e} \frac{\delta G}{\delta u_e^b} \nabla^{[a} u_e^{b]} + u_e^b \nabla^a \frac{\delta G}{\delta \Upsilon^b} + \frac{\delta G}{\delta \Upsilon^b} \nabla^b u_e^a \right] \\ & + \frac{\delta F}{\delta s_{(n)}} \nabla^b \left[s_{(n)} \frac{\delta G}{\delta \Upsilon^b} \right] + \sum_\alpha \frac{\delta F}{\delta \rho_\alpha} \nabla^b \left[\frac{\delta G}{\delta u_\alpha^b} + \rho_\alpha \frac{\delta G}{\delta \Upsilon^b} \right] \\ & + \sum_\alpha \frac{\delta F}{\delta u_\alpha^a} \left[\nabla^a \frac{\delta G}{\delta \rho_\alpha} + u_\alpha^b \nabla^a \frac{\delta G}{\delta \Upsilon^b} + \frac{\delta G}{\delta \Upsilon^b} \nabla^b u_\alpha^a \right] \\ & + \frac{\delta F}{\delta \Upsilon^a} \Bigg[s_e \nabla^a \frac{\delta G}{\delta s_e} + \rho_e \nabla^a \frac{\delta G}{\delta \rho_e} + \nabla^b \left(u_e^a \frac{\delta G}{\delta u_e^b} \right) - \frac{\delta G}{\delta u_e^b} \nabla^a u_e^b \\ & + \sum_\alpha \left(\rho_\alpha \nabla^a \frac{\delta G}{\delta \rho_\alpha} + \nabla^b \left[u_\alpha^a \frac{\delta G}{\delta u_\alpha^b} \right] - \frac{\delta G}{\delta u_\alpha^b} \nabla^a u_\alpha^b \right) \\ & + s_{(n)} \nabla^a \frac{\delta G}{\delta s_{(n)}} + \Upsilon^b \nabla^a \frac{\delta G}{\delta \Upsilon^b} + \nabla^b \left(\Upsilon^a \frac{\delta G}{\delta \Upsilon^b} \right) \Bigg] \Bigg\}. \end{aligned} \tag{28}$$

The representation of the bracket in Eq. (27) is non-canonical. To establish that it is a Poisson bracket, it is necessary to verify that the Jacobi "identity,"

$$[E, [F, G]] + [F, [G, E]] + [G, [E, F]] = 0, \tag{29}$$

is satisfied for arbitrary smooth E, F, and G. The bracket (F, G) is known to satisfy the Jacobi identity because it is the direct sum of the Poisson bracket for an ordinary perfect fluid [20] and the Poisson bracket for a mixture of superfluids given by Holm and Kupershmidt [5] (up to simple algebraic changes of variables). Thus, (F, G) is a Poisson bracket. In the Appendix we discuss how the Jacobi identity can be verified for brackets like $[F, G]$ that are constructed by adding terms to a Poisson bracket. We have carried out this computation and have determined that $[F, G]$ does, in fact, satisfy the Jacobi identity whenever $K_{\alpha\beta\gamma}$ satisfies the following conditions:

$$0 = s_{(n)} \frac{\partial K_{\alpha\beta\gamma}}{\partial s_{(n)}} + \sum_{\sigma} \rho_\sigma \frac{\partial K_{\alpha\beta\gamma}}{\partial \rho_\sigma} + K_{\alpha\beta\gamma}, \tag{30}$$

$$0 = \frac{\partial K_{\alpha\beta\gamma}}{\partial \rho_\mu} + \sum_{\sigma} K_{\mu\alpha\sigma} K_{\sigma\beta\gamma}, \tag{31}$$

$$0 = \sum_{\sigma} (K_{\mu\alpha\sigma} K_{\sigma\beta\gamma} - K_{\mu\beta\sigma} K_{\sigma\alpha\gamma}). \tag{32}$$

We have also shown that Eq. (30) is a necessary condition for Eq. (29) to hold. We suspect that Eqs. (31) and (32) are also necessary conditions, although we have been unable to prove this. That there exist non-trivial solutions to Eqs. (30)–(32) can be illustrated for the simple "diagonal" case in which $K_{\alpha\beta\gamma}$ is proportional to Kronecker deltas: $K_{\alpha\beta\gamma} \propto \delta_{\alpha\gamma}\delta_{\beta\gamma}$. In this case the general solution to Eqs. (30)–(32) is given by

$$K_{\alpha\beta\gamma} = \frac{\delta_{\alpha\gamma}\delta_{\beta\gamma}}{\rho_\alpha + \kappa_\alpha s_{(n)}}, \tag{33}$$

where the κ_α are arbitrary constants. We note that the case $\kappa_\alpha = 0$ is the generalization to mixtures of the "momentum representation" bracket introduced by Holm and Kupershmidt [21] for a single component "non-rotating" superfluid, written here in a somewhat different choice of variables.

III. The Electromagnetic-Superfluid Interaction

In this section we investigate the coupling of the charged components of the superfluid mixture to the electromagnetic field. The most efficient way to accomplish this coupling uses the procedure, based on the Hamiltonian formulation of the equations, that is discussed by Holm and Kupershmidt [5]. In this approach, modeled after the electromagnetic coupling to a charged particle, the physical momenta in the problem are replaced by their "canonical" counterparts (which are formed by adding terms to the physical momenta that are proportional to the electromagnetic vector potential). The Hamiltonian is rewritten in terms of these "canonical" variables, but its value (for a given fluid state) is modified only by the addition of the terms needed to describe the energy of the electromagnetic field. The Poisson bracket is unchanged (when written in terms of these "canonical" variables) except for the addition of the terms needed to describe the dynamics of the electromagnetic field itself. This procedure has been described as "minimal coupling," since it limits the electromagnetic-interaction terms in the fluid equations to those obtained by appropriately replacing the physical momenta of the uncharged theory with the corresponding "canonical" momenta. The fluid equations generated in this way interact with the electromagnetic field only through forces that are qualitatively similar to the Lorentz force. These equations guarantee that the appropriate conservation laws are satisfied identically.

The state of the superfluid mixture is described by the dynamical variables s_e, ρ_e, v_e^a, $s_{(n)}$, ρ_α, v_α^a, and P^a. Each of the vector fields among these dynamical variables is proportional to the physical momentum density of its corresponding constituent particles: P^a is the total momentum density of the fluid; v_α^a is proportional to the momentum density of the superfluid particles (or Cooper pairs) of species α (appropriately averaged when λ_α^a is non-zero); and v_e^a is proportional to the momentum density of the ordinary fluid. Thus, it is appropriate to define the following "canonical" variables:

$$\Upsilon^a = P^a + \left[a_e \rho_e + \sum_\alpha a_\alpha \rho_\alpha \right] A^a, \tag{34}$$

$$u_\alpha^a = v_\alpha^a + a_\alpha A^a, \tag{35}$$

$$u_e^a = v_e^a + a_e A^a, \tag{36}$$

where A^a is the electromagnetic vector potential and $a_\alpha = q_\alpha / m_\alpha c$ is the charge to mass ratio of species α divided by the speed of light [22].

The next step in this minimal coupling procedure is to add to the Hamiltonian the terms that describe the energy of the electromagnetic field. The standard macroscopic expression for the variation of this energy is

$$\mathbf{d}U_{\text{EM}} = \frac{1}{4\pi} \left[E^a \, \mathbf{d}D_a + H^a \, \mathbf{d}B_a \right]. \tag{37}$$

where D^a is the displacement, B^a is the magnetic induction $[B^a \equiv (\nabla \times \mathbf{A})^a]$, and where this expression serves as the definitions of E^a and H^a, the electric and magnetic fields. When vortices are present in the superfluid there exist microscopic electrical currents circulating about each vortex. The vortex averaged equations neglect these currents and the magnetic energy that is associated with them. While this energy could be included implicitly in the definition of H^a given in Eq. (37), we find it to be more convenient to include an expression for this additional energy explicitly:

$$\mathbf{d}U_{\text{EM}} = \frac{1}{4\pi} \left[E^a \, \mathbf{d}D_a + H^a \, \mathbf{d}B_a \right] + \sum_\alpha a_\alpha \lambda_\alpha^a \, \mathbf{d}B_a. \tag{38}$$

This additional energy term, when combined with the term $\lambda_\alpha^a \, \mathbf{d}\omega_{\alpha a}$ that was included in the expression for the fluid energy in Eq. (25), results in the following expresion for the internal energy associated with the vortices: $\lambda_\alpha^a \, \mathbf{d}(\omega_{\alpha a} + a_\alpha B_a)$. Since the vortex averaged version of the London equation (3) dictates that $\omega_\alpha^a + a_\alpha B^a$ is proportional to the density of vortices, it is appropriate that the variation of this combination yields the internal vortex energy λ_α^a. We note that the term, $a_\alpha \lambda_\alpha^a \, \mathbf{d}B_a$, is also consistent with the expression for the magnetic energy density of a flux line lattice as computed by de Gennes and Matricon [23]. When the additional terms in Eq. (38) are added to the expression for the variation of the fluid Hamiltonian in Eq. (25) and when the result is re-expressed in terms of the "canonical" variables of Eqs. (34)–(36), the following expression for the variation of the total Hamiltonian is obtained:

$$\begin{aligned}\delta H = \int d^3x \Big\{ & T_e \delta s_e + (\mu_e - \tfrac{1}{2}|\mathbf{v}_{(n)}|^2 - a_e v^b_{(n)} A_b)\, \delta\rho_e + \rho_e(u^b_e - a_e A^b - v^b_{(n)})\, \delta u_{eb} \\ & + T_{(n)}\, \delta s_{(n)} + \sum_\alpha (\mu_\alpha - \tfrac{1}{2}|\mathbf{v}_{(n)}|^2 - a_\alpha v^b_{(n)} A_b)\, \delta\rho_\alpha + v^b_{(n)}\, \delta\Upsilon_b \\ & + \sum_\alpha [P^b_\alpha + (\nabla\times\boldsymbol{\lambda}_\alpha)^b]\, \delta u_{\alpha b} + \frac{1}{4\pi} E^b\, \delta D_b + \left[\frac{1}{4\pi}(\nabla\times\mathbf{H})^b - \frac{1}{c} J^b\right] \delta A_b \Big\}, \end{aligned} \tag{39}$$

where the total macroscopic electrical current, J^a, is defined by

$$J^a = a_e c \rho_e v^a_e + \sum_\alpha a_\alpha c(\rho_\alpha v^a_{(n)} + P^a_\alpha). \tag{40}$$

The final step in this minimal-coupling procedure is to construct the appropriate Poisson bracket. By assumption, the fluid portions of the bracket are the same (when written in terms of the "canonical" variables) as the uncharged-superfluid bracket, $[F, G]$, of Eq. (27). One simply adds to $[F, G]$, the standard Poisson bracket for the electromagnetic field:

$$\{F, G\} = [F, G] + 4\pi c \int d^3x\, g^{ab} \left(\frac{\delta F}{\delta A^a} \frac{\delta G}{\delta D^b} - \frac{\delta F}{\delta D^a} \frac{\delta G}{\delta A^b} \right). \tag{41}$$

Since $\{F, G\}$ is defined as the direct sum of $[F, G]$ and the standard bracket for the electromagnetic field, it will satisfy the Jacobi identity whenever $[F, G]$ does.

Having specified the desired form of the Hamiltonian in Eq. (39), along with the bracket, $\{F, G\}$ in Eq. (41), the time evolution of any quantity is determined via Hamilton's equations:

$$\partial_t F = -\{F, H\}. \tag{42}$$

In particular, the time evolutions of the dynamical variables s_e, ρ_e, v^a_e, $s_{(n)}$, ρ_α, v^a_α, P^a, A^a, and D^a are determined. The resulting evolution equations for the mass and entropy densities are unchanged from their uncharged analogues, Eqs. (4)–(7) (i.e., the corresponding conservation laws). In contrast the total fluid momentum density, P^a, is no longer conserved as it was in Eq. (8). This quantity now evolves as

$$\partial_t P^a + \nabla_b \pi^{ab} = \sigma E^a + \left[\left(\frac{1}{c}\mathbf{J} + \sum_\alpha a_\alpha \nabla\times\boldsymbol{\lambda}_\alpha \right) \times \mathbf{B} \right]^a, \tag{43}$$

where P^a is the total fluid momentum density of Eq. (9), π^{ab} is the fluid stress tensor of Eq. (18), and the total electrical charge density σ is defined by

$$\sigma = a_e c\rho_e + \sum_\alpha a_\alpha c\rho_\alpha. \tag{44}$$

Note that the total fluid momentum responds to a Lorentz force in which the electric current includes contributions of the form $ca_\alpha \nabla \times \boldsymbol{\lambda}_\alpha$. This additional current can be associated with the microscopic motions of the particles of species α which circulate about the vortices. In effect $a_\alpha \boldsymbol{\lambda}_\alpha$ acts as a magnetization of the fluid. The momentum evolution equation (43) can also be written in a form in which the Lorentz force on its right side involves only the macroscopic current J^a. This is accomplished by redefining the stress tensor (including the pressure) in Eq. (18) by making the substitution $\boldsymbol{\omega}_\alpha \rightarrow \boldsymbol{\omega}_\alpha + a_\alpha \mathbf{B}$. Since the combination $\boldsymbol{\omega}_\alpha + a_\alpha \mathbf{B}$ represents the density of vortices, by Eq. (3), this re-expressed version of the stress tensor might be considered to be the more natural one.

The equations for the evolution of v^a_α and v^a_e, Eqs. (10) and (11), are also transformed by the addition of electromagnetic terms. In particular, the forces $\mathbf{F}_\alpha$ and $\mathbf{F}_e$ become

$$\begin{aligned} \mathbf{F}_\alpha = & -\mathbf{v}_\alpha \times \boldsymbol{\omega}_\alpha + ca_\alpha \mathbf{E} + \mathbf{v}_{(n)} \times (\boldsymbol{\omega}_\alpha + a_\alpha \mathbf{B}) \\ & + \sum_{\beta\gamma} K_{\alpha\beta\gamma}(\mathbf{P}_\beta + \nabla \times \boldsymbol{\lambda}_\beta) \times (\boldsymbol{\omega}_\gamma + a_\gamma \mathbf{B}) \end{aligned} \tag{45}$$

and

$$\mathbf{F}_e = a_e(c\mathbf{E} + \mathbf{v}_e \times \mathbf{B}). \tag{46}$$

While the ordinary fluid responds to the standard Lorentz force, the force on the superfluid is rather more complicated. For now let us observe only that this force is influenced profoundly by the presence of the new vorticity preserving forces that are proportional to $K_{\alpha\beta\gamma}$. We note that this force reduces to the expression given by Holm and Kupershmidt [5] for the case $K_{\alpha\beta\gamma} = \lambda^a_\alpha = 0$. We also note that this force agrees with that of Vardanyan and Sedrakyan [4] only when the vortex-free London equation, $\boldsymbol{\omega}_\alpha + a_\alpha \mathbf{B} = 0$, is satisfied or when $K_{\alpha\beta\gamma}$ has the value given in Eq. (57) and the mass density tensor, $\rho_{\alpha\beta}$ of Eq. (13), is diagonal. Their equations do not appear to be consistent with energy conservation under any other circumstances.

The Hamiltonian formalism also produces the equations for the evolution of the dynamical electromagnetic fields, A^a and D^a. These are simply the time-dependent Maxwell equations,

$$\partial_t A^a = -cE^a \tag{47}$$

and

$$\partial_t D^a = c(\nabla \times \mathbf{H})^a - 4\pi J^a. \tag{48}$$

The divergence of Eq. (48) guarantees that the constraint equation,

$$\nabla_a D^a = 4\pi\sigma, \tag{49}$$

is preserved as the fluid evolves. We note that the definition of the magnetic field H^a given in Eq. (38) allows the equations for electromagnetic fields to be written in a form that includes only the sources, σ and J^a, associated with the macroscopic motions of the fluid.

It is enlightening to consider the special case of a simple fluid of point charges for which the electromagnetic energy density assumes the form $U_{\mathrm{EM}} = (\mathbf{D}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{B})/8\pi$. In this case the electromagnetic fields are related by the expressions

$$E^a = D^a \tag{50}$$

and

$$H^a = B^a - 4\pi \sum_\alpha a_\alpha \lambda_\alpha^a. \tag{51}$$

For this case the Maxwell equation (48) can be re-expressed in terms of E^a and B^a:

$$\partial_t E^a = c(\nabla\times\mathbf{B})^a - 4\pi J^a - 4\pi c \sum_\alpha a_\alpha(\nabla\times\boldsymbol{\lambda}_\alpha)^a. \tag{52}$$

We note that the current source in this form of the equation (including the contributions $c a_\alpha \nabla\times\boldsymbol{\lambda}_\alpha$ from the microscopic vortices) is the one that appears in the expression for the Lorentz force that acts on the total momentum of the fluid in Eq. (43).

Hamilton's equations, (42), determine the evolution of all physical quantities, including the total momentum and energy of the combined superfluid–electromagnetic system. We note that the equation for the evolution of the momentum, Eq. (43), can be rewritten in a form that makes the conservation of the total momentum self-evident:

$$0 = \partial_t\left[P^a + \frac{1}{4\pi c}(\mathbf{D}\times\mathbf{B})^a\right] + \nabla_b\left\{\pi^{ab} - U_{\mathrm{EM}}\, g^{ab} - (g^{ac}g^{bd} - g^{ab}g^{cd})\left[\frac{1}{4\pi}(E_c D_d + H_c B_d) + \sum_\alpha a_\alpha \lambda_{\alpha c} B_d\right]\right\}. \tag{53}$$

Similarly, the equation for the evolution of the total energy of the system can be written in the form of a conservation law:

$$0 = \partial_t(U + U_{\mathrm{EM}}) + \nabla_b\left[U^b + \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})^b + c\sum_\alpha a_\alpha(\mathbf{E}\times\boldsymbol{\lambda}_\alpha)^b\right]. \tag{54}$$

That these conservation laws are satisfied is unrelated to the issue of whether the bracket $\{F, G\}$ in Eq. (41) satisfies the Jacobi identity or not. Thus, the evolution equations presented here guarantee momentum and energy conservation for any values of $K_{\alpha\beta\gamma}$ even if they cannot be expressed as a rigorous Hamiltonian system. For simple fluid systems in which the electromagnetic fields satisfy Eqs. (50)–(51), the expressions for the electromagnetic energy flux, stress tensor, and Poynting vector in these conservation laws reduce to their standard forms in terms of E^a and B^a [24]. In this special case the total stress tensor is symmetric whenever the fluid stress tensor π^{ab} is symmetric.

Up to this point we have conducted our discussion of the electromagnetic coupling to mixtures of superfluids without restricting the values of the coefficients $K_{\alpha\beta\gamma}$ that appear in the superfluid force, Eq. (45). Ultimately, the choice of these coefficients must be based on experimental criteria, or at the very least on a microscopic model of the material. We conclude this section by suggesting a natural choice for the $K_{\alpha\beta\gamma}$, from the viewpoint of the macroscopic analysis developed here. Since these coefficients participate in the superfluid–electromagnetic coupling, we choose them to ensure that this coupling has the form of an appropriate Lorentz force. In analogy with the force that acts on the total momentum of the fluid, Eq. (43), it seems natural to require that $\mathbf{F}_\alpha$ take the form of the Lorentz force with an electric current arising from the macroscopic flow of the superfluid condensate plus the microscopic circulation of these particles about the vortices. The macroscopic current associated with the particles of species α that have condensed into the superfluid state is given by

$$\mathbf{J}_\alpha^{(s)} = ca_\alpha(\rho_\alpha^{(s)}\mathbf{v}_{(n)} + \mathbf{P}_\alpha), \tag{55}$$

where the superfluid component of the mass density is denoted $\rho_\alpha^{(s)}$. The superfluid force equation will reduce to the desired form,

$$\mathbf{F}_\alpha = \frac{1}{\rho_\alpha^{(s)}} [\rho_\alpha^{(s)}(\mathbf{v}_{(n)} - \mathbf{v}_\alpha) + \mathbf{P}_\alpha + \nabla \times \boldsymbol{\lambda}_\alpha) \times \boldsymbol{\omega}_\alpha + ca_\alpha \mathbf{E} + \frac{1}{c\rho_\alpha^{(s)}} (\mathbf{J}_\alpha^{(s)} + ca_\alpha \nabla \times \boldsymbol{\lambda}_\alpha) \times \mathbf{B}, \tag{56}$$

when $K_{\alpha\beta\gamma}$ is chosen as

$$K_{\alpha\beta\gamma} = \frac{1}{\rho_\alpha^{(s)}} \delta_{\alpha\gamma} \delta_{\beta\gamma}. \tag{57}$$

This choice generalizes to mixtures of charged superfluids the interaction proposed by Volovik and Dotsenko [25] and Khalatnikov and Lebedev [26]. It is interesting to note that the argument given for selecting this interaction was different in each of these investigations. We chose this interaction so that the superfluid velocity would respond to an appropriate Lorentz force law. The macroscopic analysis of the uncharged fluid equations by Khalatnikov and Levedev [26] chose an interaction equivalent to Eq. (57) so that the force on the superfluid velocity would be independent of the normal fluid velocity. In contrast, Volovik and Dotsenko [25] base their choice of this interaction on a microscopic analysis of the fluid equations and the underlying dynamics of the fluid vortices (see also Rasetti and Regge [27]). We also note that the choice of $K_{\alpha\beta\gamma}$ in Eq. (57) does not in general have the form, Eq. (33), needed for the bracket to satisfy the Jacobi identity. Except for the case of a zero-temperature fluid (when $s_{(n)}=0$ and $\rho_\alpha^{(s)}=\rho_\alpha$) these fluid equations do not, therefore, have a rigorous Hamiltonian formulation. Volovik and Dotsenko [25] indicate, however, that a Hamiltonian formulation can be recovered (even for non-zero temperatures) if the superfluid component of the mass density is treated as an independent dynamical variable. While this may be appropriate under certain circumstances (e.g., near the superfluid transition temperature [28]), we have taken the more traditional approach of assuming that $\rho_\alpha^{(s)}$ is given in terms of the dynamical variables by an appropriate equation of state.

The superfluid force, Eq. (56), can be re-expressed in a form that illustrates more clearly the physical significance of its various terms. Let us consider the case when the superfluid mass currents $\mathbf{P}_\alpha$ are given by the Andreev and Bashkin [3] expression, Eq. (13). If we define the density of normal particles of species α as the coefficient of $\mathbf{v}_{(n)}$ in the mass current, then the superfluid density $\rho_\alpha^{(s)}$ is related to the mass density matrix $\rho_{\alpha\beta}$ by,

$$\rho_\alpha^{(s)}=\sum_\beta \rho_{\alpha\beta}. \tag{58}$$

Using this relation, the superfluid force can be written as

$$\begin{aligned}\rho_\alpha^{(s)}\mathbf{F}_\alpha=&\sum_\beta \rho_{\alpha\beta}(\mathbf{v}_\beta-\mathbf{v}_\alpha)\times\boldsymbol{\omega}_\alpha+(\nabla\times\boldsymbol{\lambda}_\alpha)\times(\boldsymbol{\omega}_\alpha+a_\alpha\mathbf{B})\\&+ca_\alpha\rho_\alpha^{(s)}\mathbf{E}+\frac{1}{c}\,\mathbf{J}_\alpha^{(s)}\times\mathbf{B}.\end{aligned} \tag{59}$$

In this expression, the first term is a mutual drag force between the flow of one species of superfluid past another. This term vanishes if the mass density matrix is diagonal. The next term is analogous to the force, introduced by Hall [17], to describe the elasticity of the vortices. When the vortex energy $\boldsymbol{\lambda}_\alpha$ is taken to be proportional to $\boldsymbol{\omega}_\alpha+a_\alpha\mathbf{B}$, this term is the natural charged-fluid generalization of Bekarevich and Khalatnikov's [11] expression for this force. The last terms are the Lorentz force with an electric current produced by the macroscopic flow of superfluid particles. We note that the current that appears in this Lorentz force contains Fermi-liquid effects (by way of the mass density matrix $\rho_{\alpha\beta}$) analogous to those computed by Easson and Pethick [29] to describe non-superfluid protons in a neutron-star interior. This expression also reduces to the standard Lorentz force when there is only a single component charged superfluid [30].

The expression, Eq. (57), for the coefficients $K_{\alpha\beta\gamma}$ could be generalized slightly by setting

$$K_{\alpha\beta\gamma} = \left(\frac{1}{\rho_\alpha^{(s)}} - \beta'_\alpha\right) \delta_{\alpha\gamma} \delta_{\beta\gamma}. \tag{60}$$

The functions β'_α correspond to the Hall and Vinen mutual friction coefficient β' discussed by Bekerevich and Khalatnikov [11]. Abrikosov, Kemoklidze, and Khalatnikov [31] argue, in agreement with our macroscopic analysis, that this coefficient must be set to zero for charged species in order for the theory to predict what they considered to be the correct dispersion relation for helicon waves. The theory of Nozières and Vinen [32] obtains the same dispersion relation for these waves while Bardeen and Stephen [33] obtain a more general dispersion relation (implying a non-zero value for β'). Jones [34] has argued that the theory of Nozières and Vinen [32] is more applicable to the proton superfluid of neutron-star interiors and implicitly that β'_α should be zero for the case of primary interest to us. For uncharged species the situation is less clear. Experimental measurements on He II show that β'_α is small ($\rho_\alpha^{(s)}\beta'_\alpha \ll 1$) but not zero for that system [35]. Nevertheless setting this coefficient to zero appears to be a good approximation, especially for temperatures far below the critical temperature. The coefficients β'_α must vanish (for charged or uncharged species) when the temperature vanishes; otherwise the superfluid force, Eq. (45), would contain terms proportional to the normal velocity, $\mathbf{v}_{(n)}$, which is not well defined in this limit. A more complete discussion of these mutual friction effects can be found in Sonin [8].

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APPENDIX: Modifying Poisson Brackets

In this appendix we investigate the conditions under which terms can be added to a Poisson bracket and have the result continue to satisfy the Jacobi identity. Consider the Poisson bracket,

$$(F, G) = \int d^3x \, \frac{\delta F}{\delta z^A} \left[X^{AB} \left(\frac{\delta G}{\delta z^B} \right) \right]. \tag{61}$$

In this expression F and G are arbitrary functions of the fields z^A, and the index A runs over the complete collection of these fields. The operator X^{AB} acts to the right (on the argument in parenthesis) and is anti-symmetric in the sense that

$(F, G) = -(G, F)$. Summation over repeated indices A, B, C, etc. is assumed. We also assume that this bracket satisfies the Jacobi identity,

$$0 = \sum_{(EFG)} (E, (F, G)), \tag{62}$$

where the indicated sum is to be performed by adding the terms obtained by cyclically permuting the arbitrary smooth functions E, F, and G.

Next we modify the bracket (F, G) as

$$[F, G] = (F, G) + \int d^3x \, \frac{\delta F}{\delta z^A} \left[Y^{AB} \left(\frac{\delta G}{\delta z^B} \right) \right]. \tag{63}$$

The operator Y^{AB} is also anti-symmetric so that $[F, G] = -[G, F]$. We wish to investigate the conditions under which the combined bracket, $[F, G]$, satisfies the Jacobi identity. A straightforward calculation yields the identity:

$$\sum_{(EFG)} [E, [F, G]] = -\sum_{(EFG)} \iint d^3x \, d^3x' \left\{ \left[Y^{AB} \left(\frac{\delta E}{\delta z^B} \right) \right] \frac{\delta F}{\delta z^C} \left[\frac{\delta X^{CD}}{\delta z^A} \left(\frac{\delta G}{\delta z^D} \right) \right] + \left[(X^{AB} + Y^{AB}) \left(\frac{\delta E}{\delta z^B} \right) \right] \frac{\delta F}{\delta z^C} \left[\frac{\delta Y^{CD}}{\delta z^A} \left(\frac{\delta G}{\delta z^D} \right) \right] \right\}. \tag{64}$$

The combined bracket, $[F, G]$, satisfies the Jacobi identity if and only if this expression vanishes for all smooth E, F, and G.

In general, it requires an extremely tedious calculation to determine whether or not the Jacobi identity is satisfied for a given bracket. In some cases, however, one is interested in determining whether a relatively simple addition to a given Poisson bracket results in a bracket that also satisfies the Jacobi identity. In that case, Eq. (64) provides a relatively simple method of checking the Jacobi identity. When the operator X^{AB} is rather complicated compared to Y^{AB}, Eq. (64) is reasonably easy to evaluate. It involves only the "cross terms" between these operators plus terms that are quadratic in Y^{AB}. In contrast it is necessary to evaluate all of the terms that are quadratic in X^{AB} in order to verify the Jacobi identity for the original bracket.

The Poisson bracket for a mixture of superfluids falls into this general category. The bracket (F, G) defined in Eq. (28) is quite complicated but it is known to satisfy the Jacobi identity (based on Lie-algebra arguments). The combined bracket $[F, G]$ of Eq. (27) involving $K_{\alpha\beta\gamma}$ is a relatively trivial modification of the original bracket. Thus Eq. (64) provides a relatively simple way to check whether the Jacobi identity is satisfied. We have evaluated this identity for the brackets defined in Eqs. (27) and (28) with the result,

$$
\begin{aligned}
\sum_{(EFG)} & [E,[F,G]] \\
&= \sum_{(EFG)} \sum_{\mu\nu} \int d^3x \left\{ 2 \sum_{\gamma} \left[s_{(n)} \frac{\partial K_{\mu\nu\gamma}}{\partial s_{(n)}} + \sum_{\alpha} \rho_\alpha \frac{\partial K_{\mu\nu\gamma}}{\partial \rho_\alpha} + K_{\mu\nu\gamma} \right] \nabla^{[c} u_\gamma^{d]} \frac{\delta E}{\delta u_\mu^c} \frac{\delta F}{\delta u_\nu^d} \nabla^b \frac{\delta G}{\delta \Upsilon^b} \right. \\
&\quad + 2 \sum_{\alpha\beta\gamma} \left[\delta_{\alpha\beta} \frac{\partial K_{\mu\nu\gamma}}{\partial \rho_\alpha} + K_{\beta\mu\alpha} K_{\alpha\nu\gamma} \right] \nabla^{[c} u_\gamma^{d]} \frac{\delta E}{\delta u_\mu^c} \frac{\delta F}{\delta u_\nu^d} \nabla^b \frac{\delta G}{\delta u_\beta^b} \\
&\quad + 2 \sum_{\alpha\beta\gamma} [K_{\nu\beta\alpha} K_{\alpha\mu\gamma} - K_{\nu\mu\alpha} K_{\alpha\beta\gamma}] \nabla^{[c} u_\gamma^{d]} \frac{\delta E}{\delta u_\mu^d} \frac{\delta F}{\delta u_\nu^b} \nabla^b \frac{\delta G}{\delta u_\beta^c} \\
&\quad + 2 \sum_{\alpha\beta\gamma} [K_{\alpha\nu\gamma} \nabla^b K_{\beta\mu\alpha} - K_{\beta\nu\alpha} \nabla^b K_{\alpha\mu\gamma}] \nabla^{[c} u_\gamma^{d]} \frac{\delta E}{\delta u_\mu^c} \frac{\delta F}{\delta u_\nu^d} \frac{\delta G}{\delta u_\beta^b} \\
&\quad \left. + \sum_{\alpha\beta\gamma} [K_{\mu\nu\alpha} K_{\alpha\beta\gamma} - K_{\mu\beta\alpha} K_{\alpha\nu\gamma}] \nabla^c \nabla^d u_\gamma^b \frac{\delta E}{\delta u_\mu^c} \frac{\delta F}{\delta u_\nu^d} \frac{\delta G}{\delta u_\beta^b} \right\}. \qquad (65)
\end{aligned}
$$

This expression clearly vanishes if the coefficient of each term in the integrand vanishes separately. Four conditions are sufficient to guarantee that the integral vanishes for all smooth E, F, and G: the three conditions in Eqs. (30)–(32) plus the additional condition,

$$
0 = \sum_{\alpha} (K_{\alpha\nu\gamma} \nabla^a K_{\beta\mu\alpha} - K_{\beta\nu\alpha} \nabla^a K_{\alpha\mu\gamma}). \qquad (66)
$$

Since Eq. (66) is a consequence of Eqs. (30)–(32), it is not an independent condition. Only the first term in the integral in Eq. (65) involves the variations with respect to Υ^a. Since the entire integral must vanish for arbitrary smooth functions E, F, and G, it follows that the coefficient of this first term must vanish separately. Thus Eq. (30) is a necessary condition. All of the remaining terms involve only variations with respect to u_α^a. We have performed numerous integrations by parts of the expression given here, but have not been able to produce a form of this equation from which the remaining necessary conditions may easily be extracted. The weakest conditions that we have found, however, are Eqs. (30)–(32); and we suspect that they are in fact the necessary conditions.

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