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THE OSCILLATIONS OF RAPIDLY ROTATING NEWTONIAN STELLAR MODELS

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ABSTRACT

A method is described for solving the equations that govern the oscillations of rapidly rotating inhomogeneous Newtonian stellar models. A covariant reformulation of the general pulsation equations is presented which reduces them to a single scalar equation for a single potential. From this potential all of the properties of the normal mode may be deduced. The techniques developed to solve this complicated eigenvalue problem numerically are described, and solutions to this equation are presented for the $l = m$ f-modes of rigidly rotating stellar models having polytropic equations of state.

Subject headings: hydrodynamics — stars: interiors — stars: pulsation — stars: rotation

I. INTRODUCTION

An astrophysical problem of fundamental importance whose general solution has not been understood up to now is that of solving the equations that govern the normal-mode oscillations of rapidly rotating, inhomogeneous Newtonian stellar models. While these equations have been solved analytically for the special case of uniform-density stellar models (Bryan 1889; Chandrasekhar 1969), the normal-mode equations for more realistic, inhomogeneous models must be solved numerically. (Indeed, even the equilibrium equations for rapidly rotating inhomogeneous stellar models must be solved numerically.) Numerical solutions to the general pulsation equations have been obtained to date only in the special case of the axisymmetric modes (Clement 1981). In this paper we describe how this problem may be solved in general, and we present the numerical solutions of these equations for the $l = m$ f-modes of rapidly rotating stellar models having polytropic equations of state.

The traditional way of representing the equations that describe the pulsations of rapidly rotating stellar models is in fact a significant obstacle to finding solutions. The dynamical variable in the standard representation of the equations is the Lagrangian displacement vector ξ^a. The standard equations are a set of three second-order partial differential equations for ξ^a and in addition the second-order (Poisson's) equation for the perturbed gravitational potential $\delta\Phi$. Together these equations constitute an eighth-order system for the four dependent variables (the three components of ξ^a and $\delta\Phi$). The problem of attempting to solve this system directly (even numerically) is formidable. To our knowledge, Clement's (1981) analysis of the axisymmetric modes is the only direct solution to have been made of the general case of these equations. Instead, progress has been made in the development of indirect methods of estimating the frequencies of these modes. Variational principles have been developed (Clement 1964; Lynden-Bell and Ostriker 1967; Friedman and Schutz 1978; Ipser and Managan 1985; Managan 1985) which can be used to estimate the normal-mode eigenfrequencies. The frequencies for an assortment of the f- and lower lying p-modes having various (small) values of the spherical harmonic indices l and m have been determined by Managan (1986) using these variational techniques.

A need exists, nevertheless, to develop the techniques that will allow the complete exact solution to the equations for the pulsations of rapidly rotating stars. In a variety of astrophysical contexts a detailed knowledge of the frequencies *and* the eigenfunctions describing the fluid motions in a mode is desired. For example, when studying the millisecond pulsars (Backer *et al.* 1982; Fruchter, Stinebring, and Taylor 1988; Kristian *et al.* 1989), one needs to know the normal-mode eigenfunctions, as well as the exact eigenfrequencies, in order to assess the effects of gravitational radiation reaction and of viscous dissipation on the evolution and stability of the star. Also, in the case of objects like the pulsating X-ray sources (see Joss and Rappaport 1984 and references therein) it may be important to know these eigenfunctions in order to understand the interactions between the star and the surrounding accretion disk. It is not possible to estimate these eigenfunctions accurately using the variational principle techniques that have been developed to date. Furthermore, without a knowledge of the exact frequencies for at least some rapidly rotating inhomogeneous Newtonian stellar models it is difficult to evaluate the accuracy of the frequencies computed with the variational principle techniques.

Our main purpose in this paper is to describe in detail a method for solving the general normal-mode equations of rapidly rotating Newtonian stellar models. This method, already described briefly in Ipser and Lindblom (1989), is based on a reformulation of the equations of stellar pulsation (Ipser and Managan 1985; Managan 1985) in terms of a single scalar potential, δU, from which all the other properties of the normal mode (including the Lagrangian displacement ξ^a) can be deduced. The reformulation yields a relatively simple fourth-order system of equations (either a single fourth-order equation for δU alone or a pair of second-order equations for δU and $\delta\Phi$) whose numerical solution (for both the frequencies and the eigenfunctions) is relatively straightforward and efficient.

Our discussion in this paper, the first in a series on the oscillations of rapidly rotating Newtonian stellar models, will proceed according to the following outline. In § II the basic properties of rapidly rotating Newtonian stellar models are reviewed, and the equations for the oscillation of these stars are reformulated in terms of a single scalar potential, δU. The discussion in this section differs from earlier treatments by presenting the equations in a covariant form. In § III the details of the numerical methods that we employ to solve the oscillation equations are presented. In particular, the angular differencing scheme (the differential version of Gaussian quadrature) that we use is presented. In § IV results are presented of our numerical evaluation of the $l = m$ f-modes of uniformly rotating stellar models based on polytropic equations of state.

In a series of future papers we plan to use and extend the methods described in this paper to study other aspects of the oscillations of rapidly rotating Newtonian stellar models. In particular we plan to study the effects of dissipation (both gravitational radiation reaction and viscosity) on the normal modes; we plan to study the properties of the $l = -m$ f-modes and the possibility of a viscosity-driven secular instability in these modes (as exists in the uniform-density Maclaurin spheroids; cf. Roberts and Stewartson 1963); we plan to investigate the dipole ($l = 1$) p_1-modes and determine whether a viscous secular instability (as suggested by Lindblom and Splinter 1989) can exist in these modes; we plan to investigate the sensitivity of the frequencies and eigenfunctions of these modes to more realistic neutron matter equations of state; and we plan to calibrate the accuracy with which the variational principles are capable of computing the frequencies and eigenfunctions of these modes.

II. THE FUNDAMENTAL EQUATIONS

a) Introduction

In this paper we restrict our attention to the study of stars composed of material that can be described as a fluid in which the effects of dissipation (i.e., viscous and thermal conduction) are sufficiently small that they can be neglected in the first approximation. For simplicity we also restrict our attention to stars in which Newtonian physics is adequate to describe the dynamics of the stellar fluid (i.e., characteristic velocities including the sound speed must be small compared with the speed of light), and to stars in which Newtonian gravitation theory is an adequate description (i.e., the gravitational fields must be suitably weak and gravitational radiation reaction effects must be negligible). Under these assumptions the equations which describe the dynamical evolution of an arbitrary state of the star are the standard Newtonian fluid equations:

$$\partial_t \rho + \nabla_a(\rho v^a) = 0, \tag{1}$$

$$\partial_t v^a + v^b \nabla_b v^a = -\nabla^a(h - \Phi) \equiv -\nabla^a U , \tag{2}$$

$$\nabla^a \nabla_a \Phi = -4\pi G \rho , \tag{3}$$

where h is defined by the integral

$$h(p) = \int_0^p \frac{dp'}{\rho(p')} . \tag{4}$$

The quantities ρ and p are the mass density and pressure of the fluid, which are assumed to satisfy a barotropic equation of state, $\rho = \rho(p)$; v^a is the velocity of the fluid; Φ is the gravitational potential; G is Newton's constant. The derivative operator ∂_t is the partial derivative with respect to time, while ∇_a is the standard Euclidean covariant derivative (i.e., partial derivatives in Cartesian coordinates). Tensor indices are raised and lowered with the Euclidean metric g_{ab} (the identity matrix in Cartesian coordinates) and its inverse g^{ab}. The potential $U = h - \Phi$ defined in equation (2) will be useful in our later discussion.

b) The Equilibrium Stellar Models

We consider in this paper the small linear oscillations of stellar models about their equilibrium configurations. Such equilibria are the time-independent solutions of the fluid equations (1)–(4). We limit our considerations here to the case of axisymmetric equilibrium states in which the fluid velocity has the form $v^a = \varpi^2 \Omega(\varpi) \nabla^a \varphi \equiv \Omega \phi^a$. Throughout this paper we will refer to the standard spherical coordinates as r, θ, and φ (with the polar axis aligned with the rotation axis of the star); the cylindrical radial coordinate is $\varpi = r \sin\theta$. The vector ϕ^a introduced here satisfies Killing's equation: $0 = \nabla_a \phi_b + \nabla_b \phi_a$. The angular velocity of the fluid, Ω, must (as an integrability condition for the time-independent eq. [2]) depend only on ϖ. Under these assumptions the equations which govern the structure of an equilibrium stellar model reduce to the following:

$$c = h(p) - \Phi - \Psi(\varpi) , \tag{5}$$

$$\nabla^a \nabla_a \Phi = -4\pi G \rho(p) , \tag{6}$$

where

$$\psi(\varpi) = \int_0^\varpi \Omega^2(\varpi')\varpi' \, d\varpi' ; \tag{7}$$

equation (5) holds in the interior of the star, and c is a constant. These equations are solved iteratively using well-understood numerical techniques (Stoeckly 1965). We comment here on only one nuance of this procedure: we describe a method for computing a sequence of rotating stellar models of *fixed* mass but having a range of angular velocities.

Equations (5) and (6) are solved by first fixing the desired angular velocity distribution of the star, $\Omega(\varpi)$, and then choosing some initial estimates for the pressure p_0, the gravitational potential Φ_0, and the constant c_0 (for example, by using the structure of a nonrotating star, or perhaps a previously determined structure of a slightly less rapidly rotating star). These estimates are improved to the refined values $p_0 + \delta p$, $\Phi_0 + \delta\Phi$, and $c_0 + \delta c$ by solving the following linear equations for $\delta\Phi + \delta c$ and then δp:

$$\nabla^a\nabla_a(\delta\Phi + \delta c) + 4\pi G\rho_0\left(\frac{d\rho}{dp}\right)_0(\delta\Phi + \delta c) = -\nabla^a\nabla_a\Phi_0 - 4\pi G\rho_0\left[1 + \left(\frac{d\rho}{dp}\right)_0(\Phi_0 + c_0 - h_0 + \Psi)\right], \tag{8}$$

$$\delta p = \rho_0(\delta\Phi + \delta c) + \rho_0(\Phi_0 + c_0 - h_0 + \Psi) . \tag{9}$$

These equations do not determine the value of the constant δc. It is customary simply to set $\delta c = 0$. This effectively fixes the value of the gravitational potential at the intersection of the rotation axis and the surface of the star. This choice is not necessary, however, and it may be considered more desirable to fix some other physical quantity along the sequence of stellar models being constructed. We find it convenient to fix the total mass of the star. For this choice δc can be determined by requiring that the monopole component of the revision in the gravitational potential vanish. This condition can be enforced by computing δc with the simple integral formula in terms of the known potential $\delta\Phi + \delta c$:

$$\delta c = \frac{1}{4\pi}\int_{r=R}(\delta\Phi + \delta c)\sin\theta\, d\theta\, d\varphi . \tag{10}$$

This integral may be performed over the surface of any sphere, $r = R$, which lies entirely outside the support of the fluid. Once the choice of δc is made, the estimate of the equilibrium solution is updated to $p_0 + \delta p$, $\Phi_0 + \delta\Phi$, and $c_0 + \delta c$. This procedure is iterated until the resulting changes δp, $\delta\Phi$, and δc become sufficiently small and equations (5) and (6) are satisfied to the desired degree of accuracy.

c) *The Pulsation Equations*

The equations for the evolution of small oscillations of a star are deduced by linearizing equations (1)–(3) about an equilibrium stellar model (which satisfies equations [5]–[7]). We will denote the (Eulerian) perturbation in a quantity q by the notation δq. In the equations that follow, any quantity not prefaced by δ is assumed to be the value of that quantity evaluated in the equilibrium model. We assume that all of the perturbated quantities have sinusoidal dependence in the coordinates t and φ: $\delta q = \delta q(r, \theta)e^{i\omega t + im\varphi}$, where ω is the frequency of the mode and m is an integer. We begin by linearizing Euler's equation (2) about an arbitrary equilibrium state; the resulting equation is given by

$$iQ_{ab}^{-1}\delta v^b \equiv [i(\omega + m\Omega)g_{ab} + 2\nabla_b v_a - \phi_a\nabla_b\Omega]\delta v^b = -\nabla_a\delta U . \tag{11}$$

The tensor Q_{ab}^{-1} can (generically) be inverted to obtain an expression for the velocity perturbation in terms of the potential δU:

$$\delta v^a = iQ^{ab}\nabla_b\delta U . \tag{12}$$

The following expression for Q^{ab} can be obtained by inverting the expression for Q_{ab}^{-1} given in equation (11):

$$Q^{ab} = \frac{1}{\omega + m\Omega}\left[\lambda g^{ab} + (1 - \lambda)z^a z^b + \frac{i\lambda}{\omega + m\Omega}(\phi^a\nabla^b\Omega - 2\Omega\nabla^a\phi^b)\right] . \tag{13}$$

In this expression z^a is the unit Cartesian vector field parallel to the rotation axis of the star; ϕ^a is a Killing vector proportional to the equilibrium velocity of the star, $v^a = \Omega\phi^a$ (which satisfies $\nabla_a\phi_b = \epsilon_{abc}z^c$, with ϵ_{abc} the totally antisymmetric tensor); and $\lambda = (\omega + m\Omega)^2/[(\omega + m\Omega)^2 - 4\Omega^2 - \varpi\partial_\varpi\Omega^2]$. The necessary and sufficient conditions that Q_{ab}^{-1} have an inverse are that $\lambda^{-1} \neq 0$ and $\omega + m\Omega \neq 0$. This form of the equations will be nonsingular, therefore, except at isolated values of the angular velocity. For the $l = m$ modes studied in this paper, the equations were found to be nonsingular for all values of the angular velocity in rigidity rotating polytropes. We note that in the case of real frequencies, ω, and uniform rotation, $\nabla_a\Omega = 0$, the tensor Q^{ab} is Hermitian, $Q^{ab} = Q^{*ba}$, and covariantly constant, $0 = \nabla_c Q^{ab}$. The Lagrangian displacement vector ξ^a is related to the velocity perturbation (for perturbations with t and φ dependence $e^{i\omega t + im\varphi}$) by the expression

$$\xi_a = -i\left[\frac{g_{ab}}{\omega + m\Omega} - \frac{i\phi_a\nabla_b\Omega}{(\omega + m\Omega)^2}\right]\delta v^b . \tag{14}$$

Thus the Lagrangian displacement (and consequently all of the other fluid perturbation quantities) can be expressed in terms of the potential δU by making use of equations (12) and (13):

$$\xi^a = \Xi^{ab}\nabla_b\delta U \equiv \frac{1}{(\omega + m\Omega)^2}\left[\lambda g^{ab} + (1 - \lambda)z^a z^b - \frac{2i\lambda\Omega}{\omega + m\Omega}\nabla^a\phi^b - \frac{\lambda\partial_\varpi\Omega^2}{\varpi(\omega + m\Omega)^2}\phi^a\phi^b\right]\nabla_b\delta U . \tag{15}$$

Having reduced the perturbed Euler equation to the form given in equation (12), we turn now to the perturbed versions of the mass conservation law (eq. [1]) and the perturbed gravitational potential equation (eq. [3]):

$$0 = \nabla_a(\rho Q^{ab}\nabla_b\delta U) + (\omega + m\Omega)\rho\frac{d\rho}{dp}(\delta U + \delta\Phi) , \tag{16}$$

$$0 = \nabla^a\nabla_a\delta\Phi + 4\pi G\rho\frac{d\rho}{dp}(\delta U + \delta\Phi) . \tag{17}$$

These expressions have been simplified by using equation (12) to eliminate the velocity perturbations, δv^a; and the perturbed mass density has been replaced by the expression $\delta\rho = \rho(d\rho/dp)(\delta U + \delta\Phi)$ in terms of the potentials δU and $\delta\Phi$. Equations (16) and (17) are the master equations that determine the properties of the oscillations of rapidly rotating Newtonian stellar models. They are a fourth-order system of partial differential equations for the two potentials δU and $\delta\Phi$. These are real equations for the real functions $\delta U(r, \theta)$ and $\delta\Phi(r, \theta)$ when the perturbed potentials have the assumed form $\delta U = \delta U(r, \theta)e^{i\omega t + im\varphi}$ and $\delta\Phi = \delta\Phi(r, \theta)e^{i\omega t + im\varphi}$. Once appropriate boundary conditions have been specified, these equations form an eigenvalue problem with the frequency of the mode, ω, acting as the eigenvalue. The first derivation of these equations for the general case of interest here (for inhomogeneous differentially rotating stars) was given by Ipser and Managan (1985) and Managan (1985). The potential δU had also been introduced by Poincaré (1885), who derived the special case of equation (16) for stars of uniform density and rotation. Despite the obvious elegance and simplicity of this form of the perturbation equations, this formalism seems to have been generally ignored (although see Cartan 1922) until it was rediscovered by Ipser and Managan (1985).

The system of equations (16) and (17) can be reduced further to a single fourth-order equation for the single potential δU by solving equation (16) for $\delta\Phi$ and then by inserting the resulting expression in equation (17):

$$0 = \nabla^c\nabla_c\left[\frac{dp}{d\rho}\,\frac{\nabla_a(\rho Q^{ab}\nabla_b\,\delta U)}{\rho(\omega + m\Omega)} + \delta U\right] + \frac{4\pi G}{\omega + m\Omega}\,\nabla_a(\rho Q^{ab}\nabla_b\,\delta U)\,. \tag{18}$$

The solutions to this equation determine the entire structure of the pulsations of rapidly rotating Newtonian stellar models.

To complete the mathematical specification of the eigenvalue problem for the normal modes of rapidly rotating Newtonian stellar models, equations (16) and (17) (or equivalently eq. [18]) must be supplemented with appropriate boundary conditions. For the case of the perturbed gravitational potential, $\delta\Phi$, the appropriate boundary condition is that the solutions of equation (17) should fall to zero at the appropriate rate as $r \to \infty$. For a perturbation having angular dependence $e^{im\varphi}$ the gravitational potential must satisfy (for $|m| \geq 1$)

$$\lim_{r\to\infty} r^{|m|}\delta\Phi = 0\,. \tag{19}$$

(For the case $m = 0$ the correct boundary condition is obtained by setting $m = 1$ in eq. [19].) The boundary condition on the potential δU arises from the requirement that the thermodynamic function $h(p)$, which vanishes on the surface of the equilibrium configuration, should also vanish on the perturbed surface of the oscillating star. This condition requires that $\delta h + \xi^a\nabla_a h = 0$ on the surface of the equilibrium stellar model. Using the expression $\delta h = \delta U + \delta\Phi$ and equations (5) and (15), we find that this boundary condition can be written as

$$\delta U + \delta\Phi + \Xi^{ab}\nabla_a(\Phi + \Psi)\nabla_b\,\delta U = 0\,. \tag{20}$$

Equation (20) is to be imposed on the surface of the star. We note that while δU has no physical meaning outside the star, it can be extended into the exterior of the star simply by imposing equation (20) in that region. While this extension has no physical significance, it makes it more convenient to evaluate δU inside the star if it is taken to have *some* smooth extension into the exterior region. This particular extension is a natural one to impose.

d) The Variational Principle

When analyzing the modes of a system (either numerically or analytically), it is extremely helpful to have available a variational principle expression for the frequencies of that system. In the case of our analysis here, in which we seek to solve exactly the pulsation equations for rapidly rotating stellar models, we find it helpful to use the variational principle in two different ways. In the first place we use it to obtain an initial estimate of the frequency of a mode and in the second place to estimate the accuracy of our computations once the complete eigenfunction has been determined. A variational principle for the frequencies of the modes of rapidly rotating stellar models has been derived previously in terms of the potential δU by Ipser and Managan (1985) and Managan (1985). We present here a simpler derivation in terms of the covariant representation of the pulsation equations developed here.

For the purposes of analyzing the variational principle, it will be helpful to replace the potential δU in our equations by the related potential $\delta V = \delta U/(\omega + m\Omega)$. It is easy to show that equation (16), when expressed in terms of the potential δV, has the form

$$0 = \mathscr{L}_\omega(\delta V) \equiv \nabla_a(\rho H^{ab}\nabla_b\,\delta V) + (\omega + m\Omega)\rho\,\frac{d\rho}{dp}\,[(\omega + m\Omega)\delta V + \delta\Phi(\delta V)]\,. \tag{21}$$

In this expression the tensor H^{ab} is related to the tensor Q^{ab} that appears in equation (16) and is given by the expression

$$H^{ab} = \lambda g^{ab} + (1 - \lambda)z^a z^b + \frac{\lambda\nabla^c\Omega\nabla_c(\varpi^2\Omega)}{\varpi^2(\omega + m\Omega)^2}\,\phi^a\phi^b - \frac{i\lambda}{\omega + m\Omega}\,(2\Omega\nabla^a\phi^b - \phi^a\nabla^b\Omega + \phi^b\nabla^a\Omega)\,. \tag{22}$$

Note that H^{ab} is Hermitian for real values of the frequency: $H^{ab} = H^{*ba}$. In equation (21) the perturbed gravitational potential $\delta\Phi(\delta V)$ is considered to be the function of the potential δV that is determined by solving the perturbed gravitational potential equation (i.e., eq. [17])

$$\nabla^a\nabla_a\,\delta\Phi + 4\pi G\rho\,\frac{d\rho}{dp}\,\delta\Phi = -4\pi G(\omega + m\Omega)\rho\,\frac{d\rho}{dp}\,\delta V \tag{23}$$

for given δV.

Let δV and $\delta \bar{V}$ be independent potentials having angular dependences $e^{im\varphi}$ that satisfy the boundary conditions implicit in equations (19) and (20). We also assume (probably unnecessarily) that the density ρ of the equilibrium stellar model vanishes on the surface of the star. Under these conditions the symmetry of the operator $\mathscr{L}_\omega$ acting on these functions is determined by evaluating the integral

$$\int \delta\bar{V}^* \mathscr{L}_\omega(\delta V) d^3x = \int \left[-\rho H^{ab} \nabla_a \delta\bar{V}^* \nabla_b \delta V + \rho \frac{d\rho}{dp} (\omega + m\Omega)^2 \delta\bar{V}^* \delta V + \frac{1}{4\pi G} \nabla^a \delta\Phi(\delta\bar{V}^*) \nabla_a \delta\Phi(\delta V) - \rho \frac{d\rho}{dp} \delta\Phi(\delta\bar{V}^*) \delta\Phi(\delta V) \right] d^3x . \quad (24)$$

The right-hand side of equation (24) is obtained by performing two integrations by parts, using the boundary conditions and equation (23). Since the tensor H^{ab} is Hermitian for real frequencies, it follows that the operator $\mathscr{L}_\omega$ is symmetric in the sense that

$$\int \delta\bar{V}^* \mathscr{L}_\omega(\delta V) d^3x = \int \delta V [\mathscr{L}_\omega(\delta\bar{V})]^* d^3x . \quad (25)$$

Now consider the expression

$$0 = \int \delta V^* \mathscr{L}_\omega(\delta V) d^3x \quad (26)$$

as an equation for the frequency, ω, in terms of the potential δV. As a consequence of the symmetry of the operator $\mathscr{L}_\omega$, this equation yields a value for ω that is stationary with respect to infinitesimal variations of δV precisely when δV is the normal-mode eigenfunction with ω the corresponding eigenvalue. Equation (26) is, therefore, the variational principle expression for the frequency of the mode.

III. THE NUMERICAL METHOD

a) Spherical Coordinates

Even the most rapidly rotating stellar models are spherical to a first approximation, since the ratio of the equatorial to the polar radii in these models is never larger than about 2/1. Therefore, spherical coordinates are most naturally adapted to analyzing the full range of rotating stellar models, from spherical nonrotating stars to the oblate rapidly rotating models. In order to find explicit solutions of the pulsation equations (16) and (17) subject to the boundary conditions, equations (19), (20), we must have explicit representations of these equations in the coordinate system in which the equations are to be solved. It is straightforward to deduce the following expressions for equations (16), (17), and (20) in spherical coordinates:

$$[\Lambda^{rr}\partial_r^2 + \Lambda^{r\mu}\partial_\mu\partial_r + \Lambda^{\mu\mu}\partial_\mu^2 + \Lambda^r\partial_r + \Lambda^\mu\partial_\mu + \Lambda]\delta U(r, \mu) = -(\omega + m\Omega)^2 \delta\Phi(r, \mu) , \quad (27)$$

$$\left[\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\mu^2 - \frac{2\mu}{r^2}\partial_\mu - \frac{m^2}{r^2(1-\mu^2)} + 4\pi G\rho \frac{d\rho}{dp} \right] \delta\Phi(r, \mu) = -4\pi G\rho \frac{d\rho}{dp} \delta U(r, \mu) , \quad (28)$$

$$[\Xi^r\partial_r + \Xi^\mu\partial_\mu + \Xi]\delta U(r, \mu) = -(\omega + m\Omega)^2 \delta\Phi(r, \mu) . \quad (29)$$

In these expressions we have introduced the angular coordinate $\mu = \cos\theta$; and partial derivatives with respect to r and μ are denoted ∂_r and ∂_μ. The functions Λ^{rr}, $\Lambda^{r\mu}$, and $\Lambda^{\mu\mu}$ are proportional to the corresponding components of the tensor Q^{ab} given in equation (13). The functions Λ^r, Λ^μ, and Λ involve various derivatives of the equilibrium density ρ and the components of Q^{ab}. Similarly, the functions Ξ^r, Ξ^μ, and Ξ involve various components of the tensor Ξ^{ab}. Explicit expressions for Λ^{rr}, $\Lambda^{r\mu}$, $\Lambda^{\mu\mu}$, Λ^r, Λ^μ, Λ, Ξ^r, Ξ^μ, and Ξ are given in Appendix A. Each of these functions is real, making equations (27)–(29) a real system of equations for the functions $\delta U(r, \mu)$ and $\delta\Phi(r, \mu)$.

The remaining boundary condition, equation (19), is the requirement that $\delta\Phi \to 0$ as $r \to \infty$. In numerical work it is only possible to work with a finite range of coordinates, so it is necessary to reformulate this boundary condition in terms of quantities that can be evaluated at finite r. This condition is equivalent to the statement that in the exterior of the star $\delta\Phi$ can be represented as a sum over its multiple components containing only negative powers of r. We enforce this condition by imposing the requirement that the radial derivative of $\delta\Phi$ be given by the appropriate sum over these multipole components:

$$0 = \partial_r \delta\Phi(r, \mu) + \sum_{l \geq |m|} \frac{(l+1)(2l+1)(l-m)!}{2r(l+m)!} P_l^m(\mu) \int_{-1}^{1} \delta\Phi(r, \mu') P_l^m(\mu') d\mu' . \quad (30)$$

This linear integral-differential constraint on the potential $\delta\Phi(r, \mu)$ can be imposed on the surface of any r = constant sphere in the exterior of the stellar model.

The variational principle expression for the frequency of a mode, equation (26), will be a useful tool in our numerical work. The explicit representation of this equation for general differentially rotating stars is given in Appendix A in terms of spherical coordinates. This expression is in general a rather complicated function of the frequency whose roots would have to be located numerically. For the case of the pulsations of uniformly rotating stellar models (the case of particular interest to us) this expression simplifies considerably and can be written as a quartic polynomial in the frequency. Consider the polynomial $F(\sigma)$ defined as

$$F(\sigma) = a_4\sigma^4 - (4\Omega^2 a_4 + a_3 + a_2)\sigma^2 - 2\Omega a_1 \sigma + 4\Omega^2 a_3 . \quad (31)$$

The constants a_1, a_2, a_3, and a_4 are defined as integrals which involve the potential δU and $\delta\rho = \rho(d\rho/dp)(\delta U + \delta\Phi)$:

$$a_1 = \int 2m\rho\, \delta U(r\partial_r \delta U - \mu\partial_\mu \delta U)dr\, d\mu\ , \tag{32}$$

$$a_2 = \int \rho\left[(1-\mu^2)(r\partial_r \delta U - \mu\partial_\mu \delta U)^2 + \frac{m^2}{1-\mu^2}\,\delta U^2\right]dr\, d\mu\ , \tag{33}$$

$$a_3 = \int \rho[\mu r\partial_r \delta U + (1-\mu^2)\partial_\mu \delta U]^2 dr\, d\mu\ , \tag{34}$$

$$a_4 = \int r^2 \delta\rho\, \delta U\, dr\, d\mu\ . \tag{35}$$

The variational principle expression for the pulsation frequencies, ω, of a uniformly rotating star can be expressed in terms of this polynomial as

$$F(\omega + m\Omega) = 0\ . \tag{36}$$

We note that the coefficients satisfy the inequalities $a_3 \geq 0$ and $a_2 \geq |a_1|$. Since $F(0) = 4\Omega^2 a_3 \geq 0$ and $F(\pm 2\Omega) = -4\Omega^2(a_2 \pm a_1) \leq 0$, it follows that either two or four roots of this polynomial must lie in the range $-2\Omega \leq \omega + m\Omega \leq 2\Omega$. If the coefficient $a_4 \geq 0$ (as is the case for the $l = m$ f-modes studied by us), then this polynomial has two roots (one positive and one negative) in the range $|\omega + m\Omega| \geq 2\Omega$. These bounds on the locations of the roots of $F(\sigma)$ make it easy to determine them numerically. We find that the physical root of this equation (the one corresponding to the actual eigenvalue of the system) satisfies $\omega + m\Omega \geq 2\Omega$ in the $l = m$ f-modes.

b) The Finite-Difference Equations

The various functions that describe the equilibrium stellar models [e.g., $\rho(r, \mu)$ and $\Phi(r, \mu)$] and the functions that describe the pulsations of these stars [e.g., $\delta U(r, \mu)$ and $\delta\Phi(r, \mu)$] are computed on a finite grid of points. This grid, chosen to reflect the approximate spherical symmetry of this problem, consists of N evenly spaced points along each of $2L - 1$ radial spokes emanating from the center of the star. These points are located at the radii r_α for $\alpha = 1, N$. The spokes are located at the angles, $\mu_i = \cos\theta_i$, which correspond to the zeros of the Legendre polynomial of order $2L - 1$: $P_{2L-1}(\mu_i) = 0$. We note that the angles θ_i become evenly spaced in the limit of large values of L. They are in fact very evenly spaced for moderate values of L. For $L = 5$ the angular separation of the spokes is uniform to within 0°.1.

The functions that describe the equilibria and the pulsations of rotating stellar models have definite parity (either even or odd) with respect to reflections about the equator $\mu = 0$. Consequently, these functions need only be evaluated on the L spokes which lie in the quadrant with $\mu \geq 0$. Because of the approximate spherical symmetry of rotating stellar models, it is possible to obtain sufficiently accurate representations of this problem using a grid with a relatively small number of spokes, $L \approx 10$, and a larger number of points along each spoke, $N \approx 500$.

In order to obtain numerical solutions to the equations that describe the pulsations of rotating stellar models, it is necessary to have expressions for the derivatives and the integrals of functions whose values are known only on the points in the coordinate grid. In our analysis we use a grid containing many radial points but only a small number of spokes. It is appropriate under these circumstances to use low-order approximations for the radial derivatives and integrals while using more sophisticated, higher order expressions for the angular derivatives and integrals. Since the radial spokes have been chosen to lie at the angles which are the zeros of an appropriate Legendre polynomial, it is possible to use Gaussian quadrature (see, e.g., Abramowitz and Stegun 1964) to obtain accurate expressions for the angular integrals of functions which may occur in the problem. For functions of odd parity the angular integrals are zero, for even functions these integrals may be expressed as the sum

$$\int_{-1}^{1} f^+(r, \mu)d\mu \approx \sum_{i=1}^{L} w_i f^+(r, \mu_i)\ . \tag{37}$$

This expression for the integral of a function f^+ is exact if f^+ is a polynomial in μ of order $4L - 4$ (or less). The constants w_i have a simple expression in terms of the derivatives of Legendre polynomials and have been tabulated for many values of L (see, e.g., Abramowitz and Stegun 1964). The integrals of functions in the radial direction are performed using a simple trapezoidal rule.

To evaluate the derivatives of functions whose values are given only on the discrete points of our coordinate grid, we use a variety of finite-difference approximations. For the first and second radial derivatives we use the standard three-point difference formulae (see, e.g., Abramowitz and Stegun 1964). We use the centered or the off-centered formulae depending on whether the point is in the interior or on the boundary of the model. For derivatives in the angular directions we use the higher order difference formulae which are the analogs of the Gaussian quadrature equations. These formulae are specifically constructed to approximate the derivatives of functions of given parity and given azimuthal dependence $e^{im\varphi}$. The formulae are designed to evaluate exactly the derivatives of a function which can be written as a (sufficiently small finite) sum of associated Legendre functions P_l^m. The following expressions represent the first angular derivatives and the angular part of the Laplace operator:

$$\partial_\mu f^\pm(r, \mu_i) \approx \sum_{j=1}^{L} D_{ij}^\pm f^\pm(r, \mu_j)\ , \tag{38}$$

$$\left[(1-\mu^2)\partial_\mu^2 - 2\mu\partial_\mu - \frac{m^2}{1-\mu^2}\right]f^\pm(r, \mu_i) \approx \sum_{j=1}^{L} \Delta_{ij}^\pm f^\pm(r, \mu_j)\ . \tag{39}$$

The second μ-derivative can be determined from the appropriate linear combination of equations (38) and (39). These expressions presume that the function $f^{\pm}(r, \mu)$ is the projection of the function $f^{\pm} = f^{\pm}(r, \mu)e^{im\varphi}$ which has definite parity: $f^{\pm}(r, \mu) = \pm f^{\pm}(r, -\mu)$. The constants $D_{ij}^{\pm}$ and $\Delta_{ij}^{\pm}$ can be represented as sums involving the associated Legendre functions P_l^m. These expressions are derived and the specific formulae for the constants $D_{ij}^{\pm}$ and $\Delta_{ij}^{\pm}$ are given in Appendix B.

The asymptotic boundary condition on the perturbed gravitional potential, equation (30), is imposed on the set of grid points having the largest value of r: $r = r_N$. The angular integrals contained in this condition can be converted to sums using equation (37), so that the asymptotic boundary condition may be cast in the following approximate form:

$$\partial_r \delta\Phi^{\pm}(r_N, \mu_i) + \frac{1}{r_N} \sum_{j=1}^{L} E_{ij}^{\pm} \delta\Phi^{\pm}(r_N, \mu_j) \approx 0 . \tag{40}$$

The constants $E_{ij}^{\pm}$ are evaluated explicitly in Appendix B.

Using the finite-difference approximations for the derivatives of $\delta U(r, \mu)$ and $\delta\Phi(r, \mu)$, equations (27) and (28) along with the boundary conditions equations (29) and (40) become a system of linear algebraic equations. Let a and b denote indices that label each of the points in our two-dimensional grid. [We take $a = (\alpha - 1)L + i$ to denote the point located at (r_α, μ_i) but this choice is not necessary.] The algebraic equations which result from imposing the finite-difference expression for equation (27) in the interior of the star and the finite-difference representation of equation (29) in the exterior of the star can be expressed schematically as

$$\sum_b A_a{}^b \delta U_b = -(\omega + m\Omega)^2 \delta\Phi_a . \tag{41}$$

We note that imposing equation (29) in the exterior of the star allows us to define δU smoothly in this region. This extension of δU allows the use of the standard angular difference formulae (i.e., eqs. [38] and [39]) even near the boundary of the star. Since these derivatives may involve the values of δU both inside and outside the star, it would be necessary to employ special difference formulae if δU were not extended smoothly. In principle, the frequencies and eigenfunctions determined in this way could depend on the nonphysical extension of δU. The agreement which we obtain between the values of the frequency determined directly from equation (41) and those obtained from the variational principle equation (36) suggest, however, that the extension of δU into the exterior of the star via equation (29) is an acceptable numerical artifice.

In a similar fashion the gravitational potential equation (28) together with the boundary condition equation (40) can be combined into a single linear algebraic equation of the form

$$\sum_b B_a{}^b \delta\Phi_b = \Theta(\rho_a)\delta U_a , \tag{42}$$

where $\Theta(\rho_a)$ is zero outside the star and unity inside. Equation (42) is constructed by imposing the finite-difference form of equation (28) on each grid point except those at the edge of the grid (i.e., at $r = r_N$). The boundary condition equation (40) is imposed on these last grid points. We note that the matrices $A_a{}^b$ and $B_a{}^b$ depend on the structure of the equilibrium stellar model, and that $A_a{}^b$ depends explicitly on the eignvalue ω as well. If the structure of the indices a and b is chosen reasonably, the matrices $A_a{}^b$ and $B_a{}^b$ will be sparse and band-diagonal. For our choice [$a = (\alpha - 1)L + i$] the matrix $A_a{}^b$ has $4L$ nonvanishing codiagonals, while $B_a{}^b$ has $2L$.

While it is sufficient to impose the boundary condition for δU, equation (29), only on the grid points which lie in the exterior region of the star, it is really more appropriate to ensure that this condition is imposed on the real surface of the star. This can be accomplished in a reasonably simple manner. Let $\Xi[\delta U]_{\alpha,i}$ denote the operator on the left-hand side of equation (29) evaluated at the grid point (r_α, μ_i). We wish this operator, when evaluated at the true radius of the star, $R(\mu_i)$, to yield the value of the potential $\delta\Phi$ evaluated at this same point. When expressed in terms of quantities that can be evaluated on the fixed grid of points, this condition is (to lowest order)

$$[R(\mu_i) - r_{\eta-1}]\Xi[\delta U]_{\eta,i} - [R(\mu_i) - r_\eta]\Xi[\delta U]_{\eta-1,i} = -(\omega + m\Omega)^2\{[R(\mu_i) - r_{\eta-1}]\delta\Phi(r_\eta, \mu_i) - [R(\mu_i) - r_\eta]\delta\Phi(r_{\eta-1}, \mu_i)\} , \tag{43}$$

where r_η is the last grid point inside the surface of the star on the given spoke μ_i. Equation (43) can be enforced on the grid point (r_η, μ_i) instead of imposing equation (27) at that point. The error in imposing the boundary condition in this way varies with the step size h as h^2, while the simple method of imposing the boundary condition only on the points in the exterior of the star has an error which varies as h.

c) *Solving the Eigenvalue Problem*

The eigenvalue problem to determine the frequencies of the modes of rapidly rotating Newtonian stellar models has been reduced to the system of *algebraic* equations (41) and (42). These equations can be solved using reasonably standard numerical techniques. In Ipser and Lindblom (1989) we described briefly how these equations could be solved by converting them to the single algebraic equation

$$\sum_b \sum_c B_a{}^b A_b{}^c \delta U_c = -(\omega + m\Omega)^2 \Theta(\rho_a)\delta U_a . \tag{44}$$

The eigenvectors, δU_a, and eigenvalues, $\omega + m\Omega$, of this problem can be determined very efficiently using standard methods (see, e.g., Wilkinson 1965). The standard techniques work here even though in this problem the matrix $A_a{}^b$ depends on the eigenvalue. The numerical results presented in Ipser and Lindblom (1989) were obtained by solving the equations in this way.

We describe here an alternate method for solving this eigenvalue problem which is more efficient both in terms of the amount of time required to converge to a solution and in terms of the amount of memory required to implement the algorithm. We begin by choosing an initial guess, $\delta U^{(i)}$, for the function $\delta U(r, \mu)$. For nonrotating stars we find that the estimate $\delta U(r, \mu) = r^l P_l^m(\mu)$ (for $l > 1$) is an adequate initial approximation to the f-mode eigenfunction. For rotating stellar models we use as an initial guess the function $\delta U(r, \mu)$ computed for a stellar model having a slightly smaller angular velocity. Given this initial estimate $\delta U^{(i)}$, we solve equation (42) (which does not depend on the eigenvalue ω) for the potential $\delta\Phi(r, \mu)$. Given these initial estimates $\delta U^{(i)}$ and $\delta\Phi^{(i)}$ we use the variational principle (eq. [A12] or, for uniformly rotating stars, eq. [36]) to determine the initial estimate $\omega^{(i)}$ of the frequency of the mode. Knowing the frequency of the mode, we can solve equation (42) for a refined estimate $\delta U^{(i+1)}$ of the potential. This procedure can be iterated.

After two or three iterations of the algorithm described above, we find that convergence can be improved by changing the method of updating the eigenvalue estimate. Instead of using the variational principle, we update the eigenvalue by attempting to minimize the average change in the eigenfunction from one iteration to the next. We define an average value $\langle \delta U \rangle$ of the eigenfunction. (One example of such an average is the square root of the absolute value of the integral defined in eq. [35].) The change in this average is monitored from one iteration to the next by computing the ratio $z^{(i)} \equiv \langle \delta U^{(i-1)} \rangle / \langle \delta U^{(i)} \rangle$. We update the eigenvalue by anticipating the value that will make this ratio in the next iteration as close to unity as possible:

$$\omega^{(i+1)} = \omega^{(i)} + s(1 - z^{(i)}) \frac{\omega^{(i)} - \omega^{(i-1)}}{z^{(i)} - z^{(i-1)}} . \qquad (45)$$

The constant $s \leq 1$ is a "convergence factor" chosen to make the iteration process more stable.

Solving the pair of algebraic equations (41) and (42) is more efficient than solving the single equation (44) (as in Ipser and Lindblom 1989) because of the band-diagonal structure of the matrices $A_a{}^b$ and $B_a{}^b$. These matrices are sparse, having $4L$ and $2L$ nonzero codiagonals, respectively. The product matrix $A_a{}^b B_b{}^c$ has $6L$ nonzero codiagonals. Since the time required to solve a band-diagonal linear algebraic system is proportional to the square of the number of nonzero codiagonals, it is more efficient to solve the two equations (41) and (42) rather than the single equation (44). The number of iterations required for convergence in the two methods is comparable. Solving the two equations separately also reduces the amount of memory required, since the product matrix $A_a{}^b B_b{}^c$ is not needed.

We monitor the convergence of the solution to equations (41) and (42) in a number of ways. We iterate according to the algorithm outlined above until the eigenfunctions δU, $\delta\Phi$ and the eigenvalue ω do not change significantly from one iteration to the next. (This generally takes between 5 and 10 iterations.) We check to ensure that the eigenfunctions so obtained do in fact solve equations (41) and (42) to the desired degree of accuracy (typically better than one part in 10^6). We also check that the final eigenvalue obtained with equation (45) agrees with the eigenvalue obtained by inserting the final eigenfunctions into the variational principle (i.e., eq. [A12] or eq. [36]). We typically get agreement which is better than one part in 10^3.

IV. THE $l = m$ f-MODES OF POLYTROPES

In this final section we describe the results of applying the methods developed in this paper to the study of the $l = m$ f-modes of uniformly rotating stellar models. For simplicity we have chosen to limit our study here to stellar models based on the polytropic equations of state,

$$p = \kappa \rho^{1+1/n} , \qquad (46)$$

where κ and the polytropic index n are constants. Of special interest to us are values of n near unity, since these are the best polytropic models of neutron stars. We limit our consideration here to the modes which reduce in the nonrotating limit to those having spherical harmonic indices $l = m \geq 2$. We also limit our consideration to the modes which reduce in the nonrotating limit to the f-modes (that have no nodes in the radial eigenfunctions). These are the modes that are responsible for the gravitational radiation–driven secular instability in sufficiently rapidly rotating stars (see Chandrasekhar 1970; Friedman and Schutz 1978; Friedman 1978). Our analysis here will focus on determining the properties of the adiabatic modes, leaving the study of the dissipative effects of gravitational radiation reaction and viscosity to a subsequent paper.

Using the algorithm described in § III, we have solved equations (27) and (28) together with the boundary conditions (29) and (30) to find the eigenfunctions δU and the frequencies ω of the $l = m$ f-modes. The angular velocity dependence of the frequencies of these modes, $\omega(\Omega)$, can be determined from the functions

$$\alpha_m(\Omega) = \frac{\omega(\Omega) + m\Omega}{\omega(0)} . \qquad (47)$$

These functions are displayed in Figures 1–6 for a range of different modes, $2 \leq l = m \leq 6$, and for a range of different polytropic equations of state with $n = 0$, 3/4, 1, and 5/4. The $n = 0$ results (the uniform-density Maclaurin spheroids) are based on the analytic recursion relations for the frequencies of these modes given in Lindblom (1987). The functions α_m are convenient representations of the angular velocity dependence of the frequencies of these modes because they are independent of the constant κ in the polytropic equation of state (eq. [46]) and they are independent of the total mass of the star (for polytropes). These functions are graphed as functions of the angular velocity (assumed to be uniform) in units of $(\pi G \bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the volume-averaged density of the nonrotating star having the same mass. Figure 1 illustrates how the functions α_m depend on m for a given equation of state. Figures 2–6 illustrate the dependence of these functions, α_m, on the equation of state of the stellar fluid. We note that the functions α_m (which are proportional to the frequency of the mode as measured in a frame corotating with the star) are remarkably independent of angular velocity. They are also very insensitive to the equation of state of the stellar fluid.

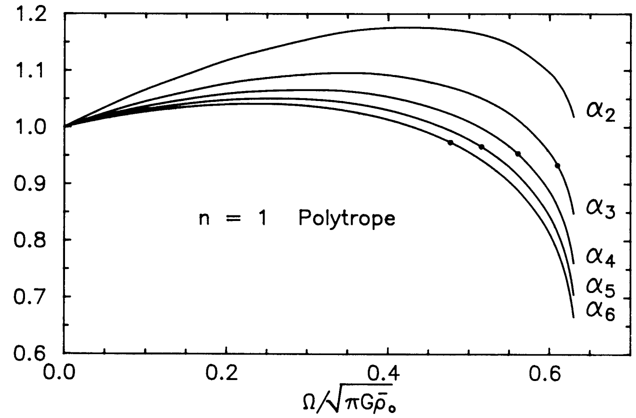

FIG. 1

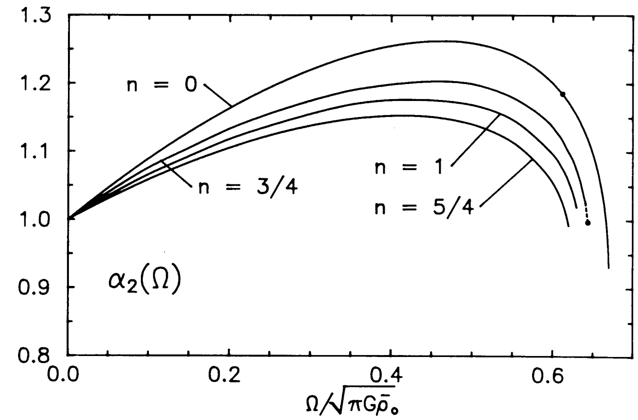

FIG. 2

FIG. 1.—Functions $\alpha_m(\Omega) = [\omega(\Omega) + m\Omega]/\omega(0)$ of the angular velocity depicted for the $l = m$ *f*-modes of uniformly rotating $n = 1$ polytropes. The angular velocities are given in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. The dots indicate the angular velocity where the frequency of that mode is zero.

FIG. 2.—Functions $\alpha_2(\Omega) = [\omega(\Omega) + 2\Omega]/\omega(0)$ of the angular velocity depicted for the $l = m = 2$ *f*-modes of uniformly rotating polytropes. The angular velocities are given in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. The dots indicate the angular velocity where the frequency of that mode is zero. The dashed portion of the $n = \frac{3}{4}$ curve represents the extrapolation to the point where the frequency vanishes.

The large dots which appear on the curves in Figures 1–6 are placed at the angular velocities, Ω_{crit}, where the frequency of that mode passes through zero: $\omega(\Omega_{\text{crit}}) = 0$. These critical angular velocities are also listed in Table 1. These are the angular velocities above which the star would be unstable to a gravitational radiation–driven secular instability if the star were inviscid. The values, τ_{crit}, of the ratio of the rotational kinetic energy to the gravitational potential energy of the star at these critical angular velocities are also listed in Table 1. Our values agree for $n = 1$ polytropes to about the 1% level with the Eulerian variational principle calculations of Managan (1985) and the Lagrangian variational principle calculations of Imamura, Friedman, and Durisen (1985). We note that we do not find a critical angular velocity for the $l = m = 2$ mode in the $n = 1$ and $n = 5/4$ polytropes. The critical angular velocity for the $l = m = 2$ mode in the $n = 3/4$ polytrope occurs very close to the termination point (where the angular velocity of the star equals the angular velocity of a particle in orbit at the equator of the star). Our results are consistent with James's (1964) conclusion that a critical angular velocity for the $l = m = 2$ mode exists only in polytropes having $n \leq 0.808$. We note that the parameter $\Omega_{\text{crit}}/(\pi G\bar{\rho}_0)^{1/2}$ is far more independent of the equation of state than the parameter τ_{crit}. Also presented in Table 1 are the frequencies of these *f*-modes in nonrotating polytropes. Our values agree to within 0.2% with the variational principle calculation of these values by Managan (1986) for $n = 1$ polytropes, Our calculations of the functions α_m also agree with the variational principle values of these functions computed by Managan (1986) for $n = 1$ polytropes to similar accuracy.

In order to evaluate the effects of gravitational radiation and viscous dissipation on the pulsations of a rotating star, a knowledge of the fluid motions associated with the pulsation are needed, as well as a knowledge of the frequency. In § II we showed that a

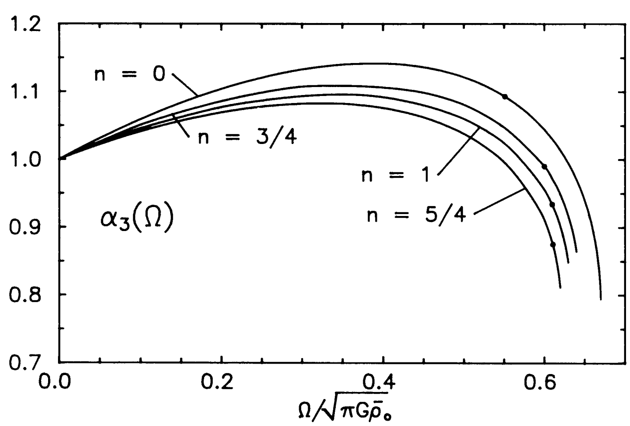

FIG. 3

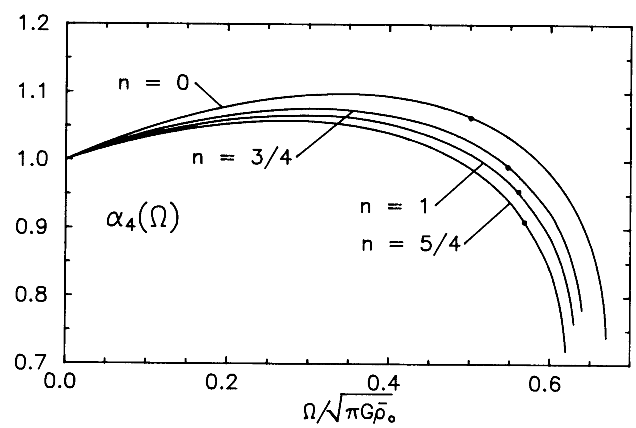

FIG. 4

FIG. 3.—Functions $\alpha_3(\Omega) = [\omega(\Omega) + 3\Omega]/\omega(0)$ of the angular velocity depicted for the $l = m = 3$ *f*-modes of uniformly rotating polytropes. The angular velocities are given in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. The dots indicate the angular velocity where the frequency of that mode is zero.

FIG. 4.—Functions $\alpha_4(\Omega) = [\omega(\Omega) + 4\Omega]/\omega(0)$ of the angular velocity depicted for the $l = m = 4$ *f*-modes of uniformly rotating polytropes. The angular velocities are given in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. The dots indicate the angular velocity where the frequency of that mode is zero.

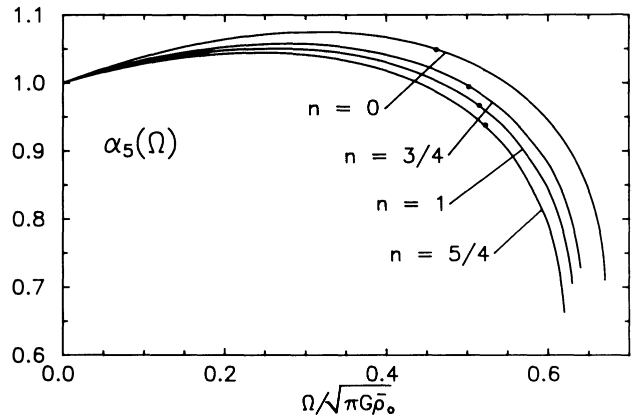

FIG. 5

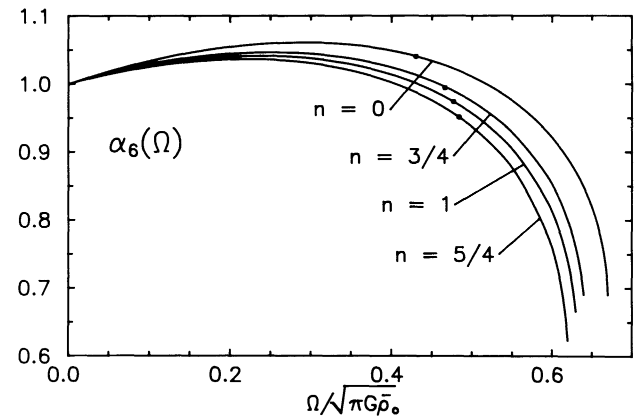

FIG. 6

FIG. 5.—Functions $\alpha_5(\Omega) = [\omega(\Omega) + 5\Omega]/\omega(0)$ of the angular velocity depicted for the $l = m = 5$ *f*-modes of uniformly rotating polytropes. The angular velocities are given in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. The dots indicate the angular velocity where the frequency of that mode is zero.

FIG. 6.—Functions $\alpha_6(\Omega) = [\omega(\Omega) + 6\Omega]/\omega(0)$ of the angular velocity depicted for the $l = m = 6$ *f*-modes of uniformly rotating polytropes. The angular velocities are given in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. The dots indicate the angular velocity where the frequency of that mode is zero.

knowledge of the potential δU was sufficient to determine the complete fluid motion in a stellar pulsation. In Figures 7–9 is illustrated the angular velocity dependence of this potential for the $l = m = 4$ mode in a sequence of uniformly rotating $n = 1$ polytropes. These potentials were computed on a grid having $L = 10$ radial spokes located at the angles $\cos\theta_i = \mu_i$ corresponding to the zeros of the Legendre polynomial $P_{19}(\mu_i) = 0$. Each curve in these figures represents the function $\delta U(r, \mu_i)$ on one of these spokes. These eigenfunctions have no zeros (except along the rotation axis) and are peaked at $\mu_1 = 0$, the equator of the star. These $l = m$ modes look very much like waves which propagate around the equator of the star in the direction opposite to the rotation of

TABLE 1

PULSATION FREQUENCIES AND CRITICAL ANGULAR VELOCITIES FOR POLYTROPES

$l = m$	n[a]	$\omega(0)$[b] / $(\pi G\bar{\rho}_0)^{1/2}$	Ω_{crit} / $(\pi G\bar{\rho}_0)^{1/2}$	Ω_{crit}[c] / Ω_{max}	τ_{crit}[d]
2.........	0	1.033	0.612	0.913	0.1375
	3/4	1.292	0.644[e]	0.994[e]	0.1298[e]
	1	1.415	...	...	...
	5/4	1.543	...	...	...
3.........	0	1.512	0.551	0.822	0.0991
	3/4	1.819	0.600	0.926	0.0866
	1	1.959	0.610	0.955	0.0800
	5/4	2.095	0.611	0.976	0.0694
4.........	0	1.886	0.501	0.748	0.0771
	3/4	2.208	0.547	0.844	0.0634
	1	2.350	0.561	0.878	0.0584
	5/4	2.481	0.568	0.908	0.0513
5.........	0	2.202	0.462	0.689	0.0630
	3/4	2.526	0.502	0.775	0.0497
	1	2.667	0.515	0.807	0.0453
	5/4	2.789	0.523	0.835	0.0398
6.........	0	2.481	0.430	0.642	0.0531
	3/4	2.812	0.466	0.719	0.0410
	1	2.939	0.477	0.747	0.0369
	5/4	3.053	0.484	0.773	0.0322

[a] The index n is the parameter in the polytropic equation of state: $p = \kappa\rho^{1+1/n}$.

[b] The frequencies and angular velocities are given in terms of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass.

[c] The maximum angular velocity of a stellar model of the same mass is denoted Ω_{max}.

[d] The quantity τ is the ratio of the rotational kinetic energy K to the gravitational potential energy W: $\tau = K/|W|$.

[e] These values are based on an extrapolation (to slightly higher angular velocity) of the frequencies which we have computed.

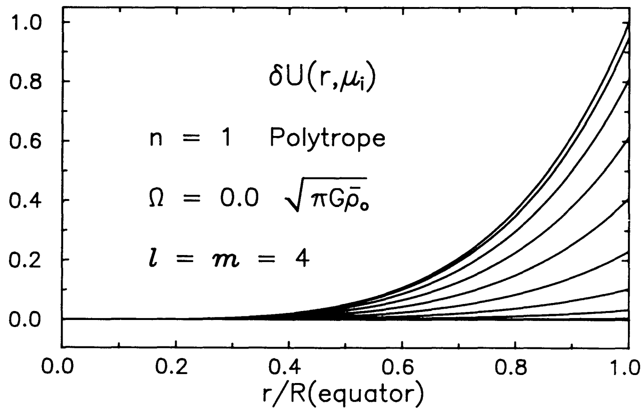

FIG. 7

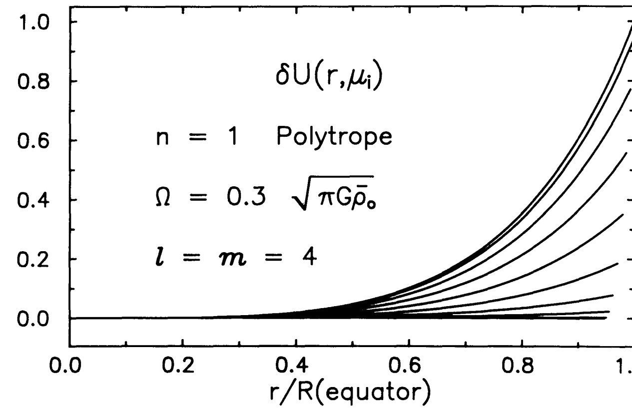

FIG. 8

FIG. 7.—Eigenfunction $\delta U(r, \mu_i)$ shown for the $l = m = 4$ f-mode of the nonrotating $n = 1$ polytrope. Each curve corresponds to a different value of μ_i. The highest curve in the figure corresponds to the equator ($\mu_i = 0$), and μ_i increases on each successively lower curve toward the value $\mu = 1$ on the rotation axis.

FIG. 8.—Eigenfuction $\delta U(r, \mu_i)$ shown for the $l = m = 4$ f-mode of the uniformly rotating $n = 1$ polytrope with angular velocity $\Omega = 0.3(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. Each curve corresponds to a different value of μ_i. The highest curve in the figure corresponds to the equator ($\mu_i = 0$), and μ_i increases on each successively lower curve toward the value $\mu = 1$ on the rotation axis.

the star. In the rotating models the curves in these figures end at the surface of the star. Thus the equatorial curves are longer than the curves for spokes near the rotation axis. These figures show that these modes tend to become more highly peaked at the surface of the star as the angular velocity of the star increases. Figures 9–11 illustrate the dependence of these modes in the index $l = m$. The eigenfunctions become more strongly peaked at the surface of the star for larger values of $l = m$. This is consistent with our knowledge that in nonrotating stars the potential is given approximately by $\delta U \approx r^l Y_{lm}$. Figures 12 and 13 together with Figure 9 illustrate the dependence of these eigenfunctions on the equation of state of the stellar fluid. These figures show that the eigenfunction tends to become more strongly peaked at the surface of the star for larger values of the polytropic index (i.e., for more compressible, softer equations of state). Finally, Figure 14 illustrates the physical components of the Lagrangian displacement vector [ξ^r and $(r \sin \theta)\xi^\varphi$] along the equatorial spoke for the $l = m = 4$ mode of two different $n = 1$ polytropes. The near-equality between the radial and axial components of the Lagrangian displacement, together with the fact that these appear out of phase, shows that the motion of a fluid particle is nearly circular motion about its equilibrium location. These graphs of the Lagrangian displacement confirm (with Figs. 7–9) that the fluid motion becomes more sharply peaked at the surface in rapidly rotating stars.

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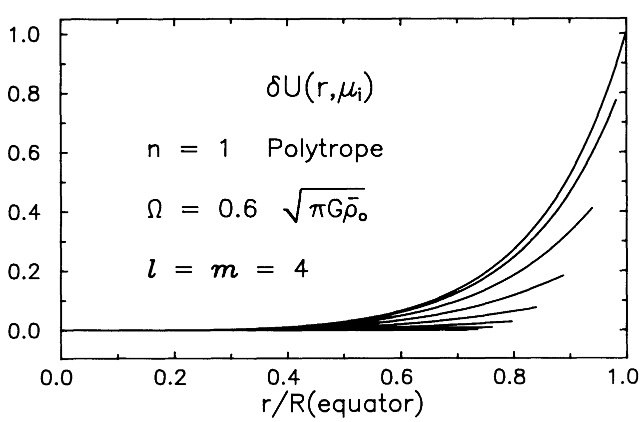

FIG. 9

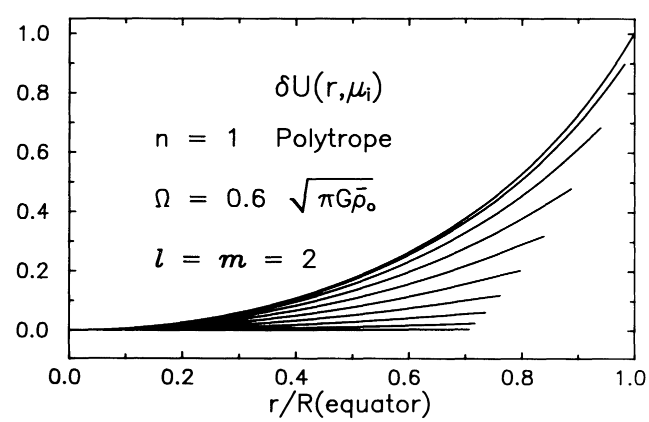

FIG. 10

FIG. 9.—Eigenfuction $\delta U(r, \mu_i)$ shown for the $l = m = 4$ f-mode of the uniformly rotating $n = 1$ polytrope with angular velocity $\Omega = 0.6(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. Each curve corresponds to a different value of μ_i. The highest curve in the figure corresponds to the equator ($\mu_i = 0$), and μ_i increases on each successively lower curve toward the value $\mu = 1$ on the rotation axis.

FIG. 10.—Eigenfunction $\delta U(r, \mu_i)$ is shown for the $l = m = 2$ f-mode of the uniformly rotating $n = 1$ polytrope with angular velocity $\Omega = 0.6(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. Each curve corresponds to a different value of μ_i. The highest curve in the figure corresponds to the equator ($\mu_i = 0$), and μ_i increases on each successively lower curve toward the value $\mu = 1$ on the rotation axis.

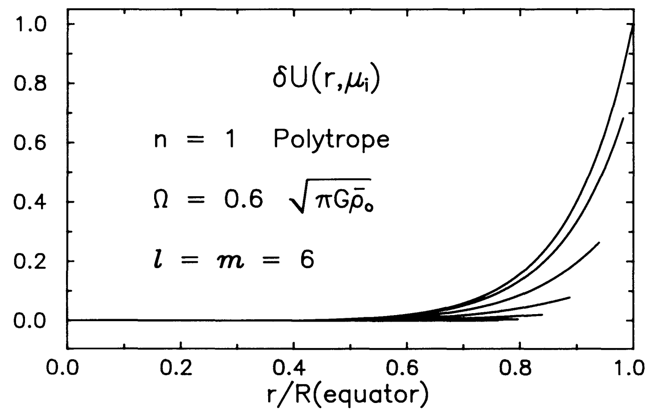

FIG. 11

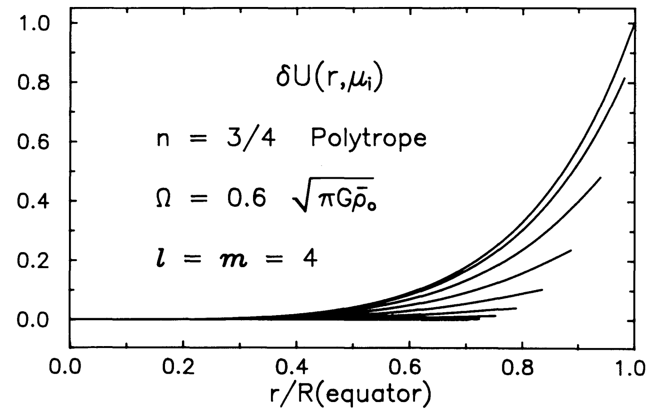

FIG. 12

FIG. 11.—Eigenfunction $\delta U(r, \mu_i)$ shown for the $l = m = 6$ f-mode of the uniformly rotating $n = 1$ polytrope with angular velocity $\Omega = 0.6(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. Each curve corresponds to a different value of μ_i. The highest curve in the figure corresponds to the equator ($\mu_i = 0$), and μ_i increases on each successively lower curve toward the value $\mu = 1$ on the rotation axis.

FIG. 12.—Eigenfunction $\delta U(r, \mu_i)$ shown for the $l = m = 4$ f-mode of the uniformly rotating $n = 3/4$ polytrope with angular velocity $\Omega = 0.6(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. Each curve corresponds to a different value of μ_i. The highest curve in the figure corresponds to the equator ($\mu_i = 0$), and μ_i increases on each successively lower curve toward the value $\mu = 1$ on the rotation axis.

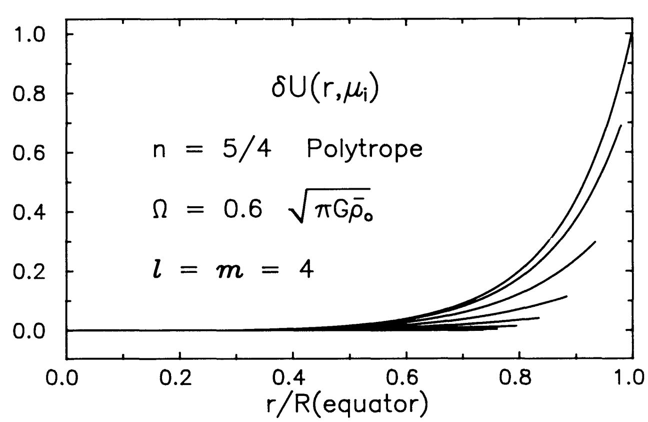

FIG. 13

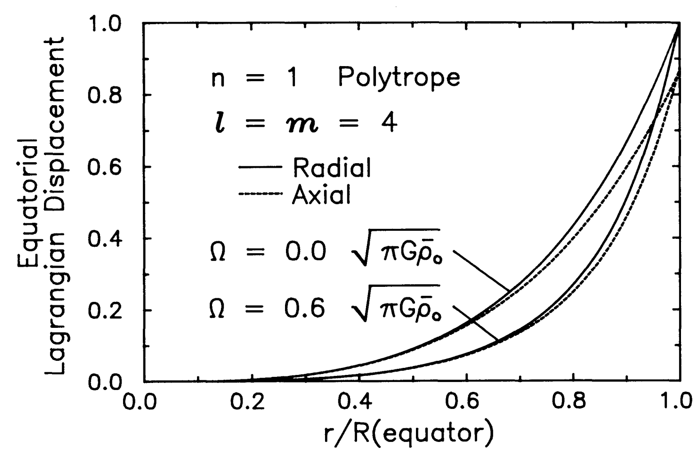

FIG. 14

FIG. 13.—Eigenfunction $\delta U(r, \mu_i)$ shown for the $l = m = 4$ f-mode of the uniformly rotating $n = 5/4$ polytrope with angular velocity $\Omega = 0.6(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass. Each curve corresponds to a different value of μ_i. The highest curve in the figure corresponds to the equator ($\mu_i = 0$), and μ_i increases on each successively lower curve toward the value $\mu = 1$ on the rotation axis.

FIG. 14.—Physical components of the Lagrangian displacement, ξ^r and $(r \sin\theta)\xi^\varphi$, for the $l = m = 4$ f-mode shown along the equatorial spokes of one nonrotating and one uniformly rotating $n = 1$ polytrope.

APPENDIX A

PULSATION EQUATIONS IN SPHERICAL COORDINATES

The equation for the potential δU (eq. [16]) has the following form when expressed in spherical coordinates r and $\mu = \cos\theta$:

$$[\Lambda^{rr}\partial_r^2 + \Lambda^{r\mu}\partial_\mu\partial_r + \Lambda^{\mu\mu}\partial_\mu^2 + \Lambda^r\partial_r + \Lambda^\mu\partial_\mu + \Lambda]\delta U(r, \mu) = -(\omega + m\Omega)^2\delta\Phi(r, \mu) . \tag{A1}$$

Partial derivatives with respect to r and μ are denoted ∂_r and ∂_μ. The functions Λ^{rr}, $\Lambda^{r\mu}$, and $\Lambda^{\mu\mu}$ are proportional to the corresponding components of the tensor Q^{ab} given in equation (13). The functions Λ^r, Λ^μ, and Λ involve various derivatives of the equilibrium density ρ and the components of Q^{ab}. Explicit expressions for these functions are given by the following:

$$\Lambda^{rr} = \frac{dp}{d\rho}[\lambda + \mu^2(1 - \lambda)] , \tag{A2}$$

$$\Lambda^{r\mu} = \frac{2}{r}\frac{dp}{d\rho}\mu(1 - \mu^2)(1 - \lambda) , \tag{A3}$$

$$\Lambda^{\mu\mu} = \frac{1}{r^2}\frac{dp}{d\rho}(1-\mu^2)[1-\mu^2(1-\lambda)] , \tag{A4}$$

$$r\rho\frac{dp}{d\rho}\Lambda^r = \rho[1-\mu^2+(1+\mu^2)\lambda] + r\partial_r\rho[\lambda+\mu^2(1-\lambda)] + \mu\partial_\mu\rho(1-\mu^2)(1-\lambda) + \rho\varpi\partial_\varpi\lambda - \frac{2m\rho\lambda\varpi\partial_\varpi\Omega}{\omega+m\Omega} , \tag{A5}$$

$$r^2\rho\frac{dp}{d\rho}\Lambda^\mu = \rho\mu[\lambda-3+3\mu^2(1-\mu)] + (1-\mu^2)\{r\mu\partial_r\rho(1-\lambda)+\partial_\mu\rho[1-(1-\lambda)\mu^2]\} - \rho\mu\varpi\partial_\varpi\lambda + \frac{2m\rho\mu\lambda\varpi\partial_\varpi\Omega}{\omega+m\Omega} , \tag{A6}$$

$$\rho\frac{dp}{d\rho}\Lambda = (\omega+m\Omega)^2\rho\frac{d\rho}{dp} + \frac{2m\Omega\lambda(r\partial_r\rho-\mu\partial_\mu\rho)}{r^2(\omega+m\Omega)} - \frac{m^2\rho\lambda}{\varpi^2} + \frac{2m\rho}{\varpi(\omega+m\Omega)}\left(\Omega\partial_\varpi\lambda+\lambda\partial_\varpi\Omega\frac{\omega-m\Omega}{\omega+m\Omega}\right) . \tag{A7}$$

In these equations we use the expression $\lambda = (\omega+m\Omega)^2/[(\omega+m\Omega)^2 - 4\Omega^2 - \varpi\partial_\varpi\Omega^2]$, $\varpi = r(1-\mu^2)^{1/2}$, and ∂_ϖ is the derivative with respect to ϖ. These expressions simplify considerably for the case of uniform rotation: $\partial_\varpi\Omega = \partial_\varpi\lambda = 0$.

When expressed in spherical coordinates, the boundary condition on δU has the following form:

$$[\Xi^r\partial_r + \Xi^\mu\partial_\mu + \Xi]\delta U(r,\mu) = -(\omega+m\Omega)^2\delta\Phi(r,\mu) . \tag{A8}$$

The functions Ξ^r, Ξ^μ, and Ξ involve various components of the tensor Ξ^{ab}. These functions are given by

$$r\Xi^r = \lambda(1-\mu^2)r^2\Omega^2 + r[\lambda+\mu^2(1-\lambda)]\partial_r\Phi + \mu(1-\mu^2)(1-\lambda)\partial_\mu\Phi , \tag{A9}$$

$$r^2\Xi^\mu = (1-\mu^2)\{-r^2\mu\lambda\Omega^2 + r\mu(1-\lambda)\partial_r\Phi + [1-\mu^2(1-\lambda)]\partial_\mu\Phi\} , \tag{A10}$$

$$\Xi = (\omega+m\Omega)^2 + \frac{2m\lambda\Omega(r^2\Omega^2+r\partial_r\Phi-\mu\partial_\mu\Phi)}{r^2(\omega+m\Omega)} . \tag{A11}$$

The function Φ in these equations is the gravitational potential of the equilibrium stellar model.

The variational principle for the frequency, equation (26), has the following form when expressed in spherical coordinates:

$$\int r^2\delta\rho\,\delta U\,dr\,d\mu = \int\frac{\rho}{(\omega+m\Omega)^2}\left\{\lambda(1-\mu^2)(r\partial_r\delta U-\mu\partial_\mu\delta U)^2 + [\mu r\partial_r\delta U+(1-\mu^2)\partial_\mu\delta U]^2 + \frac{2m\lambda\Omega}{\omega+m\Omega}(r\partial_r\delta U^2-\mu\partial_\mu\delta U^2) + \lambda m^2\frac{(\omega+m\Omega)^2-\varpi\partial_\varpi\Omega^2}{(1-\mu^2)(\omega+m\Omega)^2}\delta U^2\right\}dr\,d\mu . \tag{A12}$$

This equation is to be thought of as an expression for the frequency of the mode, ω, in terms of a trial eigenfunction δU. This expression depends on the frequency both explicitly and implicitly through λ. For the case of uniformly rotating stellar models (i.e., $\partial_\varpi\Omega = 0$) this expression simplifies considerably and is equivalent to the quartic polynomial in ω presented in equations (31)–(36).

APPENDIX B

FINITE-DIFFERENCE APPROXIMATIONS FOR ANGULAR DERIVATIVES

In this appendix we derive the formulae that approximate the angular derivatives of functions whose values are known only at the discrete set of angles $\mu_i = \cos\theta_i$, which correspond to the zeros of the Legendre polynomial $P_{2L-1}(\mu_i) = 0$. We approximate the μ- derivatives of the function $f(r,\mu)e^{im\varphi}$ as the corresponding derivatives of its multipole expansion through order $2L-2+m$. Thus, if $f(r,\mu)$ can be represented as the sum of its multipole moments,

$$f(r,\mu) = \sum_{k=0}^{\infty} f_k(r)P^m_{k+|m|}(\mu) , \tag{B1}$$

then we approximate the nth μ-derivative of f as

$$\partial^n_\mu f(r,\mu) \approx \sum_{k=0}^{2L-2} f_k(r)\partial^n_\mu P^m_{k+|m|}(\mu) . \tag{B2}$$

The multipole moments, f_k, can be determined by multiplying both sides of equation (B1) by $P^m_{k+|m|}(\mu)$ and integrating. In the analysis that follows we obtain expressions for these integrals to determine $f_k(r)$ having better accuracy than could be achieved by performing the integrals directly using the Gaussian quadrature formula.

The associated Legendre functions with $m \geq 0$ may be expressed (using Rodriques's formula) in the following form:

$$P^m_l(\mu) = \frac{(1-\mu^2)^{m/2}}{2^l l!}\frac{d^{l+m}(1-\mu^2)^l}{d\mu^{l+m}} . \tag{B3}$$

A similar expression exists for $m < 0$, since $P_l^{-m}(\mu) = [(l-m)!/(l+m)!]P^m_l(\mu)$. It follows that P^m_l is a polynomial of order $l-|m|$ multiplied by $(1-\mu^2)^{|m|/2}$. These polynomials can be expressed as linear combinations of the Legendre polynomials, and conversely the Legendre polynomials can be expressed as linear combinations of the associated Legendre functions:

$$P_l(\mu) = (1 - \mu^2)^{-|m|/2} \sum_{k=0}^{l} b_{l,k} P^m_{k+|m|}(\mu) . \tag{B4}$$

The coefficients $b_{l,k}$ can be determined using the orthogonality of the P_l^m to be

$$b_{l,k} = \frac{(2k+1)(k+|m|-m)!}{2(k+|m|+m)!} \int_{-1}^{1} (1-\mu^2)^{|m|/2} P_l(\mu) P^m_{|m|+k}(\mu) d\mu . \tag{B5}$$

These integrals can be evaluated using a number of numerical techniques. The integrands are polynomials of order $2m + l + k$. They can be evaluated, therefore, using Gaussian quadrature on a grid having $l_{\max} + m + 1$ points. For our purposes here $l_{\max} = 2L - 2$, and so the integrations, if done directly, must use a finer grid than that used to evaluate the stellar structure and pulsations. (The integrations may be done indirectly by evaluating the inverse of the matrix $b_{l,k}$. The integrals involved in determining this inverse involve lower order polynomials which may be integrated directly on the standard coordinate grid having $2L - 1$ spokes.)

We now turn to the problem of evaluating the multipole coefficients f_k defined in equation (B1). Because of the relationships between the Legendre polynomials and the associated Legendre functions established above, equation (B1) could be reexpressed as an expansion in Legendre polynomials:

$$f(r, \mu) = (1 - \mu^2)^{|m|/2} \sum_{k=0}^{\infty} f_k^\dagger(r) P_k(\mu) . \tag{B6}$$

The moments $f_k^\dagger$ are related to the multipole moments f_k by

$$f_k = \sum_{n=k}^{\infty} f_n^\dagger b_{n,k} , \tag{B7}$$

and are given by the integrals

$$f_k^\dagger(r) = (k + \tfrac{1}{2}) \int_{-1}^{1} f(r, \mu)(1 - \mu^2)^{-|m|/2} P_k(\mu) d\mu . \tag{B8}$$

The integrands in equation (B8) are polynomials of lower order (by $2m$) than would have appeared in a direct determination of the multipole moments, f_k, and, consequently, they can be determined accurately to higher order using Gaussian quadrature on a given grid of points. If $f^{\pm}(r, \mu) = \pm f^{\pm}(r, -\mu)$ is a function of definite parity, then half of the integrals in equation (B8) are identically zero. If the function has even parity, $f^+(r, \mu)$, then the moments $f_k^\dagger$ will vanish unless k is even. The nonvanishing moments can be represented (using the Gaussian integration formula, eq. [37]) as finite sums over the L spokes having $\mu_i \geq 0$:

$$f_{2k}^\dagger = \sum_{i=1}^{L} \tfrac{1}{2}(4k+1) w_i (1 - \mu_i^2)^{-|m|/2} P_{2k}(\mu_i) f^+(r, \mu_i) . \tag{B9}$$

Similarly, for functions of odd parity, only the odd components of $f_k^\dagger$ are nonzero:

$$f_{2k+1}^\dagger = \sum_{i=1}^{L} \tfrac{1}{2}(4k+3) w_i (1 - \mu_i^2)^{-|m|/2} P_{2k+1}(\mu_i) f^-(r, \mu_i) . \tag{B10}$$

The expressions for the moments $f_k^\dagger$ in equations (B9) and (B10) are used with the known constants $b_{l,k}$ and equation (B7) (truncated at $n = 2L - 2$) to determine the multipole moments f_k in terms of the values of f on the fixed angular grid: $f(r, \mu_i)$. These expressions for the multipole moments are used with equation (B2) to derive the desired expressions for the angular derivatives. These expressions have the form

$$\partial_\mu f^{\pm}(r, \mu_i) \approx \sum_{j=1}^{L} D_{ij}^{\pm} f^{\pm}(r, \mu_j) , \tag{B11}$$

$$\left[(1 - \mu^2)\partial_\mu^2 - 2\mu\partial_\mu - \frac{m^2}{1 - \mu^2}\right] f^{\pm}(r, \mu_i) \approx \sum_{j=1}^{L} \Delta_{ij}^{\pm} f^{\pm}(r, \mu_j) . \tag{B12}$$

The constants $D_{ij}^{\pm}$ and $\Delta_{ij}^{\pm}$ are given by the following equations:

$$D_{ij}^+ = \sum_{k=0}^{L-1} \sum_{n=k}^{L-1} \tfrac{1}{2}(4n+1) w_j b_{2n,2k} (1 - \mu_j^2)^{-|m|/2} P_{2n}(\mu_j) \partial_\mu P^m_{2k+|m|}(\mu_i) , \tag{B13}$$

$$D_{ij}^- = \sum_{k=0}^{L-2} \sum_{n=k}^{L-2} \tfrac{1}{2}(4n+3) w_j b_{2n+1,2k+1} (1 - \mu_j^2)^{-|m|/2} P_{2n+1}(\mu_j) \partial_\mu P^m_{2k+1+|m|}(\mu_i) , \tag{B14}$$

$$\Delta_{ij}^+ = -\sum_{k=0}^{L-1} \sum_{n=k}^{L-1} \tfrac{1}{2}(4n+1)(2k+|m|)(2k+|m|+1) w_j b_{2n,2k} (1 - \mu_j{}^2)^{-|m|/2} P_{2n}(\mu_j) P^m_{2k+|m|}(\mu_i) , \tag{B15}$$

$$\Delta_{ij}^- = -\sum_{k=0}^{L-2} \sum_{n=k}^{L-2} \tfrac{1}{2}(4n+3)(2k+|m|+1)(2k+|m|+2) w_j b_{2n+1,2k+1} (1 - \mu_j^2)^{-|m|/2} P_{2n+1}(\mu_j) P^m_{2k+1+|m|}(\mu_i). \tag{B16}$$

The even-parity versions of these equations have been given previously by Managan (1986). The derivative $\partial_\mu P_l^m(\mu)$ that appears in equations (B13) and (B14) can be reduced to an expression involving only the associated Legendre functions (if desired) using the recursion relation $(1-\mu^2)\partial_\mu P_l^m(\mu) = -\mu l P_l^m(\mu) + (l+m)P_{l-1}^m(\mu)$.

To conclude this appendix, we derive the finite-difference form of the asymptotic boundary condition on the gravitational potential. This boundary condition, equation (30), is equivalent to

$$0 = \partial_r \delta\Phi^{\pm}(r, \mu) + \sum_{k=0}^{\infty} \frac{k+|m|+1}{r} \delta\Phi_k^{\pm}(r) P_{k+|m|}^m(\mu) . \tag{B17}$$

The multipole moments $\delta\Phi_k^{\pm}(r)$ are defined as in equation (B1). These moments can be expressed in terms of the values of $\delta\Phi^{\pm}(r, \mu_i)$ as in equations (B7)–(B10). It follows that this boundary condition can be written in the form

$$\partial_r \delta\Phi^{\pm}(r, \mu_i) + \frac{1}{r} \sum_{j=1}^{L} E_{ij}^{\pm} \delta\Phi^{\pm}(r, \mu_j) \approx 0 , \tag{B18}$$

where the constants $E_{ij}^{\pm}$ are given by the following expressions:

$$E_{ij}^{+} = \sum_{k=0}^{L-1} \sum_{n=k}^{L-1} \tfrac{1}{2}(4n+1)(2k+|m|+1) w_j b_{2n,2k} (1-\mu_j^2)^{-|m|/2} P_{2n}(\mu_j) P_{2k+|m|}^m(\mu_i) , \tag{B19}$$

$$E_{ij}^{-} = \sum_{k=0}^{L-2} \sum_{n=k}^{L-2} \tfrac{1}{2}(4n+3)(2k+|m|+2) w_j b_{2n+1,2k+1} (1-\mu_j^2)^{-|m|/2} P_{2n+1}(\mu_j) P_{2k+1+|m|}^m(\mu_i) . \tag{B20}$$

The even-parity version of this boundary condition was given previously by Managan (1986).

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